EVOLUTIONARY SPECTRUM ESTIMATION FOR UNIFORMLY MODULATED PROCESSES WITH IMPROVED BOUNDARY PERFORMANCE

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ABSTRACT

The evolutionary spectrum (ES) is a time-dependent analogue of the spectrum of a stationary process. Existing estimators of the ES suffer from bias problems in the boundary region of the time-frequency domain, due to windowing effects. We propose a new estimator of the ES of a uniformly modulated process which mitigates these problems. Our estimator is based on an extrapolation of the ES in time, using an estimate of the time derivative of the ES. We apply our estimator to a simulated example of a uniformly modulated process with known ES.

***Index Terms*—** Spectral Analysis, Stochastic Processes

1. INTRODUCTION

Classical methods of time series analysis assume that the signal under study is a realization of a stationary stochastic process. However, many signals of important practical interest such as those arising in speech, seismology, climatology, and geophysics are known to exhibit sudden and sporadic behavior [1]; these are features which cannot be captured by stationary models. Therefore, nonstationarity must be allowed if one is to do realistic modeling of these (and other) types of signals. The evolutionary spectrum (ES), introduced by Priestley [2], is a time-dependent analogue of the spectrum of a stationary stochastic process, valid for a class of nonstationary processes. The ES describes the local power-frequency distribution at each instant of time, so it preserves the physical interpretation of the spectrum of a stationary stochastic process as a power-frequency distribution.

Since the introduction of the ES, several techniques have been proposed to estimate it. The evolutionary periodogram estimator [3], was one of the estimators of the Wold-Cramér evolutionary spectrum [4]. This method was based on projections of the spectrum onto the time and frequency directions using an orthonormal basis set. Thomson [1] independently introduced the high-resolution spectrogram, which is essentially the same as the evolutionary periodogram with discrete prolate spheroidal sequences chosen as the basis set [5].

All available estimators of the ES suffer from inherent problems with time and frequency resolution [3]. Furthermore, due to tapering effects, all available estimators of the ES suffer from bias at the boundaries of the time-frequency plane. We suggest an estimator for which this boundary behaviour is improved, using an estimated time derivative of the ES. We demonstrate the applicability of our estimator to uniformly modulated processes, which form a subclass of the class of oscillatory processes.

In Section 2 we review some background regarding the theory and estimation of evolutionary spectra. In Section 3, we develop an estimator of the ES with improved time resolution. In Section 4, we introduce a method to smooth the estimator from Section 3 and extrapolate the time and frequency boundaries. In Section 5, we demonstrate the performance of our estimator on a simulated uniformly modulated process.

2. EVOLUTIONARY SPECTRUM: THEORY AND ESTIMATION

2.1. Evolutionary Spectrum

The evolutionary spectrum, introduced by Priestley [2], generalizes the notion of a spectral density function to a class of nonstationary processes. In contrast with the time-invariant stationary case, the evolutionary spectrum describes the local power-frequency distribution at each time instant. In this subsection, we review the basic theory of evolutionary spectra.

Let $\mathbf{X} = \{X_t\}_{t\in\mathbb{R}}$ be a continuous-time nonstationary process. If $\mathbf{X}$ can be represented in the form

$$X_t = \int_{\mathbb{R}} M(t,f) e^{i2\pi f t} \,\mathrm{d}Z_{\mathbf{X}}(f), \quad \forall t \in \mathbb{R}, \tag{1}$$

where $\{Z_{\mathbf{X}}(f)\}_{f\in\mathbb{R}}$ is a complex-valued process with orthogonal increments and the functions $M(t,f)$ are "slowly varying," (see [2], p.147 for a precise definition) then $\mathbf{X}$ is called an *oscillatory process*. The function $M(t,f)$ can be thought of as providing time- and frequency-dependent frequency modulation. We assume that $M(t,f)$ is continuously differentiable in t. If $\mathbf{X}$ is an oscillatory process, the *time-frequency spectral density* or *evolutionary spectrum* (ES) of

$\mathbf{X}$ is the function

$$S_{\mathbf{X}}(t,f) = |M(t,f)|^2 S_{\mathbf{X}}(f),$$

where $S_{\mathbf{X}}(f)\,\mathrm{d}f = \mathbb{E}\{|\,\mathrm{d}Z_{\mathbf{X}}(f)|^2\}$ and $\mathbb{E}$ denotes the expectation operator. If $\mathbf{X}$ is an oscillatory process, and there exists a stationary process $\mathbf{Y}$ and a nonnegative function $C(t)$ whose Fourier transform exists, such that $\mathbf{X}$ can be represented as

$$X_t = \int_{\mathbb{R}} C(t) e^{i2\pi f t}\,\mathrm{d}Z_{\mathbf{Y}}(f), \quad \forall t \in \mathbb{R}, \tag{2}$$

then $\mathbf{X}$ is called a *uniformly modulated process* (UMP). Here $\{Z_{\mathbf{Y}}(f)\}_{f\in\mathbb{R}}$ is the complex-valued process with orthogonal increments appearing in the Cramér representation of $\mathbf{Y}$. The ES of a UMP $\mathbf{X}$ is therefore given by $S_{\mathbf{X}}(t,f) = |C(t)|^2 S_{\mathbf{Y}}(f)$.

