LOCAL AND GLOBAL CONVERGENCE BEHAVIOR OF NON-EQUIDISTANT SAMPLING SERIES

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ABSTRACT

In this paper we analyze the local and global convergence behavior of sampling series with non-equidistant sampling points for the Paley-Wiener space $\mathcal{PW}_\pi^1$ and sampling patterns that are made of the zeros of sine-type functions. It is proven that the sampling series are locally uniformly convergent if no oversampling is used and globally uniformly convergent if oversampling is used. Furthermore, we show that oversampling is indeed necessary for global uniform convergence, because for every sampling pattern there exists a signal such that the peak value of the approximation error grows arbitrarily large if no oversampling is used. Finally, we use these findings to obtain similar results for the mean-square convergence behavior of sampling series for bandlimited wide-sense stationary stochastic processes.

***Index Terms*—** Sampling series, sine type, non-equidistant sampling, reconstruction, stochastic process

1. INTRODUCTION AND MOTIVATION

The reconstruction of bandlimited signals from their samples is important for many applications in signal processing, communication, and information theory. The Shannon sampling series

$$f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{1}$$

with sinc-kernel is probably the most prominent example of a reconstruction process. However, it is not the only possible one. In particular for practical applications, non-equidistant sampling patterns are of interest. In this paper we analyze sampling patterns that are determined by the zeros of sine type functions.

In order to continue, we need some notations and definitions. Let $\hat{f}$ denote the Fourier transform of a function f, where $\hat{f}$ is to be understood in the distributional sense. $L^p(\mathbb{R})$, $1 \leq p < \infty$, is the space of all pth-power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$, and $L^\infty(\mathbb{R})$ is the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. For $\sigma > 0$ and $1 \leq p \leq \infty$ we denote by $\mathcal{PW}_\sigma^p$ the Paley-Wiener space of signals f with a representation $f(z) = 1/(2\pi) \int_{-\sigma}^{\sigma} g(\omega) \,\mathrm{e}^{iz\omega} \, \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\pi, \pi]$. If $f \in \mathcal{PW}_\sigma^p$ then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\sigma^p$, $1 \leq p < \infty$, is given by $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi) \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^p \, \mathrm{d}\omega)^{1/p}$.

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We analyze the local and global convergence behavior of the sampling series

$$\sum_{k=-N}^{N} f(t_k) \phi_k(t), \tag{2}$$

where ϕ_k, $k \in \mathbb{Z}$, are certain reconstruction functions, determined by sine-type functions, and $\{t_k\}_{k \in \mathbb{Z}}$ is the sequence of real sampling points. Without loss of generality we assume that $t_0 = 0$, and that the sequence of zeros is ordered strictly increasingly, i.e.,

$$\ldots < t_{-N} < \ldots < t_{-1} < t_0 < t_1 < \ldots < t_N < \ldots. \tag{3}$$

Before we proceed, we define functions of sine type and state several of their key properties. For further details and proofs we would like to refer the reader to [1] and [2].

Definition 1. An entire function f of exponential type π is said to be of sine type if

(i) the zeros of f are separated, and

(ii) there exist positive constants A, B, and H such that
$$A \,\mathrm{e}^{\pi|y|} \leq |f(x+iy)| \leq B \,\mathrm{e}^{\pi|y|}$$
whenever x and y are real and $|y| \geq H$.

Example 1. $\sin(\pi z)$ is a function of sine type and its zeros are $t_k = k$, $k \in \mathbb{Z}$.

The class of functions of sine type is very large. In section 4.2 we will present one possibility to construct such functions. Furthermore, functions of sine type have many interesting properties. One concerns their behavior outside circles centered around the zeros of the function, another the distribution of their zeros. Since we will need these properties later, we state them in Lemmas 1 and 2.

Lemma 1. *Let f be a function of sine type, whose zeros $\{\lambda_k\}_{k \in \mathbb{Z}}$ are ordered increasingly according to their real parts. Then we have*

$$\inf_{n \in N} |\lambda_{n+1} - \lambda_n| \geq \underline{\delta} > 0 \tag{4}$$

and

$$\sup_{n \in N} |\lambda_{n+1} - \lambda_n| \leq \overline{\delta} < \infty \tag{5}$$

for some constants $\underline{\delta}$ and $\overline{\delta}$.

