INSTANTANEOUS FREQUENCY RATE ESTIMATION FOR HIGH-ORDER POLYNOMIAL-PHASE SIGNAL

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ABSTRACT

For a high-order polynomial-phase signal (PPS), instantaneous frequency rate (IFR), which is defined as the second derivative of the phase, is estimated by using an estimator with only a second-order nonlinearity. Compared to high-order phase function (HPF), the proposed IFR estimator presents improved performance including smaller mean-squared error (MSE) and lower SNR threshold. Statistical analysis via a multivariate first-order perturbation analysis is derived for the estimate bias and MSE. Numerical results verify our analytical results.

Index Terms- Polynomial phase signal, parameter estimation.

1. INTRODUCTION

In radar, sonar, and communications, instantaneous frequency rate (IFR) reveals the rate of change in the velocity, i.e., acceleration, of a moving target [1]. Estimation of the IFR is encountered in many cases. In synthetic aperture radar (SAR), modeling of echoes by incorporating time-varying acceleration was considered in [2]. A detailed investigation of the influence of the target time-varying acceleration on a ground moving-target indication was presented in [2]. For a polynomial-phase signal (PPS),

$$s(n) = Ae^{j\phi(n)} = Ae^{j\sum_{i=0}^{p} a_{i}n^{i}},$$
(1)

where p is the known order of the PPS, A is the constant amplitude, $\phi(n)$ is the instantaneous phase (IP) and $\{a_i\}_{i=0}^p$ are unknown phase parameters, respectively, the IFR, denoted as $\Omega(n)$, is defined as the second-order derivative of the IP [3]:

$$\Omega(n) = \frac{d^2 \phi(n)}{dt^2} = \sum_{i=2}^p i(i-1)a_i n^{i-2}.$$
 (2)

Depending on the order of the PPS, there are generally three cases to address the IFR:

• p = 2 (i.e, a linear frequency-modulated (FM) signal): the IFR reduces to the well-known chirp-rate [4];

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- p = 3 (i.e., a quadratic FM signal): the IFR is linearly proportional to time n (cf.(2)), and can be estimated by cubic phase function (CPF) [3];
- p > 3 (i.e., a high-order PPS): the IFR is generally a nonlinear function of time n (cf.(2)), and can be estimated by high-order phase function (HPF) [5]:

$$H_q(n,\omega) = \sum_{m=-M}^{M} \prod_{l=1}^{q/2} \left[s(n+d_l m) s(n-d_l m) \right]^{(r_l)} e^{-j\omega m^2},$$

where q is the order of the HPF, 2M + 1 is the window length, d_l denotes the lag-coefficient, r_l is used to impose complex conjugation if $r_l = 1$, and ω denotes the index in the IFR domain. Note that the HPF with q = 2 reduces to the CPF [3]. For the high-order PPS, the HPF order q and two coefficient sets d and r are chosen to assure the HPF is centered along the IFR of the signal (see Proposition 1 of [6]).

For the high-order PPS, the HPF is generally involved in highorder nonlinearity. For example of a fourth-order PPS, the HPF order should be greater than six, i.e., $q \ge 6$. This high-order nonlinearity results in high mean-squared error (MSE) and high SNR threshold for the IFR estimate. In this paper, an IFR estimator with only a second-order nonlinearity is proposed for the high-order PPS. Analytical results via a multidimensional first-order perturbation analysis show that the proposed IFR estimator is asymptotically unbiased and presents lower MSE than the HPF-based IFR estimates at high SNR. Moreover, it provides much lower SNR threshold than the HPF-based one for the high-order PPS.

The rest of this paper is organized as follows. Section 2 introduces the proposed IFR estimator and provides the asymptotic bias and MSE of the proposed estimator. Numerical results and conclusions are provided in Section 3 and Section 4.

