APPLICATION OF CHARACTERISTIC FUNCTION TO DETECTION IN SINUSOIDAL INTERFERENCE PLUS GAUSSIAN NOISE

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ABSTRACT

In this work, a detector scheme for the detection of signal in a group of non-Gaussian narrowband interferences and white Gaussian noise is developed. Since there exists no closed-form probability distribution for this type of disturbance modeling, the key innovation lies in the use of characteristic function rather than the probability distribution to both design and implement the detector. Parameter estimation is performed at first step to find the unknown disturbance parameters. The utilized detector uses these parameters to form an approximately Gaussian distributed test statistic based on the empirical characteristic function of received data. Performance of the detector is investigated by means of both analytical and Monte Carlo simulations.

***Index Terms*—** Estimation, Detection in non-Gaussian noise, Interference suppression, characteristic function.

1. INTRODUCTION

Coherent signal detection in a mixture of noise and other disturbances is presently of much interest in various radar applications. Many traditionally designed detectors use an uncorrelated Gaussian model in order to accomplish the detection problem, whether or not the disturbance is actually normally distributed [1, 2]. However, there are many radar applications in which this sort of modeling may not be realistic. Hence, non-Gaussian modeling is currently under investigation for applications where the Gaussianity assumption of disturbance may not be applicable. The main focus of this work is to consider the problem of coherent signal detection in non-Gaussian disturbance specified by a group of narrowband sinusoidal interferences. We first motivate the interference model with results from Helstrom [3]. Unless under very simple assumptions, there exists no closed-form probability density function (PDF) for this type of data modeling. Thus, in order for the coherent detection of signal to be performed, an ordinary likelihood ratio test based upon the received data PDF, would not be applicable. In this regard, J. Ilow and D. Hatzinakos [4] introduced a novel detector based on the empirical characteristic function (ECF) which exploits the correlation structure of the ECF evaluated at a finite number of points, by some approximation. The considered detector is extensively useful in detection problems for which standard approaches based upon the PDF of data are inaccurate or difficult to implement. Our aim here is to first derive a closed-form expression for the characteristic function (CF) of observations and then estimate its unknown parameters via two proper estimation methods based upon the ECF of a secondary set of data. The estimated parameters are further utilized in the design of a discrete-time detector which in turn incorporates the ECF of received primary data in order to form the desired test statistic. We then analyze and optimize the performance of the designed detector and compare it with some other conventional detectors.

The paper is organized as follows. In section II, a closed-form expression for the characteristic function of the considered narrowband disturbance is obtained and estimation of the corresponding unknown parameters is performed. The detector structure is derived in section III. Section IV is devoted to performance assessment of the proposed detector and some further simulation results. Concluding remarks are drawn in section V.

2. DISTURBANCE MODEL AND ESTIMATION OF ITS UNKNOWN PARAMETERS

In this section, we first introduce a suitable model for the characteristic function of data samples and then estimate its unknown parameters using two estimation procedures. As described in Helstrom [3], a certain narrowband process has the form $\Re\{v'(t)\exp(j\Omega t)\}$, with Ω the carrier frequency and $v'(t)$ the complex envelope of the process. Sampling the envelope at time t and considering the real part yields the following random variable

$$V = A\cos(\theta_0) + \sum_{k=1}^{M} b_k \cos(\theta_k) + N. \tag{1}$$

Here A is the signal amplitude, θ_0 the signal phase that we can assume here to be equal to zero, the M sinusoidal interference amplitudes $\{b_k\}$ are independent real random sequences,

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the phases $\{\theta_k\}$ are iid random variables and uniformly distributed over $[0, 2\pi]$ and N is a sample of narrowband zero-mean white Gaussian noise with variance equal to σ^2. The first term in (1) we call the signal, the second and third terms the sum of sinusoidal interferences and noise. Helstrom [3] stated that the CF, i.e. the Fourier transform of PDF, of variable V in (1) can be expressed as

$$\begin{aligned}\Phi(u) &= \mathbb{E}\{\exp(jVu)\} \\ &= J_0(Au)\exp(-\frac{\sigma^2 u^2}{2})\prod_{k=1}^{M}\mathbb{E}\{J_0(b_k u)\}\end{aligned} \quad (2)$$

where $J_0(\cdot)$ represents the zeroth order Bessel function and $\mathbb{E}\{\}$ indicates an expected value with respect to the distribution of interference amplitudes. By assuming the independence of interference and noise, and that the interference amplitudes are independent zero-mean Normal variables with variances equal to σ_b^2, (2) is derived as

$$\Phi(u) = J_0(Au)\exp\left(-\frac{\sigma^2 u^2}{2} - \frac{\sigma_b^2 M u^2}{4}\right) I_0^M(\frac{\sigma_b^2 u^2}{4}) \quad (3)$$

