NEW DISCRETE INVERSE S TRANSFORM WITH LEAST SQUARE ERROR IN FILTERING

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ABSTRACT

The S transform is a useful linear time-frequency distribution and has been applied to various fields. Since the inverse S transform is an over-determined problem, there exist several different algorithms, which result in different filtering effects. This paper models the discrete S transform in matrix form and proposes a new inverse S transform algorithm with least square error in filters. This paper also compares the new inverse S transform with the previous ones.

***Index Terms*—** S transform, time-frequency analysis, time-varying filter

1. INTRODUCTION

Time-frequency analysis, which can analyze a signal in both the time and frequency domains simultaneously, is a powerful tool for signal processing. Researchers have proposed many kinds of algorithms for time-frequency analysis, including linear, bilinear, and the other ones. One advantage of the linear time-frequency analysis is its high efficiency and flexibility in the time-frequency filters, which can be achieved by multiplying the time-frequency distribution by a weighting function. Two of the most well-known linear time-frequency distributions are the short-time Fourier transform (STFT) and wavelet transform (WT) [1].

The S transform (ST) can be considered as a hybrid of the STFT and WT [2]. The ST has a form similar to the STFT except that the window's width in the ST can be changed with frequency. Therefore, the ST can provide the progressive resolution like that in the WT. Besides, the ST uses the time-frequency axis rather than the time-scale axis used in the WT. Therefore, the interpretation of the frequency information in the ST is more straightforward than in the WT. The ST has been shown useful and applied to various fields including medical imaging [3], geophysics [4], and electrical engineering.

Because the inverse S transform (IST) is an over-determined problem, there exist several different algorithms, which may result in different filtering effects. Stockwell *et al.*'s algorithm was efficient but suffers from the problem of time leakage in filters [2]. Schimmel and Gallart's algorithm provided better time localization in filters but contained the reconstruction errors [5]─[9]. The time-frequency filter's performance is significantly influenced by the inverse algorithm. Since the ST is a powerful time-frequency distribution, it is important to find an IST algorithm that will provide satisfactory performance in the filters.

In this paper, the discrete ST is modeled in matrix form and a new IST is proposed that has least square error in the time-frequency filters.

This paper is organized as follows. Section 2 briefly reviews and discusses the STs and ISTs. The new IST is derived in Section 3, along with the experiments in Section 4. The performance of the ISTs are discussed and concluded in Section 5.

2. REVIEW OF ST AND ITS INVERSE

2.1. S Transform

The ST, derived by Stockwell *et al.* [2], of a time series $u(t)$ is

$$S(\tau, f) = \int_{-\infty}^{\infty} u(t) w(\tau - t, f) e^{-i2\pi ft} dt \tag{1}$$

where the $w(\tau, f)$ is usually a Gaussian window

$$w(\tau - t, f) = \frac{|f|}{k\sqrt{2\pi}} e^{\frac{-f^2(\tau-t)^2}{2k^2}} \quad , k > 0. \tag{2}$$

τ and t are time variables and f is a frequency variable. The parameter k can be used to adjust the width of the window and the tiling in the time-frequency domain. All k is set to unity in this paper unless otherwise stated. The Gaussian window can be replaced by other windows as long as the area under it is equal to 1 [10].

To simplify the computation, Stockwell *et al.* proposed another equivalent form of the ST:

$$S(\tau, f) = \int_{-\infty}^{\infty} U(\alpha + f) e^{-\frac{2\pi^2 k^2 \alpha^2}{f^2}} e^{i2\pi\alpha\tau} d\alpha \,. \tag{3}$$

where $U(\alpha)$ is the Fourier transform of $u(t)$. Therefore, the ST can be derived in the frequency domain.

2.2. Stockwell *et al.*'s Inverse S Transform

Since the area under the window, as a function of τ, is equal to 1 for every fixed f, an efficient IST is derived by Stockwell *et al.*.

$$\int_{-\infty}^{\infty} S(\tau, f)d\tau = \int_{-\infty}^{\infty} u(t)\left[\int_{-\infty}^{\infty} w(\tau - t, f)d\tau\right]e^{-i2\pi ft}dt = U(f) \cdot \quad (4)$$

$$u_1(t) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} S(\tau, f)d\tau\right]e^{i2\pi ft}df . \quad (5)$$

The Stockwell *et al.*'s IST during filtering is

$$u_{f1}(t) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} S(\tau, f)F(\tau, f)d\tau\right]e^{i2\pi ft}df . \quad (6)$$

where the $F(\tau, f)$ is the time-frequency filter.