2. PROPOSED IFR ESTIMATOR

2.1. IFR Estimation for High-Order PPS

To avoid the highly nonlinear transformation of the HPF, we consider a bilinear transformation s(n+m)s(n-m) for a PPS with an arbitrary order p and, using the binomial expansion, observe that

$$s(n+m)s(n-m) = A^{2}e^{j2\phi(n)+j\sum_{l=1}^{L}\frac{2\phi(2l)(n)m^{2l}}{(2l)!}}$$

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where $L = \lfloor p/2 \rfloor$ and $\phi^{(2l)}(n)$ denotes the 2*l*-th derivative of the $\phi(n)$. It is seen that the resulting phase is a polynomial in m with even-order, and each even-order term in m is associated with the corresponding even-order derivative of the IP up to a constant, including the second derivative of the IP which is the IFR $\Omega(n) = \phi^{(2)}(n)$. In order to obtain these phase derivatives, we apply a multidimensional match filter in m to the above bilinear transformation

$$B_L(n,\Psi) = \sum_{m=-M}^{M} s(n+m) s(n-m) e^{-j \sum_{l=1}^{L} \omega_l m^{2l}}, \quad (3)$$

where $\Psi \triangleq [\omega_1, \omega_2, \cdots, \omega_L]^T$ denotes a set of indexes. When L = 1, the proposed function reduces to the CPF in [3]. Once ω_l matches $\frac{2\phi^{(2l)}(n)}{(2l)!}$, the magnitude of $B_L(n, \Psi)$ reaches its maximum and the $\phi^{(2l)}(n), l = 1, \cdots, L$ can be estimated by locating the peak. Note that the *L* phase derivatives include the IFR information $\phi^{(2)}(n)$. Therefore, for a given time *n*, the *L* estimates of the even-order derivatives of the IP are

$$\left[\hat{\Omega}(n),\cdots,\frac{2\hat{\phi}^{(2L)}(n)}{(2L)!}\right]^{T} = \arg\max_{\Psi}|B_{L}(n,\Psi)|^{2}.$$
 (4)

2.2. Asymptotic Bias and MSE

In this section, the perturbation of the noise to the estimates is quantified as a function of the SNR, the number of samples N, and the window parameter M. The detailed analysis using a multivariate first-order approximation is presented in Appendix, and the results are summarized below.

Proposition 1: For a *p*th-order PPS corrupted by a white Gaussian noise with mean zero and variance σ^2 , the asymptotic bias and the MSE of the *L* phase-derivative estimates using (4) are given by:

$$E\left\{\delta\omega_{L\times1}\right\} = \mathbf{0}_{L\times1},\tag{5}$$

$$E\left\{\left(\delta\omega_{l}\right)^{2}\right\} = \frac{\left(1 + \frac{1}{2\mathrm{SNR}}\right)}{4M^{4l+1} \cdot \mathrm{SNR}} \left[\boldsymbol{\Delta}^{-1}\right]_{ll}, l = 1, \cdots, L, \quad (6)$$

where SNR is defined as A^2/σ^2 and Δ is an $L \times L$ matrix:

$$\left[\mathbf{\Delta}\right]_{i_1 i_2} = \frac{i_1 i_2}{\left(2i_1 + 1\right) \left(2i_2 + 1\right) \left(2i_1 + 2i_2 + 1\right)}.$$
 (7)

From *Proposition 1*, for all *L* estimates, the proposed estimator is asymptotically unbiased, and the MSEs of the estimate are independent of the phase parameter $\{a_i\}_{i=0}^p$ of the PPS. At high SNR, the MSEs for all estimates are approximately proportional to SNR⁻¹, while the MSEs are proportional to SNR⁻² at low SNR. From (6), the estimates present the MSEs inversely proportional to *M*. In other words, the larger the window length, the lower the MSE. As such, for a given SNR, the minimum MSE of the estimator is determined by the maximum window length available at time *n*, which leads to the following proposition.

