Steady-state Performance Analysis for Adaptive Filters with Error Nonlinearities

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***Abstract*—A unified approach to the steady-state mean square error (MSE) and tracking performance analyses for real and complex adaptive filtes with error nonlinearities is developed. Some general closed-form analytical expressions for the steady-state performances are given. Our analyses are based on Taylor series expansion and and so-called complex Brandwood-form series expansion (BSE). Under these general explicit expressions, some well-known adaptive filters can be viewed as special cases. In addition, the closed-form analytical expressions for the steady-state performance for real and complex least-mean *p*-power (LMP) algorithm with different choices of parameter *p* are also given. A mass of simulations show the accuration of our analyses.**

***Index Terms*—adaptive filters, steady-state analysis, mean-square error, tracking performance, Taylor series expression.**

I. Introduction

The performance of an adaptive filter is generally measured in terms of its transient behavior and its steady-state behavior. There have been numerous works in the literature on the performance of adaptive filters with many creationary results and approaches [1]-[9]. In most of these literatures, the steady-state performance is often obtained as a limiting case of the transient behavior. However, most adaptive filters are inherently nonlinear and time-variant systems. The nonlinearities in the update equations tend to lead to difficulties in the study of their steady-state performance as a limiting case of their transient performance [6]. Using the energy conservation relation, references [5, 6] rederived the steady-state performance for a large class of adaptive filters, such as LMS algorithm, LMMN algorithm, and so on, which bypassed the difficulties encountered in obtaining steady-state results as the limiting case of a transient analysis. While it is generally observed that most works for analyzing the steady-state performance study individual algorithms separately, the energy conservation approach of [5] allows for a unified framework that applies to different algorithms.

The main contribution in this article is to use the energy conservation approach [5] to derive a general expression for the steady-state performance of adaptive filters with error nonlinearities. Rather than obtaining a limiting case of the transient behavior, our analyses are based on Taylor series expansions and *so-called* complex Brandwood-form series expansion (BSE). BSE was derived by G. Yan in [7] under Brandwood's derivation operators with respect to the complex-valued variable and its conjugate, and was used to analyze the MSE for Bussgang algorithm (BA) in noiseless environments [8].

II. System Model

The stochastic gradient approach for adaptive filters with error nonlinearities can be modeled by [5, 6, 8, 9]

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \mu \mathbf{u}_i^* f\left(e_i, e_i^*\right), \tag{1}$$

$$e_i = d_i - \mathbf{u}_i \mathbf{w}_i, \tag{2}$$

$$d_i = \mathbf{u}_i \mathbf{w}_o + v_i, \tag{3}$$

where μ is step-size, $\mathbf{u}_i$ is row input vector, $\mathbf{w}_o$ is unknown column vector that we wish to estimate, d_i is scalar-valued noisy measurement, v_i accounts for both measurement noise and modeling errors, and $f\left(e_i, e_i^*\right)$ denotes memoryless nonlinearity function acting upon the error e_i and its complex conjugate e_i^*. Different choices for $f\left(e_i, e_i^*\right)$ result in different adaptive algorithms. For example, Table Ⅰ defines $f\left(e_i, e_i^*\right)$ for many well-known special cases of (1) with error nonlinearities [6].

TABLE I EXAMPLES FOR $f\left(e_i, e_i^*\right)$

Algorithms	$f\left(e_i, e_i^*\right)$
LMS	e_i
LMF	$\lvert e_i\rvert^2 e_i$
LMMN	$e_i\left(\delta + \bar{\delta}\lvert e_i\rvert^2\right)$
LMP	$\lvert e_i\rvert^{p-2} e_i$

III. Steady-state performance for adaptive filters

Define so-call *a priori* estimation error to be $e_a(i) = \mathbf{u}_i \tilde{\mathbf{w}}_i$, and $\tilde{\mathbf{w}}_i = \mathbf{w}_o - \mathbf{w}_i$ to be the weight-error vector. Under (2) and (3), the relation between e_i and $e_a(i)$ can be expressed as

$$e_i = e_a(i) + v_i. \tag{4}$$

The steady-state MSE for an adaptive filter can be written as $\zeta_{MSE} = \lim_{i\to\infty} \mathrm{E}\lvert e_i\rvert^2$. To get ζ_{MSE}, we use the following two assumptions:

A.1: *The noise sequence* $\{v_i\}$ *with zero-mean and variance* $\sigma_v^2 \neq 0$ *is independent and identically distributed (i.i.d) and statistically independent of the regressor sequence* $\{\mathbf{u}_i\}$.