The disadvantage of this IST is that the imposed time localization of the filter may not be correctly inherited by the output time series.

2.2. Schimmel and Gallart's Inverse S Transform

To improve the problem of time leakage in (6), Schimmel and Gallart proposed another algorithm [5].

$$u_2(t) = k\sqrt{2\pi}\int_{-\infty}^{\infty}\frac{S(t, f)}{|f|}e^{i2\pi ft}df \quad (7)$$

$$u_{f2}(t) = k\sqrt{2\pi}\int_{-\infty}^{\infty}\frac{S(t, f)F(t, f)}{|f|}e^{i2\pi ft}df . \quad (8)$$

where (7) is the normal IST and (8) is the IST with a filter $F(t, f)$. Because this IST integrates only over the frequency, the time localization of the filter will be correctly inherited by the output time series.

However, the output time series contains a reconstruction error [6]–[9].

$$u_2(t) = u(t) * I(t) \quad (9)$$

where "∗" denotes the convolution and

$$I(t) = \int_{-\infty}^{\infty} e^{\frac{-f^2t^2}{2k^2}}e^{i2\pi ft}df . \quad (10)$$

It was shown that this reconstruction error decreased with increasing k [9]. Therefore (7) and (8) can regarded as satisfactory approximations when $k \geq 1$. An alternative is to retrieve the correct signal by applying the deconvolution of $I(t)$ for $u_2(t)$ or $u_{f2}(t)$ [6].

3. DISCRETE IST WITH LEAST SQUARE ERROR

The discrete ST equations are:

$$S[m,n] = \sum_{p=0}^{N-1} u[p]\frac{|n|}{kN\sqrt{2\pi}}e^{-\frac{n^2(m-p)^2}{2k^2N^2}}e^{-\frac{i2\pi pn}{N}} , \quad n \neq 0 \quad (11)$$

or

$$S[m,n] = \sum_{l=-N/2}^{N/2-1} U[l+n]e^{-\frac{2\pi^2k^2l^2}{n^2}}e^{\frac{i2\pi lm}{N}} , \quad n \neq 0 \quad (12)$$

$$S[m,0] = \frac{1}{N}\sum_{p=0}^{N-1} u[p]. \quad (13)$$

where p and m are the time variables and n is the frequency variable. Although the continuous time STs (1) and (3) are equivalent to each other, the discrete STs (11) and (12) are slightly different [9]. Usually the size of $S[m,n]$ is $N \times N$ because the FT is used.

Since the ST is a linear transform, it can be modeled in matrix form. Let **u** be a $N \times 1$ vector, **s** be a $N^2 \times 1$ vector, and **T** be a $N^2 \times N$ matrix representing the time series, unfolded time-frequency distribution, and transformation matrix of the ST, respectively. The $((i-1)\times N+1)$th to $(i\times N)$th elements of the vector **s** correspond to the elements in the ith column of the matrix $S[m,n]$. Therefore the discrete ST can be modeled as

$$\mathbf{s} = \mathbf{T}\mathbf{u}. \quad (14)$$

The matrix **T** can be derived as follows. Let $\mathbf{T_i}$ represent the ith column vector of the matrix **T**. When **u** is the Kronecker delta function $\delta(p\text{-}i)$, the time-frequency distribution vector **s** is equivalent to the column vector $\mathbf{T_i}$. In other words, the column vector $\mathbf{T_i}$ is the discrete ST of the Kronecker delta function $\delta(p\text{-}i)$. Therefore, from (11) the element T_{ab} of the matrix **T** is

$$T_{ab} = \frac{|\gamma|}{kN\sqrt{2\pi}}e^{-\frac{\gamma^2(v-b)^2}{2k^2N^2}}e^{-\frac{i2\pi b\gamma}{N}} . \quad (15)$$

where γ is the smallest non-negative integer that satisfies $\gamma \equiv a \pmod N$ and $\nu = \lfloor a/N \rfloor$ is the largest integer smaller than or equal to (a/N). The transformation matrix **T** of (12) is slightly different. The closed-form solution of this matrix is too complex to be found. The alternatives are calculating the matrix numerically from (12) or using (15) as an approximation.

The new IST can be derived from (14):

$$\mathbf{u} = \mathbf{T}^{+}\mathbf{s} \quad (16)$$

where "+" denotes the pseudo inverse and $\mathbf{T}^{+} = (\mathbf{T}^{*}\mathbf{T})^{-1}\mathbf{T}^{*}$ where $\mathbf{T}^{*}$ is the conjugate transpose of the matrix **T**.