Lemma 2. *Let f be a function of sine type. For each $\epsilon > 0$ (in particular for $\epsilon = \underline{\delta}/2$) there exists a number $C_1 > 0$ such that*

$$|f(x+iy)| \geq C_1 \,\mathrm{e}^{\pi|y|}$$

outside the circles of radius ϵ centered at the zeros of f.

Furthermore, there is an important connection between the set of zeros $\{t_k\}_{k \in \mathbb{Z}}$ of sine-type functions, the basis properties of the system of exponentials $\{\mathrm{e}^{i\omega t_k}\}_{k \in \mathbb{Z}}$, and complete interpolating sequences [1, pp. 143–144].

Lemma 3. *If $\{t_k\}_{k\in\mathbb{Z}}$ is the set of zeros of a function of sine type, then the system $\{e^{i\omega t_k}\}_{k\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\pi,\pi]$, and $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$.*

Definition 2. We say that a sequence $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$ if the interpolation problem $f(t_k) = c_k$, $k \in \mathbb{Z}$, has exactly one solution $f \in \mathcal{PW}_\pi^2$ for every sequence $\{c_k\}_{k\in\mathbb{Z}}$ satisfying $\sum_{k=-\infty}^{\infty}|c_k|^2 < \infty$.

Since the sequence of sampling points $\{t_k\}_{k\in\mathbb{Z}}$ is a complete interpolating sequence for $\mathcal{PW}_\pi^2$, it follows by definition that, for each $k \in \mathbb{Z}$, there is exactly one function $\phi_k \in \mathcal{PW}_\pi^2$ that solves the interpolation problem

$$\phi_k(t_l) = \begin{cases} 1 & l = k \\ 0 & l \neq k. \end{cases} \tag{6}$$

Moreover, the product $\phi(z) = \lim_{N\to\infty} \prod_{|k|\leq N} (1 - z/t_k)$ converges uniformly on $|z| \leq R$ for all $R < \infty$ and ϕ is an entire function of exponential type π [2]. As a consequence,

$$\phi_k(t) = \frac{\phi(t)}{\phi'(t_k)(t-t_k)} \tag{7}$$

is the unique function in $\mathcal{PW}_\pi^2$ that solves the interpolation problem (6).

Lemma 3 implies that if ϕ is a function of sine type with zeros $\{t_k\}_{k\in\mathbb{Z}}$ then $\{\phi_k\}_{k\in\mathbb{Z}}$, where ϕ_k is given by (7), is a Riesz basis for $\mathcal{PW}_\pi^2$.

Lemma 4. *If $\{\phi_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{PW}_\pi^2$ then there exist positive constants A and B such that for all $M, N \in \mathbb{N}$ and arbitrary scalars c_k we have*

$$A \sum_{k=-M}^{N} |c_k|^2 \leq \int_{-\infty}^{\infty} \left| \sum_{k=-M}^{N} c_k\, \phi_k(\tau) \right|^2 \mathrm{d}\tau \leq B \sum_{k=-M}^{N} |c_k|^2. \tag{8}$$

Equation (8) is important for the convergence behavior of the series (2) for signals $f \in \mathcal{PW}_\pi^2$. If $\{\phi_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{PW}_\pi^2$ then, by virtue of equation (8), one has

$$\lim_{N\to\infty} \left\| f - \sum_{k=-N}^{N} f(t_k)\phi_k \right\|_\infty = 0 \tag{9}$$

for all signals $f \in \mathcal{PW}_\pi^2$. We will need (9) in the proof of Theorem 1.

Example 2. The Shannon sampling series is a special case of the general sampling series that are considered in this paper. Let $\phi(t) = \sin(\pi t)$ with zeros $t_k = k$, $k \in \mathbb{Z}$. Then $\phi'(t_k) = \pi\cos(\pi t_k) = \pi(-1)^k$ and

$$\phi_k(t) = \frac{\phi(t)}{\phi'(t_k)(t-t_k)} = \frac{(-1)^k \sin(\pi t)}{\pi(t-t_k)} = \frac{\sin\big(\pi(t-k)\big)}{\pi(t-k)}$$

is the well known sinc-kernel of the Shannon sampling series.