A.2: *The a priori estimation error* $e_a(i)$ *with zero-mean is independent of* v_i. *And for complex-valued cases, it satisfies the circularity condition, namely,* $\mathrm{E}e_a^2(i) = 0$.

The above assumptions are popular, which are commonly used in the steady-state performance analyses for most of adaptive algorithms [5]. Then, the steady-state MSE can be rewritten as

$$\zeta_{MSE} = \sigma_v^2 + \zeta_{EMSE}, \tag{5}$$

where ζ_{EMSE} is the excess mean square error (EMSE), defined by

$$\zeta_{EMSE} = \lim_{i\to\infty} \mathrm{E}\left|e_a(i)\right|^2. \tag{6}$$

Observing from (5), we can find that getting ζ_{EMSE} is equivalent to getting the MSE.

A. Nonlinear equations of $e_a(i)$

In order to solve for the steady-state performance analysis for adaptive algorithms, we first introduce two nonlinear equations of $e_a(i)$ based on the energy conservation relation during an equalizer update. In a stationary environment, as described in [8, 9], the nonlinear equations of $e_a(i)$ can be written as

$$\mathrm{E}\varphi\left(e,e^*\right) = \mu \mathrm{E}\left[\|\mathbf{u}\|^2 q\left(e,e^*\right)\right], \tag{7}$$

where

$$\varphi\left(e,e^*\right) = 2\,\mathrm{Re}\left[e_a^* f\left(e,e^*\right)\right],\quad q\left(e,e^*\right) = \left|f\left(e,e^*\right)\right|^2. \tag{8}$$

Here, for the ease of reading, the index 'i' is omitted at steady state, that is

$$e_a(i) \to e_a, \mathbf{u}_i \to \mathbf{u}, v_i \to v,\ e_i \to e \qquad i \to \infty$$

Similarly, in a nonstationary environment, a wildly used first-order random-walk model is used to get the tracking performance [5, 6]. The model assumes that $\mathbf{w}_o$ appearing in (3) undergoes random variations of the form

$$\mathbf{w}_{o,i+1} = \mathbf{w}_{o,i} + q_i, \tag{9}$$

where q_i denotes some random perturbation.

A.3: *The stationary sequence $\{q_i\}$ is i.i.d., zero-mean, with covariance matrix $E\{q_i q_i^*\} = Q$, which is independent of the regressor sequences $\{\mathbf{u}_i\}$.*

As described in [5, 6], under A.3, we have

$$\mathrm{E}\varphi\left(e,e^*\right) = \mu^{-1}\mathrm{Tr}(Q) + \mu \mathrm{E}\left[\|\mathbf{u}\|^2 q\left(e,e^*\right)\right] \tag{10}$$

The above equation will be used to get the tracking performances. Comparing (7) with (10), we see that there are actually minor differences between mean-square analysis and tracking analysis.

B. Steady-state performance

At steady-state, since the behavior of e_a in the limit is likely to be less sensitive to the input data when the adaptive filter is long enough, the following assumption can be used to obtain the steady-state EMSE for adaptive filters, i.e.,

A.4: $\|\mathbf{u}\|^2$ *is independent of* e_a.

This assumption is referred to as the *separation principle* in [5]. Under A.4, (11) and (14) can be rewritten as

$$\mathrm{E}\varphi\left(e,e^*\right) = \mu \mathrm{Tr}(\mathbf{R}_u)\mathrm{E}q\left(e,e^*\right), \tag{11}$$

and

$$\mathrm{E}\varphi\left(e,e^*\right) = \mu^{-1}\mathrm{Tr}(Q) + \mu \mathrm{Tr}(\mathbf{R}_u)\mathrm{E}q\left(e,e^*\right), \tag{12}$$

respectively, where $\mathrm{Tr}(\mathbf{R}_u) = \mathrm{E}\|\mathbf{u}\|^2$.