The new IST in filters is

$$\mathbf{u_f} = \mathbf{T}^{+}(\mathbf{F}\mathbf{s}). \quad (17)$$

where **F** is the $N^2 \times N^2$ filter matrix in which the weighting function is set on the diagonal elements and zero value is set on the others. Another method is to impose the filter before transforming the matrix $S[m,n]$ into the vector **s**.

The back transformed signals from the normal ISTs (5) and (16) are equivalent since the time-frequency distribution is not modified. However, because the ISTs (6), (8) and (17) have different filtering effects, the back transformed signals after filtering are different. From the developed theory in the field of linear algebra, it is known that the back transformed signal from (17) will have the least square error in the time-frequency domain, which means that the mean square of $(\mathbf{T}\mathbf{u_f} - \mathbf{F}\mathbf{s})$ is smaller than or equal to that of $(\mathbf{T}\mathbf{u} - \mathbf{F}\mathbf{s})$ for all time series **u**. This is an important property in evaluating the

	Example in Sec. 4.1		Example in Sec. 4.2	
	MSE_{TF}	MSE_T	MSE_{TF}	MSE_T
IST (6)	0.5777	**0.0298**	1.1267	0.0656
IST (8)	0.5434	0.0655	1.0773	0.0781
Proposed IST (17)	**0.4907**	0.0355	**0.9349**	**0.0482**

Table 1. The MSE_{TF} and MSE_T of the back transformed signals for the examples in Sec. 4.1 and 4.2. The minimum MSE_{TF} and MSE_T were highlighted by using bold font.

performance of the ISTs in filters. The IST (17) will provide the optimal solution in the least-squares sense.

4. EXPERIMENTS

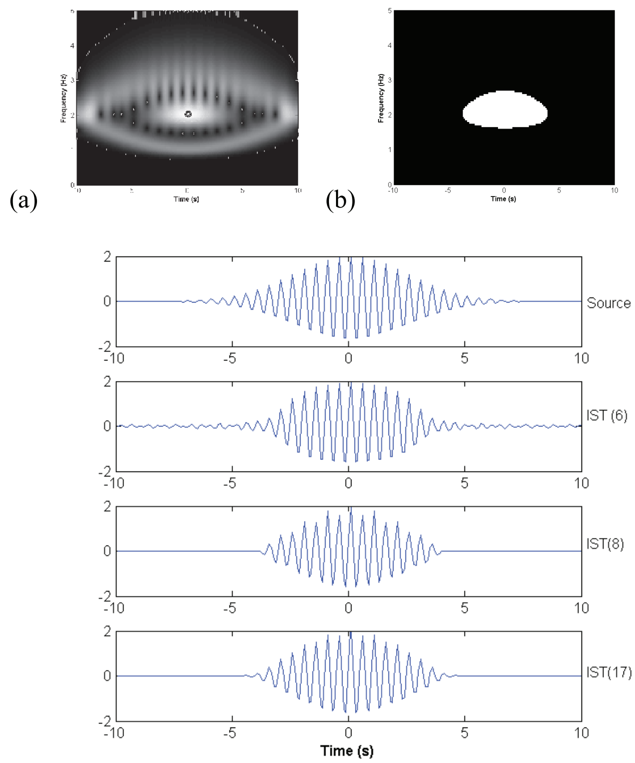
Consider the synthetic signal: $s'(t) = s(t) + n(t)$, where $s(t)$ is the source signal and $n(t)$ is the noise. The objective is to retrieve the source signal $s(t)$ from the noisy signal $s'(t)$ by filtering in the time-frequency domain. The mean-square-error (MSE) of the time series is defined as: $MSE_T = \frac{1}{N}\sum_{p=0}^{N-1}\left(u_f[p] - u[p]\right)^2$. The MSE of the time-frequency distribution is defined as: $MSE_{TF} = \frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\left(S_f[m,n] - S[m,n]\right)^2$, where $S_f[m,n]$ is the ST of the filtered time series $u_f(p)$. The MSEs of the retrieved signals from the three ISTs are compared in this section.