Proposition 2: For a *p*th-order noisy PPS and a given SNR, the asymptotic MSEs of the L phase-derivative estimates at time n is minimized to

$$E\left\{(\delta\omega_{l})^{2}\right\} = \frac{\left(1 + \frac{1}{2\text{SNR}}\right)}{4\left(\frac{N-1}{2} - |n_{0} + \frac{N-1}{2} - n|\right)^{4l+1} \cdot \text{SNR}} \left\{[\mathbf{\Delta}]^{-1}\right\}_{ll}$$

where $n \in \{n_0, n_0 + 1, \dots, n_0 + N - 1\}$ and n_0 is the initial time (sample) index. Historically, two cases have been considered for the initial time index n_0 , i.e., $n_0 = 0$ [7, 8] and $n_0 = -(N - 1)/2$

(assume N is odd) [3, 5], respectively. *Proposition 2* is straightforward from *Proposition 1* by using the maximum value of M which is subject to

$$n_0 \le (n \pm M) \le n_0 + N - 1.$$
 (8)

From *Proposition 2*, the minimum MSE across time *n* for the estimates is achieved at the middle point of observations, i.e., $n = n_0 + (N-1)/2$.

2.3. Examples of L = 1 and L = 2

2.3.1. The PPS with order p = 2 and p = 3

Since $L = \lfloor p/2 \rfloor$, we use the $B_L(n, \Psi)$ with L = 1, which is also the CPF [5]

$$B_1(n,\omega_1) = \sum_m x(n+m)x(n-m)e^{-j\omega_1m^2}.$$
 (9)

The MSE of the IFR estimate using $B_1(n, \omega_1)$ can be obtained by setting L = 1 in *Proposition 2*. In this case, the matrix Δ reduces to a scalar 1/45. Note that the CPF in [5] considered the case of $n_0 = -(N-1)/2$. As a result, the MSE of the IFR estimate in *Proposition 2* reduces to

$$E\left\{\left(\delta\omega_{1}\right)^{2}\right\} = \frac{45\left(1 + \frac{1}{2\mathrm{SNR}}\right)}{4\left(\frac{N-1}{2} - |n|\right)^{5} \cdot \mathrm{SNR}}.$$
(10)

which agrees with the derived results in [5] (i.e., (40) of [5]).

2.3.2. The PPS with order p = 4 and p = 5

In this case, we propose to use the $B_L(n, \Psi)$ with L = 2:

$$B_2(n,\Psi) = \sum_m x(n+m)x(n-m)e^{-j(\omega_1m^2 + \omega_2m^4)}.$$
 (11)

From (4), for a given *n*, the indexes for the peak of $|B_2|^2$ are $\omega_1 = \Omega(n)$ and $\omega_2 = \phi^{(4)}(n)/12$. Compared to the HPF-based IFR estimator [5, 9], the proposed estimator is involved in only a second-order nonlinearity, while the HPF involves a sixth-order nonlinearity for the fourth-order or fifth-order PPS. By setting L = 2, we have

$$\mathbf{\Delta} = \begin{bmatrix} \frac{1}{45} & \frac{2}{105} \\ \frac{2}{105} & \frac{4}{225} \end{bmatrix} \Rightarrow \mathbf{\Delta}^{-1} = \begin{bmatrix} \frac{2205}{-4725} & -\frac{4725}{-4725} \\ -\frac{4725}{8} & \frac{11025}{16} \end{bmatrix}, \quad (12)$$

and the MSEs of both estimates in Proposition 1 reduce to

$$E\left\{\left(\delta\omega_{1}\right)^{2}\right\} = 137.8 \frac{\left(1 + \frac{1}{2\mathrm{SNR}}\right)}{M^{5} \cdot \mathrm{SNR}},\tag{13}$$

$$E\left\{\left(\delta\omega_{2}\right)^{2}\right\} = 172.26 \frac{\left(1 + \frac{1}{2\text{SNR}}\right)}{M^{9} \cdot \text{SNR}}.$$
(14)

Compared to the asymptotic MSE of the HPF-based IFR estimate [9]

$$E\left\{\left(\delta\omega_{1}\right)^{2}\right\}_{\mathrm{HPF}} \approx \frac{207.7}{M^{5} \cdot \mathrm{SNR}},\tag{15}$$

the MSE of the proposed IFR estimate in (13) is about 50.72% less at high SNR. At low SNR, the HPF-based MSEs for the IFR estimate vary approximately in proportional to SNR^{-6} (see Section III and IV in [10] and Section III in [5]), whereas the proposed IFR estimator presents MSEs proportional to SNR^{-2} (cf. (13)). In other words, the proposed IFR estimator exhibits much lower SNR threshold than the



Fig. 1. MSEs of the proposed and HPF-based IFR estimators with M = 32 and M = 64 for a fourth-order PPS.