Lemma 1 If $\varphi\left(e,e^*\right)$ is defined by (8), then

$$\varphi\left(v,v^*\right) = 0,\quad \varphi^{(2)}_{e,e^*}\left(v,v^*\right) = 2\,\mathrm{Re}\, f_e^{(1)}\left(v,v^*\right).$$

Here and through out this paper, $f_x^{(i)}(a)$ denotes the ith derivative of the function $f(x)$ with respect to x at the value $x = a$, and $f_{x,y}^{(i)}(a,b)$ denotes the ith partial derivative of the function $f(x,y)$ with respect to x and y at the value $(x = a, y = b)$.

Proof:

Using (8), we get $\varphi\left(v,v^*\right) = 2\,\mathrm{Re}\left[\left(e^* - v^*\right) f\left(e,e^*\right)\right]_{e=v,e^*=v^*} = 0$, and

$$\begin{aligned}
\varphi^{(2)}_{e,e^*}\left(e,e^*\right) &= \frac{\partial^2}{\partial e \partial e^*} 2\,\mathrm{Re}\left[e_a^* f\left(e,e^*\right)\right]\Big|_{e=v,e^*=v^*} \\
&= \frac{\partial}{\partial e^*}\left[(e-v)^* f_e^{(1)}\left(e,e^*\right) + f\left(e,e^*\right)^* + (e-v) f_e^{(1)}\left(e,e^*\right)^*\right]_{e=v,e^*=v^*} \\
&= f_e^{(1)}\left(e,e^*\right) + f_{e^*}^{(1)}\left(e,e^*\right)^* + (e-v)^* f^{(2)}_{e,e^*}\left(e,e^*\right) + (e-v) f^{(2)}_{e,e^*}\left(e,e^*\right)^*\Big|_{e=v,e^*=v^*} \\
&= 2\,\mathrm{Re}\, f_e^{(1)}\left(v,v^*\right).
\end{aligned}$$

Lemma 2 If $q\left(e,e^*\right)$ is defined by (8), then

$$q^{(2)}_{e,e^*}\left(v,v^*\right) = \left|f_e^{(1)}\left(v,v^*\right)\right|^2 + \left|f_{e^*}^{(1)}\left(v,v^*\right)\right|^2 + 2\,\mathrm{Re}\left[f^*\left(v,v^*\right) f^{(2)}_{e,e^*}\left(v,v^*\right)\right]$$

Proof:

Using (8), we get

$$\begin{aligned}
q^{(2)}_{e,e^*}\left(v,v^*\right) &= \frac{\partial}{\partial e^* \partial e}\left|f\left(e,e^*\right)\right|^2\Big|_{e=v,e^*=v^*} \\
&= \frac{\partial}{\partial e^*}\left[f_e^{(1)}\left(e,e^*\right) f\left(e,e^*\right)^* + f\left(e,e^*\right) f_e^{(1)}\left(e,e^*\right)^*\right]_{e=v,e^*=v^*} \\
&= \left|f_e^{(1)}\left(v,v^*\right)\right|^2 + \left|f_{e^*}^{(1)}\left(v,v^*\right)\right|^2 + 2\,\mathrm{Re}\left[f^*\left(v,v^*\right) f^{(2)}_{e,e^*}\left(v,v^*\right)\right].
\end{aligned}$$

Then, we can use the above two lemmas to get the steady-state performance for adaptive filters.