where $I_0(\cdot)$ is the zeroth order modified Bessel function of first kind. Here, A, σ, σ_b and M are considered as unknown deterministic parameters and are to be estimated via suitable methods. During the estimation part of the problem, we can assume A to be equal to zero since we use a set of secondary data to perform the estimation. In order to estimate the unknown parameters $\mathbf{P} = [\sigma, \sigma_b, M]$, we use two different methods both of which are based on minimization of a sum of squared errors, the ECF based and the moment based methods. In the former, the cited error is defined as difference between the CF expression in (3) and the ECF $\widehat{\Phi}_N(u) = \frac{1}{N}\sum_{n=1}^{N}\exp(jx_n u)$ obtained from secondary data $\{x_n\}$ in some specific frequencies $u_k\,'s$, i.e. we have

$$\min_{\mathbf{P}}\{\sum_{k=1}^{K}\|\Phi_{\mathbf{P}}(u_k) - \widehat{\Phi}_N(u_k)\|^2\} \quad (4)$$

where K is the total number of frequencies considered in the least squares method, N the total number of observations $\{x_n\}$, $\mathbf{P} = \{\sigma, \sigma_b, M\}$ the set of parameters under estimation and $\|()\|$ indicates complex Euclidean norm. In the moment-type method, we use approximations for logarithm functions of $\Phi_{\mathbf{P}}$ and $\widehat{\Phi}_N$ in (4) in order to decrease the computational burden of the minimization algorithm. In this regard, we have

$$\min_{\mathbf{P}}\{\sum_{k=1}^{K}\|\ln\Phi_{\mathbf{P}}(u_k) - \ln\widehat{\Phi}_N(u_k)\|^2\} \quad (5)$$

and with the help of Taylor series expansion we obtain for $\widehat{\Phi}_N(u)$

$$\widehat{\Phi}_N(u) = \sum_{m=0}^{\infty}\frac{\mu_m (ju)^m}{m!}, \quad \mu_m \simeq \frac{1}{N}\sum_{n=1}^{N}x_n^m \quad (6)$$

Now using Taylor series expansions of $\ln(\Phi_N(u))$ and $\ln(\widehat{\Phi}_N(u))$ and considering the first few terms, we can approximately express (5) as

$$\min_{\mathbf{P}}\sum_{k=1}^{K}\left\|\sum_{m=0}^{r}c_m u_k^m - \sum_{m=0}^{r}\hat{c}_m(ju_k)^m\right\|^2 \quad (7)$$

Here for both methods, we perform an unconstrained optimization by a gradient based routine.

3. DETECTION METHOD

We consider the problem of discrete-time binary detection for integrated square pulses in the disturbance model discussed in previous section. The hypothesis testing problem considered herein is relevant to that in active radar and sonar where it can be assumed that

$$\begin{cases}\mathbf{H_0}: & r_k = x_k \\ \mathbf{H_1}: & r_k = A + x_k\end{cases} \quad (8)$$

with $k = 1, 2, \cdots, N$. The $\mathbf{H_0}$ hypothesis corresponds to only the presence of disturbances and the $\mathbf{H_1}$ hypothesis to the presence of both target and disturbances. The perturbation components $x_k\,'s$, are univariate independent samples each of which has an unknown PDF but the corresponding CF $\Phi_{\mathbf{P}}(u)$ obtained before. We assume the parameters $\mathbf{P} = \{\sigma, \sigma_b, M\}$ to be estimated beforehand. Now, according to [4], the detection problem can be viewed as one of deciding between these two new hypotheses

$$\begin{cases}\mathbf{H_0}: & W_k = z_k \\ \mathbf{H_1}: & W_k = Au_k + z_k\end{cases} \quad (9)$$

with $k = 1, 2, \cdots, m$. Here we have

$$W_k = \arctan\left\{\frac{I_N(u_k)}{R_N(u_k)}\right\} \quad (10)$$

where $R_N(u_k)$ and $I_N(u_k)$ are respectively the real and imaginary parts of the ECF calculated using N points of data sequence. From a statistical point of view, the joint distribution of $Z_k'\ s$ are not known in general. However, from [4], it can be assumed that for rather large number of integrated pulses N, Z $= [z_1, z_2, \cdots, z_m]^T$ is approximately an m-variate zero-mean Normal random vector with covariance matrix equal to Σ_m. Under this assumption, the problem of hypothesis testing considered in (9) can be described as detection in colored Gaussian noise for the new set of data $\mathbf{W}$ and from [6] the test becomes

$$\mathbf{U}^T\Sigma_m^{-1}\mathbf{W} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{A}{2}\mathbf{U}^T\Sigma_m^{-1}\mathbf{U} \quad (11)$$