HPF-based one for the high-order PPS, which will be further verified in Section 3.

Except the IFR estimate, the proposed estimator using (11) provides additional information about the fourth phase derivative, which is $\omega_2 = \frac{2\hat{\phi}^{(4)}(n)}{(4)!} = 2a_4$ for a fourth-order PPS. From *Proposition* 2, the minimum MSE of the a_4 estimate is achieved at the middle point, i.e., $n = n_0 + (N-1)/2$,

$$E\left\{\left(\delta a_{4}\right)^{2}\right\} = \frac{E\left\{\left(\delta \omega_{2}\right)^{2}\right\}}{4}\Big|_{n=n_{0}+\frac{N-1}{2}} = \frac{22050\left(1+\frac{1}{2\text{SNR}}\right)}{N^{9}\text{SNR}}$$

Compared to CRB $\{a_4\} = \frac{22050}{N^9 \text{SNR}}$ for a fourth-order PPS, together with its unbiasedness, the proposed a_4 estimator can be said to be asymptotically efficient at high SNR.

3. NUMERICAL EXAMPLES

In this section, we provide two examples to show the MSE-versus-SNR curves for the IFR estimate and the a_4 estimate in Section 2.3.2. The measured MSE at each SNR is obtained by 500 Monte-Carlo runs. Consider a fourth-order PPS with parameters A = 1, $(a_0, a_1, a_2, a_3, a_4) = (2, 2 \cdot 10^{-2}, 1 \cdot 10^{-4}, 1 \cdot 10^{-6}, 1 \cdot 10^{-8})$, and N = 129. The IFR is measured at n = 64, which is the middle point of observations.

Fig. 1 presents the measured MSEs of the IFR estimate at n = 64 by using the proposed estimator and the HPF-based one with two windows M = 32 and M = 64. The theoretical MSE curves in *Proposition 1* with M = 64 and M = 32 are also included in this figure. From this figure, we have the following observations:

- 1. At high SNR, the measured MSEs for the proposed IFR estimator agree with their theoretical results in both cases of M = 32 and M = 64. Note that the high-SNR theoretical MSEs with M = 64 attain the CRB.
- 2. With either window length, the MSEs of the proposed estimator are generally lower than the HPF-based MSEs.
- 3. The proposed estimator presents lower SNR threshold than the HPF-based one, which is about 6 dB lower in this case.

Fig. 2 presents the measured MSEs of the a_4 estimate at n = 64 by using (11) and the high-order ambiguity function (HAF) [11]. It is observed that the proposed a_4 estimate provides 4-dB lower MSE at high SNR and about 11-dB lower SNR threshold than the HAF-based one.



Fig. 2. MSEs of the proposed and HAF-based a_4 estimators for a fourth-order PPS.

4. CONCLUSION

This paper presents an IFR estimator with a second-order nonlinearity. The asymptotical bias and MSE of the proposed IFR estimator are derived using the multivariate first-order perturbation analysis. The results show that the proposed IFR estimator is asymptotically unbiased, and provides lower MSE and SNR threshold than the HPFbased estimator. A by-product of the proposed estimator is estimation of the phase parameters. Numerical examples verify the analytical results and show that the proposed estimator outperforms other estimators for both the IFR estimation and parameter estimation.

5. APPENDIX: ASYMPTOTIC BIAS AND MSE

This analytical results are based on a multivariate first-order perturbation which is extended from the univariate first-order perturbation in [7]. We summarize the multivariate first-order perturbation as follows. The estimate bias is given by

$$E\left\{\delta\omega_{L\times 1}\right\} = -\left[\mathbf{F}_{2}\right]^{-1} E\left\{\delta\mathbf{F}_{1}\right\},\tag{16}$$