Theorem 1 Consider adaptive filters with error nonlinearities of the form (1)-(3), and suppose the assumptions A.1-A.4 are satisfied. Then, if the following condition is satisfied, i.e.,

C.1 $A > \mu B \mathrm{Tr}(\mathbf{R}_u)$,

the steady-state EMSE, tracking EMSE (TEMSE) and the optimal step-size for adaptive filters can be approximated by

$$\zeta_{EMSE} = \frac{\mu C \mathrm{Tr}(\mathbf{R}_u)}{A - \mu B \mathrm{Tr}(\mathbf{R}_u)}, \tag{13}$$

$$\zeta_{TEMSE} = \frac{\mu^{-1}\mathrm{Tr}(Q) + \mu C \mathrm{Tr}(\mathbf{R}_u)}{A - \mu B \mathrm{Tr}(\mathbf{R}_u)}, \tag{14}$$

$$\mu_{opt} = \sqrt{\left[\frac{B\mathrm{Tr}(Q)}{AC}\right]^2 + \frac{\mathrm{Tr}(Q)}{C\mathrm{Tr}(\mathbf{R}_u)}} - \frac{B\mathrm{Tr}(Q)}{AC}, \tag{15}$$

where

$$\begin{aligned}
&A = 2\,\mathrm{Re}\,\mathrm{E} f_e^{(1)}\left(v,v^*\right),\quad C = \mathrm{E}\left|f\left(v,v^*\right)\right|^2 \\
&B = \mathrm{E}\left|f_e^{(1)}\left(v,v^*\right)\right|^2 + \mathrm{E}\left|f_{e^*}^{(1)}\left(v,v^*\right)\right|^2 + 2\,\mathrm{Re}\,\mathrm{E}\left[f^*\left(v,v^*\right) f^{(2)}_{e,e^*}\left(v,v^*\right)\right]
\end{aligned} \tag{16a}$$

for complex-valued data cases, and

$$A = 2\mathrm{E} f_e^{(1)}(v),\ B = \mathrm{E}\left|f_e^{(1)}(v)\right|^2 + \mathrm{E}\left[f(v) f^{(2)}_{e,e}(v)\right],\ C = \mathrm{E}\left|f(v)\right|^2 \tag{16b}$$

for real-valued data cases, respectively.

Proof:

First, we consider the complex-valued cases. The complex BSE with respect to $\left(e_a, e_a^*\right)$ around $\left(v,v^*\right)$ for the function $\varphi\left(e,e^*\right)$ can be written as [8, 9]

$$\varphi(e,e^*)=\varphi(v,v^*)+\varphi_e^{(1)}(v,v^*)e_a+\varphi_{e^*}^{(1)}(v,v^*)e_a^* +\frac{1}{2}\left[\varphi_{e,e}^{(2)}(v,v^*)e_a^2+\varphi_{e^*,e^*}^{(2)}(v,v^*)(e_a^*)^2+2\varphi_{e,e^*}^{(2)}(v,v^*)|e_a|^2\right]+\mathrm{O}(e_a,e_a^*) \tag{17}$$

where $\mathrm{O}(e_a,e_a^*)$ denotes third and higher-power terms of e_a or e_a^*. Ignoring $\mathrm{O}(e_a,e_a^*)$ and taking expectation on both sides of (17), we get

$$\mathrm{E}\varphi(e,e^*)=\mathrm{E}\varphi(v,v^*)+\mathrm{E}\varphi_e^{(1)}(v,v^*)e_a+\mathrm{E}\varphi_{e^*}^{(1)}(v,v^*)e_a^* +\frac{1}{2}\mathrm{E}\left[\varphi_{e,e}^{(2)}(v,v^*)e_a^2+\varphi_{e^*,e^*}^{(2)}(v,v^*)(e_a^*)^2+2\varphi_{e,e^*}^{(2)}(v,v^*)|e_a|^2\right] \tag{18}$$

Under A.1 and A.2, (i.e. $\{v,e_a\}$ are mutually independent, and $\mathrm{E}e_a=\mathrm{E}e_a^2=0$) and using lemma 1, we obtain

$$\mathrm{E}\varphi(e,e^*)=A\zeta_{EMSE}, \tag{19}$$

where A is defined by (16a) and ζ_{EMSE} is defined by (6). Similarly, replacing $\varphi(e,e^*)$ in (18) by $q(e,e^*)$, we have

$$\mathrm{E}q(e,e^*)=\mathrm{E}q(v,v^*)+\mathrm{E}q_{e,e^*}^{(2)}(v,v^*)\zeta_{EMSE}. \tag{20}$$