Here for a fixed value of m, we have the design problem of covariance matrix Σ_m and the frequency vector $\mathbf{U}$. Since finding an exact expression for the covariance matrix Σ_m is so complicated, here we use an approximate approach. Based on results from [4], it can be shown that the covariance function would be approximated as

$$cov(z_i, z_j) = \frac{1}{2N} \frac{\Phi_{\mathbf{P}}(i-j) - \Phi_{\mathbf{P}}(i+j)}{\Phi_{\mathbf{P}}(i)\Phi_{\mathbf{P}}(j)} \tag{12}$$

By using Equation (3) for $\Phi_{\mathbf{P}}(u)$, equation (12) can be expressed as

$$\begin{aligned} cov(z_i, z_j) = \frac{1}{2N} I_0^{-M}(\mathbf{H}_i) I_0^{-M}(\mathbf{H}_j) \left\{ \exp(\mathbf{F}_{i,j}) I_0^M(\mathbf{L}_{i,j}) \right. \\ \left. - \exp(-\mathbf{F}_{i,j}) I_0^M(\mathbf{G}_{i,j}) \right\} \end{aligned} \tag{13}$$

where we have defined

$$\begin{cases} \mathbf{H}_i = \frac{\sigma_b^2 u_i^2}{4} \\ \mathbf{F}_{i,j} = \sigma^2 u_i u_j + \frac{\sigma_b^2 M u_i u_j}{2} \\ \mathbf{G}_{i,j} = \frac{\sigma_b^2 (u_i + u_j)^2}{4} \\ \mathbf{L}_{i,j} = \frac{\sigma_b^2 (u_i - u_j)^2}{4} \end{cases} \tag{14}$$

Once the covariance matrix Σ_m is calculated from above, we are into obtaining the set of frequencies $\{u_1, u_2, \cdots, u_m\}$. In general, there are no limitations on choice of u_k $'s$, however, an arbitrary selection of u_k $'s$ may not result in acceptable detector performance. Here, we choose u_k s such that the analytically obtained error probability P_e of the detector be minimized with respect to these frequencies. From [6] P_e can be approximated as

$$P_e = 0.5 \left[Q(\frac{\tau}{d}) + 1 - Q(\frac{\tau - \mu}{d}) \right] \tag{15}$$

where by definition

$$\begin{cases} \tau = A\mathbf{U}^T \Sigma_m^{-1} \mathbf{U}/2 \\ \mu = A\mathbf{U}^T \Sigma_m^{-1} \mathbf{U} \\ d^2 = \mathbf{U}^T \Sigma_m^{-1} \mathbf{U} \end{cases} \tag{16}$$

and $Q(\cdot)$ denotes the Q-function. We will demonstrate the analytical and Monte Carlo simulation results in section IV to support the equations derived and make comparisons.

4. SIMULATIONS AND FURTHER RESULTS

We present in this section analytical and Monte Carlo simulations to assess the performance of the estimation and detection methods described. In all of our simulations we use rectangular pulses with amplitude $A = 1$ and unless otherwise stated, the number of integrated pulses in detection N equals 10. Firstly, error performance of our estimation methods is investigated. Fig. 1 illustrates the mean square error (MSE) of the estimated parameters σ and σ_b for the ECF based (exact) and moment based (approximate) methods, which are respectively named as EBM and MBM. It is obvious that the MSE of the EBM is relatively smaller than that of the MBM. This is of course at the cost of a larger computational burden in the minimization algorithm. We observe that the approximate method leads into reasonable errors only for rather small absolute values of parameters. Fig. 2 is indicative of the advantage of ECF based detector, implemented for single frequency case (m=1), over other conventional detectors in case of error performance for almost all practical values of signal to interference and noise ratio (SINR). The linear detector or the matched filter is simply a weighted sum of input data points and is optimum only in the case of Normally distributed data. In order for the implementation of optimum detector, we are in need of an analytical closed-form PDF. Since in our case of study there is no closed-form PDF at hand, we use the Gram-Charlier A series and Edgeworth series to approximate the disturbance distribution [7]; These methods result in approximately optimum or suboptimum detectors and are thus respectively named as Suboptimal Detectors (1) and (2). Further study can be performed to analyze and optimize the performance of multiple frequency ($m > 1$) detectors since the evaluation of the optimum frequencies in this case may require sophisticated optimization procedures. In Fig. 3, we plotted the theoretical probability of error from (15) using the actual values of CF parameters in comparison with the Monte Carlo simulated P_e obtained from our detection algorithm which uses the estimated set of parameters. This shows the accuracy of our estimation methods and also the validity of the Gaussianity assumption for test statistic in (11). Fig. 4 shows the values of optimum detector frequency (for m=1) versus signal to interference and noise ratio for some different number of interferences M. For small values of M (values less than 5), choice of optimum detector frequency is rather insensitive to M and thus we can independently set the optimum detector frequency without exact knowledge of M.