and the MSE of of the *l*th estimate is

$$E\left\{\left(\delta\omega_{l}\right)^{2}\right\} = \left\{\left[\mathbf{F}_{2}\right]^{-1} E\left\{\left[\delta\mathbf{F}_{1}\right]\left[\delta\mathbf{F}_{1}\right]^{T}\right\}\left[\mathbf{F}_{2}\right]^{-1}\right\}_{ll},\quad(17)$$

where

$$\begin{split} \left[\mathbf{F}_{2}\right]_{i_{1}i_{2}} &= 2\Re\left\{\frac{\partial^{2}g_{N}\left(n,\boldsymbol{\Omega}\right)}{\partial\omega_{i_{1}}\partial\omega_{i_{2}}}g_{N}^{*}\left(n,\boldsymbol{\Omega}\right) + \frac{\partial g_{N}\left(n,\boldsymbol{\Omega}\right)}{\partial\omega_{i_{1}}}\frac{\partial g_{N}^{*}\left(n,\boldsymbol{\Omega}\right)}{\partial\omega_{i_{2}}}\right\} \\ &\left[\delta\mathbf{F}_{1}\right]_{i} = 2\Re\left\{\frac{\partial g_{N}\left(n,\boldsymbol{\Omega}\right)}{\partial\omega_{i}}\delta g_{N}^{*}\left(n,\boldsymbol{\Omega}\right) + g_{N}\left(n,\boldsymbol{\Omega}\right)\frac{\partial \delta g_{N}^{*}\left(n,\boldsymbol{\Omega}\right)}{\partial\omega_{i}}\right\}. \end{split}$$

and, according to our estimator in (4),

$$g_N(n, \Psi) = \sum_{m=-M}^{M} s(n+n)s(n-m)e^{-j\sum_{l=1}^{L}\omega_l m^{2l}}, \quad (18)$$

$$\delta g_N(n, \Psi) = \sum_{m=-M}^{M} z_{vs}(n, m) e^{-j \sum_{l=1}^{L} \omega_l m^{2l}},$$
(19)

where $z_{vs}(n,m) = s(n+m)v(n-m) + s(n-m)v(n+m) + v(n+m)v(n-m)$ denotes interference terms. Accordingly,

the maximum point for the noise-free function $g_N(n, \Psi)$ is $\mathbf{\Omega} \stackrel{\Delta}{=} \left[\phi^{(2)}(n), \cdots, \frac{2\phi^{(2L)}(n)}{(2L)!}\right]^T$ and its estimate error vector is $\delta\omega_{L\times 1} = [\delta\omega_1, \cdots, \delta\omega_L]^T$.

To derive the bias and MSE, we need the intermediate results

$$g_N^*(n, \mathbf{\Omega}) = g_N^*(n, \Psi) |_{\Psi = \mathbf{\Omega}} \approx 2A^2 e^{-j2\phi(n)} M$$
$$\frac{\partial g_N(n, \mathbf{\Omega})}{\partial \omega_{i_1}} \approx -j2A^2 e^{j2\phi(n)} \frac{M^{(2i_1+1)}}{(2i_1+1)},$$
$$\frac{\partial^2 g_N(n, \mathbf{\Omega})}{\partial \omega_{i_1} \partial \omega_{i_2}} \approx -2A^2 e^{j2\phi(n)} \frac{M^{(2i_1+2i_2+1)}}{(2i_1+2i_2+1)},$$

where we have used the approximation

$$\sum_{n=-M}^{M} m^{2k} \approx \frac{2M^{(2k+1)}}{2k+1}, (M \gg 2k) \,.$$

By inserting the above intermediate results into \mathbf{F}_2 , we have

$$[\mathbf{F}_2]_{i_1 i_2} = -\frac{32A^4 M^{(2i_1+2i_2+2)} i_1 i_2}{(2i_1+1) (2i_2+1) (2i_1+2i_2+1)}.$$
 (20)

Then, with the following results

$$\delta g_N^*\left(n,\Omega\right) = \sum_m z_{vs}^*\left(n,m\right) e^{j\sum_{l=1}^L \Omega_l m^{2l}},$$
$$\frac{\partial \delta g_N^*\left(n,\Omega\right)}{\partial \omega_{i_1}} = j\sum_m m^{2i_1} z_{vs}^*\left(n,m\right) e^{j\sum_{l=1}^L \Omega_l m^{2l}},$$