4.1. Adaptive filtering in time-frequency spectrum

The source signal was defined as: $s(t) = 2e^{-\frac{t^2}{12}}\cos(4\pi t)$. $n(t) = \cos\left[2\pi\left(t + \frac{t^3}{300}\right)\right] + \cos\left[2\pi\left(3t - \frac{t^3}{300}\right)\right]$ was the noise function. The sampling frequency was 10Hz and the time period was –10 to 10s. The time-frequency distribution of the synthetic signal $s'(t)$ was plotted in Fig. 1(a), along with the time-frequency filter in Fig. 1(b). The source signal and the back transformed ones from the ISTs (6), (8) and (17) were shown in Fig. 2(c). The MSE_{TF} and MSE_T of the back transformed signals from the ISTs were shown in Table 1.The proposed IST (17) provided the minimum MSE_{TF} and the IST (6) provided the minimum MSE_T.

4.2. Noise Reduction

The source signal was defined as: $s(t) = \cos\left(8\pi t - \frac{2\pi(t-10)^3}{15.6^2}\right)$, with a sampling frequency of 10Hz and a duration of 20s. The noise function $n(t)$ was the Gaussian noise with zero mean and variance 0.3. The time-frequency distribution of the synthetic signal $s'(t)$ was plotted in Fig. 2(a), along with the time-frequency filter in Fig. 2(b). The back transformed signals from the ISTs (6), (8) and (17) were shown in Fig. 2(c) and compared with the source signal. The MSE_{TF} and MSE_T of the back transformed signals from the ISTs were shown in Table 1.The proposed IST (17) had the minimum MSE_T and MSE_{TF}.

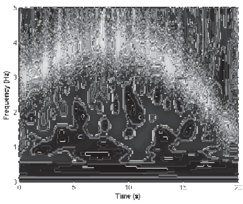
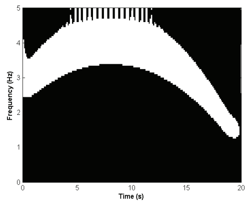


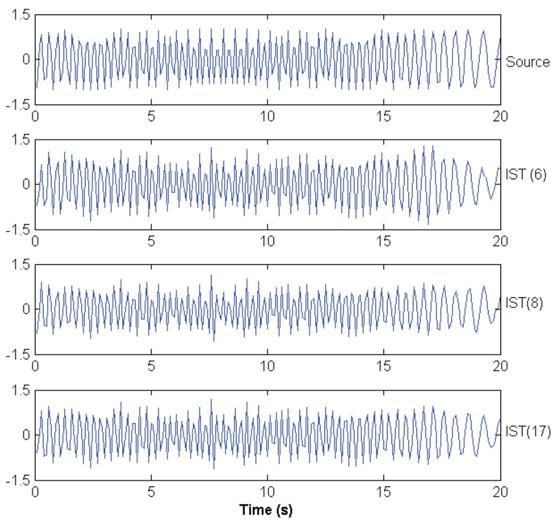
(c)

Fig. 1. (a) The time-frequency distribution of the synthetic signal for the example in Sec. 4.1. (b) The time-frequency filter. (c) The source signal and the back transformed ones from the ISTs (6), (8) and (17). Their MSE_{TF} and MSE_T were shown in Table 1. The proposed IST (17) provided the minimum MSE_{TF} and the IST (6) provided the minimum MSE_T.

5. CONCLUSION

This paper discussed the algorithms of IST. A new IST was proposed and compared with the Stockwell *et al.*'s and Schimmel and Gallart's ones. The experimental results showed that the proposed IST had least square error in the time-frequency filters. Its performance was guaranteed. The experimental results also showed that the back transformed signal that had the minimum MSE_{TF} did not necessarily had the minimum MSE_T. In fact, the MSE_T was influenced not

(a)  (b) 



(c)
Fig. 2. (a) The time-frequency distribution of the synthetic signal for the example in Sec. 4.2. (b) The time-frequency filter. (c) The source signal and the back transformed ones from the ISTs (6), (8) and (17). Their MSE_{TF} and MSE_T were shown in Table 1. The retrieved signal from the proposed IST (17) had the minimum MSE_{TF} and MSE_T.

only by the inverse algorithms but also by the time-frequency filters. How to find the optimal time-frequency filter is another problem and not discussed in this paper.

Since there are several IST algorithms, different IST can be chosen for different purpose. When the time localization is the main concern and the filter is a low-pass one, the Schimmel and Gallart's IST can provide satisfactory result, especially when *k* is large enough. When the frequency localization is the main concern and the filter is high-pass, the Stockwell *et al.* IST is a good choice. When the mean-square-error is the main concern, the proposed IST has the best performance.

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