2.2. High-Resolution Spectrogram

Let $\mathbf{X}$ be an oscillatory process, and let $X_0, X_1, \ldots, X_{N-1}$ be a discrete time sample of $\mathbf{X}$. Thomson [1] proposed the following estimator of the ES of $\mathbf{X}$, called the *high-resolution spectrogram* (HRS):

$$\hat{S}_{\mathbf{X}}(t,f) \triangleq \frac{N}{K}\left|\sum_{k=0}^{K-1}\sqrt{\lambda_k}\nu_t^{(k)}\sum_{n=0}^{N-1}\nu_n^{(k)}X_n e^{-i2\pi f n}\right|^2. \tag{3}$$

Here, $\nu_t^{(k)} \equiv \nu_t^{(k)}(N,W)$, $k \in \{0,1,\ldots,K-1\}$, is the k^{th} *discrete prolate spheroidal sequence* (DPSS) with associated eigenvalue λ_k, and $W \in (0,1/2)$ is the *analysis bandwidth* [5]. The integer K is typically chosen so that $K \leq \lfloor 2NW \rfloor - 1$, so that $\lambda_k \approx 1$ for each $k \in \{0,1,\ldots,K-1\}$. This condition on K makes the HRS essentially the same as the evolutionary periodogram estimator [3] of the Wold-Cramér ES [4]. The innovation of the HRS, then, lies essentially in the choice of the DPSSs as data tapers. The DPSSs $\nu_t^{(0)}, \ldots, \nu_t^{(K-1)}$ form a set of pairwise orthonormal data tapers which are optimally concentrated in frequency in the interval $(-W, W)$.

There are key drawbacks with the HRS estimator. First, since it uses the entire sample from $\mathbf{X}$ to produce an estimate, there are problems with its time resolution. Second, since the DPSSs tend to zero as $t \to 0$ and $t \to N-1$, the HRS estimator artificially tends to zero as $(t,f) \to (0,f)$ and $(t,f) \to (N-1,f)$, for each $f \in [-1/2, 1/2]$.

2.3. Nonstationary Quadratic-Inverse Theory

Let $\mathbf{X}$ be an oscillatory process, and let $X_0, X_1, \ldots, X_{N-1}$ be a discrete time sample of $\mathbf{X}$. The *nonstationary quadratic-inverse* (NSQI) theory, developed by Thomson [1], proposed an estimator of the ES of $\mathbf{X}$ based on the approximate linear expansion

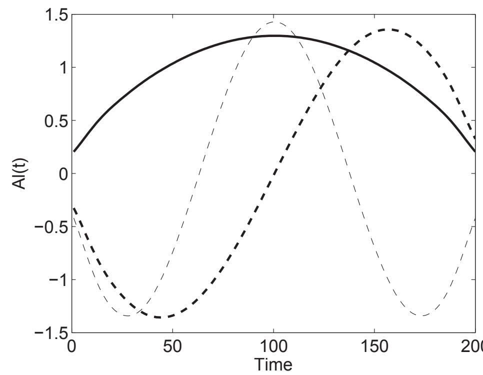

Fig. 1. Sequences $A_l(t)$; $N = 200$ and $W = \frac{1}{40}$; $l = 0$ (solid line), $l = 1$ (thick dashed line), and $l = 2$ (thin dashed line).

$$S_{\mathbf{X}}(t,f) \approx \sum_{l=0}^{L} a_l(f) A_l(t). \tag{4}$$

Here, $a_l(f)$, $l \in \{0,1,\ldots,L\}$ are unknown expansion coefficients and $A_l(t) \in \mathbb{R}$, $t \in \{0,1,\ldots,N-1\}$, $l \in \{0,1,\ldots,L\}$, are the orthonormalized solutions of the algebraic eigenvalue equation

$$\alpha_l A_l(n) = N\sum_{m=0}^{N-1}\left[\frac{\sin 2\pi W(n-m)}{\pi(n-m)}\right]^2 A_l(m) \tag{5}$$

for some fixed analysis bandwidth $W \in (0,1/2)$. It is known that the eigenvalues α_l satisfy $\alpha_l \approx \max(2NW - l/2, 0)$ for $l \in \{0,1,\ldots,\lfloor 4NW \rfloor\}$. The sequences $A_l(\cdot)$ are shown in Fig. (1) for $N = 200$ and $W = 1/40$.