Substituting (20) into the right-hand side of (11), and using Lemma 2, we have

$$\mu\mathrm{Tr}(\mathbf{R}_u)\mathrm{E}q(e,e^*)=\mu\mathrm{Tr}(\mathbf{R}_u)(C+B\zeta_{EMSE}), \tag{21}$$

where B and C are defined by (16a). Then, substituting (19) and (21) into (11), we have

$$[A-\mu B\mathrm{Tr}(\mathbf{R}_u)]\zeta_{EMSE}=\mu C\mathrm{Tr}(\mathbf{R}_u). \tag{22}$$

Since $\mu C\mathrm{Tr}(\mathbf{R}_u)\geq 0$ and $\zeta_{EMSE}\geq 0$, if the condition C.1 is satisfied, i.e., $A>\mu B\mathrm{Tr}(\mathbf{R}_u)$, we can obtain (13) for complex-valued cases.

Next, we consider the real-valued cases. Since $\{\mathbf{u},v,e_a,e\}$ are real-valued data, the equality (11) can be simplified to

$$2\mathrm{E}[e_a f(e)]=\mu\mathrm{Tr}(\mathbf{R}_u)\mathrm{E}|f(e)|^2. \tag{23}$$

The Taylor series expansion for the real estimation error signal $f(e)$ with respect to e around v can be written as

$$f(e)=f(v)+f_e^{(1)}(v)e_a+\frac{1}{2}f_{e,e}^{(2)}(v)e_a^2+\mathrm{O}(e_a), \tag{24}$$

where $\mathrm{O}(e_a)$ denotes third and higher-power terms of e_a. Substituting (24) into the left-hand side of (23) yields

$$2\mathrm{E}[e_a f(e)]=2\mathrm{E}\left[f(v)e_a+f_e^{(1)}(v)e_a^2+\mathrm{O}(e_a)\right]. \tag{25}$$

Under A.1and A.2, and neglecting $\mathrm{EO}(e_a)$, we have

$$2\mathrm{E}[e_a f(e)]=A\zeta_{EMSE}, \tag{26}$$

where A is defined by (16b). Similarly, substituting (24) into the right-hand side of (23) yields

$$\mu\mathrm{Tr}(\mathbf{R}_u)\mathrm{E}|f(e)|^2=\mu\mathrm{Tr}(\mathbf{R}_u)(C+B\zeta_{EMSE}), \tag{27}$$

where B and C are defined by (16b). Then, substituting (26) and (27) into (23), we have (22). Then, if the condition C.1 is satisfied, we can obtain (13) for real-valued cases.

Finally, substituting (19) and (21) into (12) for complex-valued cases, or substituting (26) and (27) into (12) for real-valued cases, we have

$$[A-\mu B\mathrm{Tr}(\mathbf{R}_u)]\zeta_{TEMSE}=\mu^{-1}\mathrm{Tr}(Q)+\mu C\mathrm{Tr}(\mathbf{R}_u). \tag{28}$$

Here, ζ_{EMSE} is replaced by ζ_{TEMSE}. Since $\mu^{-1}\mathrm{Tr}(Q)>0$, we can obtain (14) if the condition C.1 is satisfied. Differentiating both-hand sides of (14) with respect to μ, and letting it be zero, we get

$$\frac{\partial}{\partial\mu}\zeta_{TEMSE}\bigg|_{\mu=\mu_{opt}}=\frac{\partial}{\partial\mu}\left[\frac{\mu^{-1}\mathrm{Tr}(Q)+\mu C\mathrm{Tr}(\mathbf{R}_u)}{A-\mu B\mathrm{Tr}(\mathbf{R}_u)}\right]_{\mu=\mu_{opt}}=0. \tag{29}$$

Simplifying the above equation, we have

$$\mu_{opt}^2+\frac{2B\mathrm{Tr}(Q)}{AC}\mu_{opt}-\frac{\mathrm{Tr}(Q)}{C\mathrm{Tr}(\mathbf{R}_u)}=0. \tag{30}$$

Solving the above equality, we can obtain the optimum step-size expressed by (15). Here, we use the fact $\mu>0$. This ends the proof of Theorem 1.