5. CONCLUSIONS

The basic motivation for research in this work is to address non-Gaussian estimation and detection problems for which closed-form probability density functions does not exist or are too complex to implement. First we derived a suitable model for the characteristic function of data consisting of a sum of non-Gaussian sinusoidal interferences and Gaussian noise and estimated its unknown parameters via two different approaches. Then we used this model to form a detector based on the empirical characteristic function in order to perform the binary hypothesis testing. We analyzed the error performance for the single frequency detector and through Monte Carlo simulations, we demonstrated that it outperforms other conventional receivers. The estimation and detection schemes utilized in this paper are applicable to any disturbance model

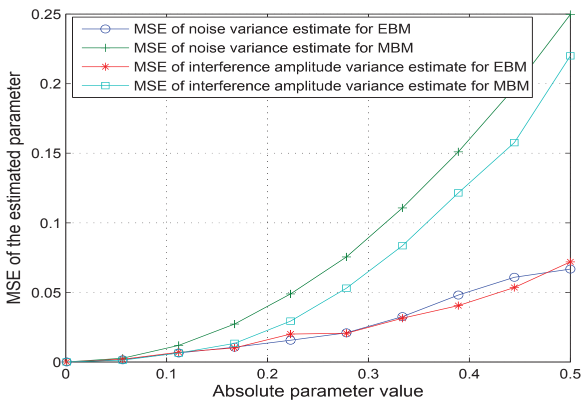

Fig. 1. Mean square errors of the estimated noise and interference variances σ, σ_b versus their actual values.

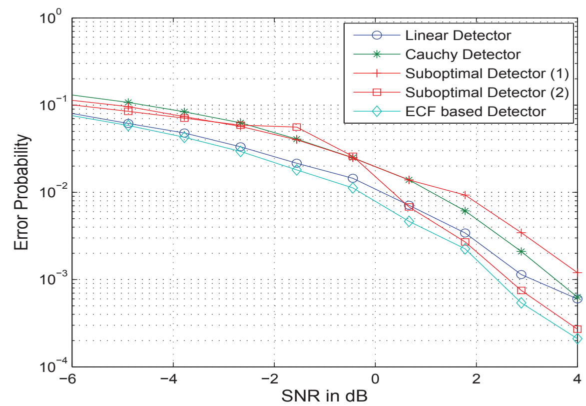

Fig. 2. Error performance of the ecf based detector and other conventional detectors for $M = 5$ versus SINR.

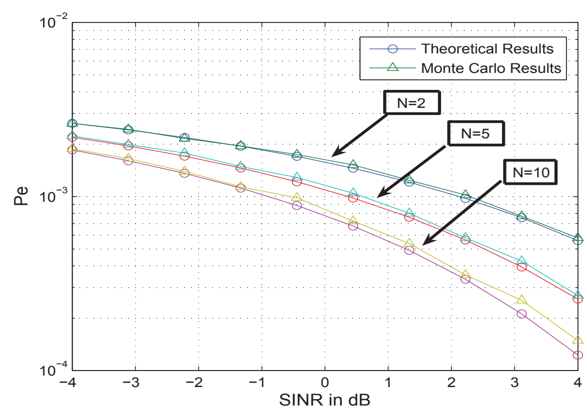

Fig. 3. Theoretical and Monte Carlo simulated error probabilities for $M = 5$ and different values of N.

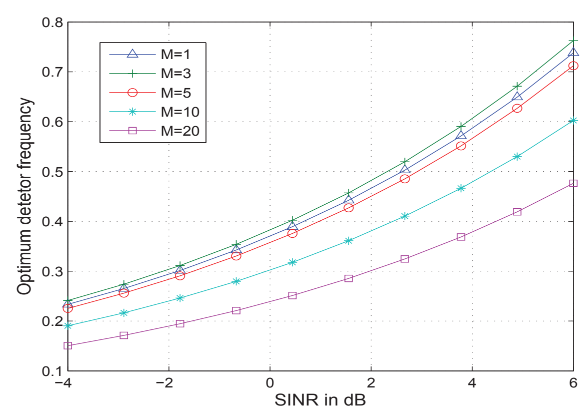

Fig. 4. Optimum detector frequencies for various values of M versus SINR.

with an analytical closed-form for the characteristic function and are robust.

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