From Eq. (4), it can be seen that if one has an estimate of the ES of $\mathbf{X}$ available, then one can estimate the expansion coefficients $a_l(f)$. Using the HRS, Thomson [1] proposes the estimators

$$\hat{a}_l(f) \triangleq \frac{K}{N\alpha_l}\sum_{t=0}^{N-1}\hat{S}_{\mathbf{X}}(t,f)A_l(t). \tag{6}$$

Thomson [1] describes the approximate relationships between the time derivatives of $S_{\mathbf{X}}(t,f)$ and the expansion coefficients $a_l(f)$. First, since the zeroth-order sequence $A_0(t)$ is approximately constant, the coefficient $\hat{a}_0(f)$ is approximately the standard multitaper spectrum estimate [5]. Similarly, since the first-order sequence $A_1(t)$ is approximately linear, $\hat{a}_1(f)$ is approximately the time-derivative of the ES.

3. SLIDING-WINDOW HRS

For the remainder of this paper we consider only uniformly modulated processes. Let $\mathbf{X}$ be a UMP and let the discrete time sample of $\mathbf{X}$ be given by $X_0, X_1, \ldots X_{N-1}$. Since $\mathbf{X}$ is a UMP, the modulation happens only in the t-direction; as a result, time resolution is of higher importance than frequency resolution. To improve the time resolution of the HRS estimator, in this section we propose the *sliding-window high-resolution spectrogram* (SWHRS).

First, define overlapping time blocks of length $B \ll N$, a compromise between time and frequency resolution, where the overlap is $B-1$. The blocks are indexed by the *base time* $b \in \{0, 1, \ldots, N-B\}$. The HRS for the b^{th} block and for $t \in \{b, b+1, \ldots, b+B-1\}$,

$$\hat{S}_{\mathbf{X},b}(t,f) \triangleq \frac{B}{K} \left| \sum_{k=0}^{K-1} \sqrt{\lambda_k} \nu_t^{(k)} \sum_{n=0}^{B-1} X_{n+b}^{(b)} \nu_n^{(k)} e^{-i2\pi fn} \right|^2,$$

where $\nu_t^{(k)}$ and K are as in Section 2.2. Due to the blocking, for each fixed f, the SWHRS produces more than one estimate of $S_{\mathbf{X}}(t,f)$. One approach is to use the b^{th} block to estimate the ES at only the middle time point of the block, denoted by $t_b \triangleq b + \lceil \frac{B}{2} \rceil$. Thus, for $t \in \{\lceil \frac{B}{2} \rceil, \lceil \frac{B}{2} \rceil + 1, \ldots, (N-1) - \lceil \frac{B}{2} \rceil\}$, the ES at (t,f) is estimated by $\hat{S}_{\mathbf{X},b}(t_b,f)$, using block $b = t - \lceil \frac{B}{2} \rceil$.

The SWHRS estimates at other (t,f) pairs, i.e., $t < \lceil \frac{B}{2} \rceil$ or $t > N-1-\lceil \frac{B}{2} \rceil$, which are referred to as *boundary regions*, are considered to be unavailable and must be estimated by other means. We consider the boundary regions as well as a modification to the ES estimate for all t in the next section.

4. TAYLOR SERIES APPROXIMATION TO THE EVOLUTIONARY SPECTRUM

In this section, we develop a technique to estimate the boundary regions using the SWHRS estimator. The technique we propose smooths the SWHRS estimate at the available time points, as well as estimating the boundary regions using the information from the smoothed SWHRS.

Let $\mathbf{X}$ be a UMP. Let $S_{\mathbf{X},b}(t,f)$ denote the ES of $\mathbf{X}$ in the b^{th} block, *i.e.*, $S_{\mathbf{X},b}(t,f) = S_{\mathbf{X}}(t,f)$ with t restricted so that $t \in \{b, b+1, \ldots, b+B-1\}$. Assume that, at a fixed frequency f, $S_{\mathbf{X},b}(t,f)$ can be well approximated by the first two terms of its Taylor series expansion around the middle time point t_b:

$$S_{\mathbf{X},b}(t,f) \approx S_{\mathbf{X},b}(t_b,f) + S_{\mathbf{X},b}^{(1)}(t_b,f)(t-t_b), \tag{7}$$

where $S_{\mathbf{X},b}^{(1)}(t_b,f)$ denotes the time derivative of $S_{\mathbf{X},b}(t,f)$ evaluated at (t_b,f).