Remarks:

1）Substituting (15) into (14) yields the minimum steady-state TEMSE.

2）In view of the step-size μ being very small, the expression (13) ~ (15) can be simplified to

$$\zeta_{EMSE}=\frac{C}{A}\mu\mathrm{Tr}(\mathbf{R}_u), \tag{31}$$

$$\zeta_{TEMSE}=\frac{\mu^{-1}\mathrm{Tr}(Q)+\mu C\mathrm{Tr}(\mathbf{R}_u)}{A}, \tag{32}$$

$$\mu_{opt}=\sqrt{\frac{\mathrm{Tr}(Q)}{C\mathrm{Tr}(\mathbf{R}_u)}}. \tag{33}$$

Substituting (33) into (32) yields the minimum steady-state TEMSE

$$\zeta_{\min}=\frac{2}{A}\sqrt{C\mathbf{Tr}(\mathbf{R}_u)\mathrm{Tr}(Q)}. \tag{34}$$

3）For real-valued cases, the steady-state EMSE expression (31) is the same as the result (see e.g. Eq. 35) in [4].

IV. STEADY-STATE PERFORMANCE FOR THE SPECIAL CASES OF ADAPTIVE FILTERS

In this section, based on Theorem 1 and Theorem 2 in Section Ⅱ, we will investigate the steady-state performances for the least-mean p-order norm (LMP) algorithm [11] with different choices of parameter p and the least-mean mixed norm (LMMN) algorithm, respectively.

A. LMP algorithm

The estimation error of LMP algorithm can be expressed as [11]

$$f(e,e^*)=|e|^{p-2}e=(ee^*)^{(p-2)/2}e \tag{35}$$

where $p>0$ is a positive integral. $p=2$ results in well-known LMS algorithm, and $p=4$ results in LMF algorithm.

Substituting (35) into (16a) and (16b), respectively, we get

$$A=2(p-1)\xi_v^{p-2},\ B=(p-1)(2p-3)\xi_v^{2p-4},\ C=\xi_v^{2p-2}, \tag{36a}$$

for real-valued cases, and

$$A=p\xi_v^{p-2},\ B=(p-1)^2\xi_v^{2p-4},\ C=\xi_v^{2p-2}, \tag{36b}$$

for complex-valued cases, where $\xi_v^k=\mathrm{E}|v|^k$. Then, under Theorem 1, the condition C.1 becomes

$$\mu\mathrm{Tr}(\mathbf{R}_u)\xi_v^{2(p-2)}<\gamma\xi_v^{p-2}, \tag{37}$$

where $\gamma=2/(2p-3)$ for real-valued cases, and $\gamma=p/(p-1)^2$ for complex-valued cases, and the steady-state performance for LMP algorithm can be obtained. Here, we only give the expression for EMSE,

$$\zeta_{EMSE}=\begin{cases}\dfrac{\mu\mathrm{Tr}(\mathbf{R}_u)\xi_v^{2p-2}}{2(p-1)\xi_v^{p-2}-(p-1)(2p-3)\mu\mathrm{Tr}(\mathbf{R}_u)\xi_v^{2(p-2)}}, & \text{real}\\[2ex] \dfrac{\mu\mathrm{Tr}(\mathbf{R}_u)\xi_v^{2p-2}}{p\xi_v^{p-2}-(p-1)^2\mu\mathrm{Tr}(\mathbf{R}_u)\xi_v^{2(p-2)}}, & \text{complex}\end{cases}. \tag{38}$$

For LMS algorithm and LMF algorithm, substituting $p = 2,4$ into (38), yields the same steady-state performance results (see e.g. Lemma 6.5.1 and Lemma 6.8.1) in [5].