2. LOCAL CONVERGENCE BEHAVIOR

A well known fact [3] about the convergence behavior of the Shannon sampling series with equidistant samples (1) is Brown's theorem, which states that for all $f \in \mathcal{PW}_\pi^1$ and $T > 0$ fixed we have

$$\lim_{N\to\infty} \left(\max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| \right) = 0.$$

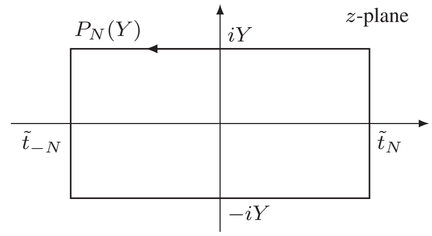

Fig. 1. Path $P_N(Y)$ in the Complex Plane.

This theorem plays a fundamental role in applications, because it establishes the uniform convergence on compact subsets of $\mathbb{R}$ for a large class of signals, namely $\mathcal{PW}_\pi^1$, which is the largest space within the scale of Paley-Wiener spaces.

In this section we prove that the uniform convergence on compact subsets of $\mathbb{R}$ still holds if non-equidistant sampling is used. In this sense Theorem 1 is an extension of Brown's theorem to non-uniform sampling.

Theorem 1. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered according to* (3)*. Furthermore, let ϕ_k be defined as in* (7)*. Then we have*

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t) \right| = 0$$

for all $T > 0$ and all $f \in \mathcal{PW}_\pi^1$.

Proof of Theorem 1. Let $T > 0$ and $f \in \mathcal{PW}_\pi^1$ be arbitrary but fixed and

$$\tilde{t}_n = \begin{cases} (t_{n+1} + t_n)/2 & \text{for } n \geq 1 \\ (t_{n-1} + t_n)/2 & \text{for } n \leq -1. \end{cases}$$

Furthermore consider, for $N \in \mathbb{N}$ and $Y > 0$, the path $P_N(Y)$ in the complex plane that is depicted in Figure 1. For all $N \in \mathbb{N}$ and $t \in \mathbb{R}$ we have the equality

$$\sum_{k=-N}^{N} f(t_k)\phi_k(t) = \frac{1}{2\pi i} \oint_{P_N(Y)} \frac{\phi(\zeta) - \phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)}\, \mathrm{d}\zeta. \tag{10}$$

Equation (10) can be easily seen by applying the method of residues. Note that by the choice of $P_N(Y)$ we have $\phi(\zeta) \neq 0$ for all $\zeta \in P_N(Y)$. Furthermore, for all $N \in \mathbb{N}$ and $t \in \mathbb{R}$ with $\tilde{t}_{-N} < t < \tilde{t}_N$, we have

$$\begin{aligned} &\frac{1}{2\pi i} \oint_{P_N(Y)} \frac{\phi(\zeta) - \phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)}\, \mathrm{d}\zeta \\ &\qquad = f(t) - \frac{1}{2\pi i} \oint_{P_N(Y)} \frac{\phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)}\, \mathrm{d}\zeta. \end{aligned} \tag{11}$$

For convenience, we introduce the abbreviation $(A_N^\phi f)(t) := \sum_{k=-N}^{N} f(t_k)\phi_k(t)$. Combining (10) and (11), it follows that

$$f(t) - (A_N^\phi f)(t) = \frac{1}{2\pi i} \oint_{P_N(Y)} \frac{\phi(t)}{\zeta - t} \frac{f(\zeta)}{\phi(\zeta)}\, \mathrm{d}\zeta \tag{12}$$

for all $N \in \mathbb{N}$ and $t \in \mathbb{R}$ with $\tilde{t}_{-N} < t < \tilde{t}_N$.

According to Lemma 1 there exist two positive constants $\overline{\delta}$ and $\underline{\delta}$ such that (4) and (5) are fulfilled. Next choose $Y_N = N\overline{\delta}$, $N \in \mathbb{N}$.