Recall from Section 2.3 that we assumed the ES has the approximate expansion shown in Eq. (4). Here we assume that a similar expansion holds for each $S_{\mathbf{X},b}(t,f)$, *i.e.*,

$$S_{\mathbf{X},b}(t,f) \approx \sum_{l=0}^{L_b} a_{l,b}(f) A_l(t), \tag{8}$$

where $a_{l,b}(f)$ are the coefficients in block b. Using the two representations of the ES from Eqs. (7) and (8), we have for each block b

$$\sum_{l=0}^{L_b} a_{l,b}(f) A_l(t) \approx S_{\mathbf{X},b}(t_b,f) + S_{\mathbf{X},b}^{(1)}(t_b,f)(t-t_b).$$

Multiplying both sides of the above equation by $A_m(t)$, summing over t and using the fact that the sequences $A_l(t)$ are orthonormal, we obtain

$$\begin{aligned} Ba_{l,b}(f) \approx{}& S_{\mathbf{X},b}(t_b,f) \sum_{t=b}^{b+B-1} A_l(t) \\ &+ S_{\mathbf{X},b}^{(1)}(t_b,f) \sum_{t=b}^{b+B-1} A_l(t)(t-t_b). \end{aligned}$$

For $l=0$ and $l=1$, the above equation simplifies to

$$S_{\mathbf{X},b}(t_b,f) \approx \frac{Ba_{0,b}(f)}{\sum_{t=b}^{b+B-1} A_0(t)} \tag{9}$$

$$S_{\mathbf{X},b}^{(1)}(t_b,f) \approx \frac{Ba_{1,b}(f)}{\sum_{t=b}^{b+B-1} A_1(t)(t-t_b)}. \tag{10}$$

To derive these two expressions, we have also used the facts $\sum_{t=b}^{b+B-1} A_0(t)(t-t_b) = 0$ and $\sum_{t=b}^{b+B-1} A_1(t) = 0$.

Now let $X_0, X_1, \ldots X_{N-1}$ be a discrete time sample of $\mathbf{X}$. Block this sample and compute $\hat{S}_{\mathbf{X},b}(t,f)$ in each block, as explained in Section 3. The coefficients $a_{0,b}(f)$ and $a_{1,b}(f)$ can be estimated as in Section 2.3 using the SWHRS estimate: For $l = 0, 1$,

$$\hat{a}_{l,b}(f) = \frac{K}{B\alpha_l} \sum_{t=b}^{b+B-1} \hat{S}_{\mathbf{X},b}(t,f) A_l(t).$$

Plugging these estimates into Eqs. (9) and (10) yields estimators of $S_{\mathbf{X},b}(t,f)$ and $S_{\mathbf{X},b}^{(1)}(t,f)$ at the middle time point t_b. Explicitly, we have

$$\hat{S}_{\mathbf{X},b,M}(t_b,f) \triangleq \frac{\frac{K}{\alpha_0} \sum_{t=b}^{b+B-1} \hat{S}_{\mathbf{X},b}(t,f) A_0(t)}{\sum_{t=b}^{b+B-1} A_0(t)} \tag{11}$$

$$\hat{S}_{\mathbf{X},b,T}^{(1)}(t_b,f) \triangleq \frac{\frac{K}{\alpha_1} \sum_{t=b}^{b+B-1} \hat{S}_{\mathbf{X},b}(t,f) A_1(t)}{\sum_{t=b}^{b+B-1} A_1(t)(t-t_b)}. \tag{12}$$

We call the estimator $\hat{S}_{\mathbf{X},b,M}(t_b,f)$ the *Modified ES Estimator* (MESE), and we call the estimator $\hat{S}_{\mathbf{X},b,T}^{(1)}(t_b,f)$ the *Time-Derivative ES Estimator* (TDESE).

As a result of blocking, the MESE and TDESE can be evaluated only at $t_b \in [\lceil \frac{B}{2} \rceil, (N-1) - \lceil \frac{B}{2} \rceil]$. In other words, the MESE and TDESE estimates are not available at times before the middle time point of the first block, t_0, and after the middle time point of the last block, t_{N-B+1}. We now describe a technique to estimate evolutionary spectra at times outside this range.