 $\delta \mathbf{F_1}$ is derived as

$$\left[\delta \mathbf{F}_{1}\right]_{i_{1}} = -4A^{2}M\Im\left\{\Gamma_{i_{1}}\right\},\tag{21}$$

where $\Im \{\cdot\}$ denotes the imaginary part of $\{\cdot\}$, and

$$\Gamma_{i_1} = e^{j2\phi(n)} \sum_m \left(m^{2i_1} - \frac{M^{2i_1}}{(2i_1 + 1)} \right) z_{vs}^* (n,m) e^{j \sum_{l=1}^L \Omega_l m^{2l}}.$$

Since $E\{z_{vs}^*(n,m)\} = 0$ for any n and m [10], we have $E\{\Im[\Gamma_{i_1}]\} = 0$ and, therefore,

$$E\left\{\left[\delta\mathbf{F}_{1}\right]_{i_{1}}\right\} = -4A^{2}ME\left\{\Im\left[\Gamma_{i_{1}}\right]\right\} = 0.$$
(22)

As a result, from (16), $E \{\delta \omega_{L \times 1}\} = \mathbf{0}_{L \times 1}$, which means all *L* estimates are asymptotically unbiased.

According to (17), we need to compute $E\left\{\left[\delta \mathbf{F}_{1}\right]\left[\delta \mathbf{F}_{1}\right]^{T}\right\}$ in order to find the asymptotic MSE. From (21), we have

$$E\left\{\left[\delta\mathbf{F}_{1}\right]_{i_{1}}\left[\delta\mathbf{F}_{1}\right]_{i_{2}}\right\} = 8A^{4}M^{2}\Re\left[E\left\{\Gamma_{i_{1}}\Gamma_{i_{2}}^{*}\right\} - E\left\{\Gamma_{i_{1}}\Gamma_{i_{2}}\right\}\right],$$

where we have used $E \{\Im[x] \Im[y]\} = 0.5\Re [E \{xy^*\} - E \{xy\}].$ Since

$$E\{z_{vs}^{*}(n,m_{1})z_{vs}(n,m_{2})\} = (2A^{2}\sigma^{2} + \sigma^{4})\delta(m_{1} + m_{2}), + (2A^{2}\sigma^{2} - \sigma^{4})\delta(m_{1} - m_{2}),$$
(23)

where $\delta(\cdot)$ denotes the Kronecker delta function, and by using some routine algorithms, we have

$$E \{ \Gamma_{i_1} \Gamma_{i_2}^* \} = 2 \left(2A^2 \sigma^2 + \sigma^4 \right) \sum_m \left(m^{2i_1} - \frac{M^{2i_1}}{(2i_1 + 1)} \right) \times \left(m^{2i_2} - \frac{M^{2i_2}}{(2i_2 + 1)} \right),$$
$$E \{ \Gamma_{i_1} \Gamma_{i_2} \} = 0.$$

Using the above results, we have

$$E\left\{\left[\delta\mathbf{F}_{1}\right]_{i_{1}}\left[\delta\mathbf{F}_{1}\right]_{i_{2}}\right\} = \frac{128A^{4}\left(2A^{2}\sigma^{2}+\sigma^{4}\right)M^{2i_{1}+2i_{2}+3}i_{1}i_{2}}{\left(2i_{1}+1\right)\left(2i_{2}+1\right)\left(2i_{1}+2i_{2}+1\right)}.$$

Comparing the above equation and (20), we notice that

$$E\left\{\left[\delta\mathbf{F}_{1}\right]\left[\delta\mathbf{F}_{1}\right]^{T}\right\} = -4M\left(2A^{2}\sigma^{2} + \sigma^{4}\right)\mathbf{F}_{2}.$$
 (24)

Finally, replacing \mathbf{F}_2 with (20) and $E\left\{\left[\delta\mathbf{F}_1\right]\left[\delta\mathbf{F}_1\right]^T\right\}$ with (24) in (17) yields the asymptotic MSE given by (6).

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