B. LMMN algorithm

The estimation error of LMMN algorithm is [1, 5]

$$f\left(e, e^*\right) = e\left(\delta + \bar{\delta}|e|^2\right), \tag{39}$$

where $0 \le \delta \le 1$ and $\bar{\delta} = 1 - \delta$. Substituting (39) into (16a) and (16b), respectively, we have

$$\begin{aligned} A &= 2\left(\delta + k_0\bar{\delta}\xi_v^2\right) \\ B &= \delta^2 + k_1\delta\bar{\delta}\xi_v^2 + k_2\bar{\delta}^2\xi_v^4 \\ C &= \delta^2\xi_v^2 + 2\delta\bar{\delta}\xi_v^4 + \bar{\delta}^2\xi_v^6 \end{aligned} \tag{40}$$

where $k_0 = 3, k_1 = 12, k_2 = 15$ for real-valued cases $k_0 = 2, k_1 = 8, k_2 = 9$ for complex-valued cases. Then, substituting $A = 2b`, C = a`, B = c`$ or $A = 2b, C = a, B = c$ into (13) - (15) yields the steady-state performance for complex and real LMMN algorithm, which coincides with the results (see e.g. Lemma 6.8.1 and Lemma 7.8.1) in [5].

V. SIMULATION RESULTS

In previous section, some well-known real and complex adaptive algorithms, such as LMS algorithm, LMF algorithm and LMMN algorithm, have shown the accuracy of the corresponding analysis results. In this section, we only give the computer simulation for the steady-state performance of real LMP algorithm with $p = 3$, which has not been involved in the previous literatures. In all the cases, an 11-tap LMP filter with tap-centered initialization is used. The variance of Gaussian noise is set $\sigma_v^2 = 0.001$, and the regressors $\{\mathbf{u}_i\}$ are generated by feeding correlated data into a tapped delay time, i.e., $u(i+1) = au(i) + \sqrt{1-a^2}s(i)$,where $\mathbf{u}_i = [u(i), u(i-1), \cdots, u(i-L+1)]$, and s_i is a unit-variance i.i.d. Gaussian random process. Here, we set $a = 0.8$.

For the different choices of step-size, Fig.1 compares the simulated and theoretical MSE results and Fig.2 compares the simulated and theoretical TMSE results with $\sigma_q = 2e-5$, where $Q = \sigma_q^2\mathbf{I}$. From these two figures, we can see that the simulated and theoretical results are matched reasonable well. In addition, Observing from the tracking figure (Fig. 2), we can find that these minimum value is in good agreement with the corresponding theoretical values, which is $\mu_{\text{opt}} = 0.0058$.

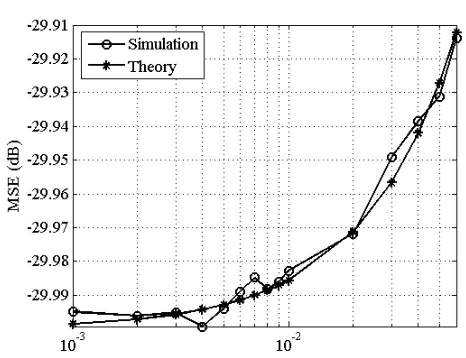

Fig. 1. Theoretical and simulated MSE curves

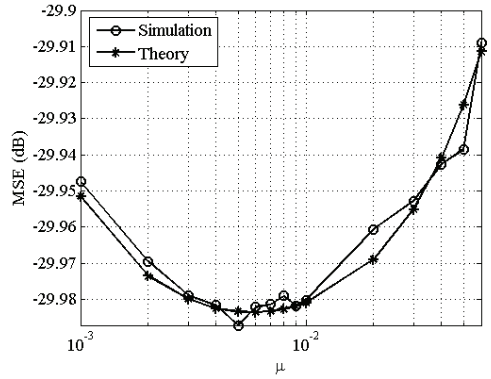

Fig. 2. Theoretical and simulated tracking MSE curves.

VI. CONCLUSIONS

This paper develops a unified approach for the steady-state performance analyses of adaptive filters with error nonlinearities based on Taylor series expansion and complex Brandwood-form series expansion. Some general closed-form analytical expressions for the steady-state performances are derived. Under these expressions, the proposed results for some well-known adaptive filters are the same as the results summarized by A. H. Sayed in [5]. In addition, the closed-form analytical expressions for the steady-state performance for real and complex LMP algorithm with different parameter *p* choices are also derived. Computational simulations show the accuration of our analyses.

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