Using the blockwise Taylor series expansion of the ES of $\mathbf{X}$ from Eq. (7), $S_{\mathbf{X},b}(t,f)$ can be extrapolated h time steps ahead by

$$\hat{S}_{\mathbf{X},b,M}(t_b \pm h, f) \approx \hat{S}_{\mathbf{X},b,M}(t_b, f) \pm h\, \hat{S}^{(1)}_{\mathbf{X},b,T}(t_b, f), \quad (13)$$

where $h \in \{1, 2, \ldots, \lceil \frac{B}{2} \rceil\}$. The final estimate of $S_{\mathbf{X}}(t,f)$ is then taken to be $\hat{S}_{\mathbf{X},b,M}(t,f)$ from Eq. (11) for $t \in [\lceil \frac{B}{2} \rceil, (N-1) - \lceil \frac{B}{2} \rceil]$ and $\hat{S}_{\mathbf{X},b,M}(t,f)$ from Eq. (13) for $t \in \{0, 1, \ldots, \lceil \frac{B}{2} \rceil - 1\}$ and $t \in \{N - \lceil \frac{B}{2} \rceil, \ldots, N-1\}$. We refer to this combination as the *Boundary-Corrected Modified ES Estimate* (BCMESE). In the next section, we demonstrate the performance of the BCMESE in estimating the ES of a simulated UMP.

5. EXAMPLE

Following an example from Priestley [2], let $\mathbf{X}$ be a UMP of the form

$$X_t = \left(2 - e^{\frac{-(t-500)^2}{2(200)^2}}\right) Y_t,$$

where $\mathbf{Y} = \{Y_t\}_{t\in\mathbb{Z}}$ is the second-order autoregressive process

$$Y_t = 0.8Y_{t-1} - 0.4Y_{t-2} + \epsilon_t.$$

Here $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is a white noise process with variance 10^4. The spectrum of $\mathbf{Y}$ is therefore

$$S_{\mathbf{Y}}(f) = \frac{10^4}{1 - 2.24\cos(2\pi f) + 1.6\cos^2(2\pi f)}.$$

The theoretical ES of $\mathbf{X}$ is

$$S_{\mathbf{X}}(t,f) = \left(2 - e^{\frac{-(t-500)^2}{2(200)^2}}\right)^2 S_{\mathbf{Y}}(f). \quad (14)$$

We generate 100 realizations of $\mathbf{X}$ of length $N = 1000$. With a compromise block size of $B = 200$, we applied the HRS estimator, SWHRS estimator, and BCMESE to each realization of $\mathbf{X}$ to estimate its ES. For each estimator, the final estimate was taken to be the average over all 100 realizations. Figure 2 shows each final estimate and the theoretical ES of $\mathbf{X}$.

6. CONCLUSION

In this paper, we proposed an estimator of the ES of a UMP. The estimator was derived using an estimate of the time derivative of the ES. The main advantage of our estimator is that it has much better boundary behaviour in the time-frequency region versus existing windowed estimates such as HRS. Moreover, our estimator has better time resolution compared with the HRS estimator [1].

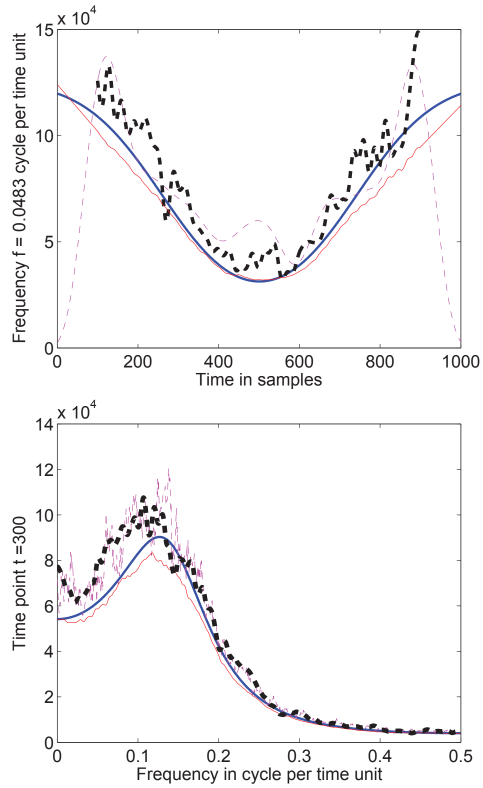

Fig. 2. Theoretical ES of $\mathbf{X}$ (thick solid blue line), HRS estimate (thin dashed pink line), SWHRS estimate (thick dashed black line), BCMESE (thin solid red line) at $f = 0.0483$ cycles per time unit (top figure) and $t = 300$ (bottom figure). The SWHRS estimate is not defined in the boundary regions.

7. REFERENCES

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