Since $|K|\underline{\delta} < |\tilde{t}_K| < (|K|+1)\overline{\delta}$, $|K| \in \mathbb{N}$, it follows that there are two positive constants C_2 and C_3 such that

$$Y_{|K|}/C_2 < |\tilde{t}_K| < Y_{|K|}/C_3 \tag{13}$$

for all $|K| \in \mathbb{N}$. Let N_0 be the smallest natural number for which $\min(t_{N_0}, |t_{-N_0}|) > T$. By using the identity (12), we obtain

$$\begin{aligned}
&|f(t) - (A_N^\phi f)(t)| \\
&\quad \le \frac{1}{2\pi}\int_{-Y_N}^{Y_N} \left|\frac{f(\tilde{t}_N + iy)}{\phi(\tilde{t}_N + iy)}\right| \frac{|\phi(t)|}{|\tilde{t}_N + iy - t|}\, \mathrm{d}y \\
&\qquad + \frac{1}{2\pi}\int_{-Y_N}^{Y_N} \left|\frac{f(\tilde{t}_{-N} + iy)}{\phi(\tilde{t}_{-N} + iy)}\right| \frac{|\phi(t)|}{|\tilde{t}_{-N} + iy - t|}\, \mathrm{d}y \\
&\qquad + \frac{1}{2\pi}\int_{\tilde{t}_{-N}}^{\tilde{t}_N} \left|\frac{f(x + iY_N)}{\phi(x + iY_N)}\right| \frac{|\phi(t)|}{|x + iY_N - t|}\, \mathrm{d}x \\
&\qquad + \frac{1}{2\pi}\int_{\tilde{t}_{-N}}^{\tilde{t}_N} \left|\frac{f(x - iY_N)}{\phi(x - iY_N)}\right| \frac{|\phi(t)|}{|x - iY_N - t|}\, \mathrm{d}x
\end{aligned} \tag{14}$$

for all $N \ge N_0$ and $t \in [-T, T]$. Next, we will upper bound the right hand side of (14). It is important that this bound is independent of N.

For all $x, y \in \mathbb{R}$, we have $|f(x+iy)| \le \mathrm{e}^{\pi|y|}\|f\|_{\mathcal{PW}_\pi^1}$. Furthermore, since ϕ is a function of sine type it follows from (4) and Lemma 2 that there exists a constant C_4 such that $|\phi(\tilde{t}_K + iy)| \ge C_4\,\mathrm{e}^{\pi|y|}$ for all $|K| \in \mathbb{N}$ and all $y \in \mathbb{R}$. Consequently, for the first term on the right hand side of (14) we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-Y_N}^{Y_N} \left|\frac{f(\tilde{t}_N + iy)}{\phi(\tilde{t}_N + iy)}\right| \frac{|\phi(t)|}{|\tilde{t}_N + iy - t|}\, \mathrm{d}y \\
&\le \frac{\|f\|_{\mathcal{PW}_\pi^1}\|\phi\|_\infty}{C_4}\frac{Y_N}{\pi(\tilde{t}_N - T)} \le \frac{\|f\|_{\mathcal{PW}_\pi^1}\|\phi\|_\infty}{C_4}\frac{\tilde{t}_N C_2}{(\tilde{t}_N - T)}
\end{aligned}$$

for all $N \ge N_0$, where we used (13) in the last inequality. Similarly, for the second term we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-Y_N}^{Y_N} \left|\frac{f(\tilde{t}_{-N} + iy)}{\phi(\tilde{t}_{-N} + iy)}\right| \frac{|\phi(t)|}{|\tilde{t}_{-N} + iy - t|}\, \mathrm{d}y \\
&\le \frac{\|f\|_{\mathcal{PW}_\pi^1}\|\phi\|_\infty}{C_4}\frac{Y_N}{\pi(|\tilde{t}_{-N}| - T)} \le \frac{\|f\|_{\mathcal{PW}_\pi^1}\|\phi\|_\infty}{C_4}\frac{|\tilde{t}_{-N}| C_2}{(|\tilde{t}_{-N}| - T)}
\end{aligned}$$

for all $N \ge N_0$. Using essentially the same estimates as before, the third and fourth term on the right hand side of (14) can both be upper bounded by $C_5\|f\|_{\mathcal{PW}_\pi^1}$ for all $N \ge N_0$. Next, we choose $N_1 \ge N_0$ such that $\max\left(\frac{\tilde{t}_{N_1}}{\tilde{t}_{N_1} - T}, \frac{|\tilde{t}_{-N_1}|}{|\tilde{t}_{-N_1}| - T}\right) \le 2$. Note that N_1 depends only on T and not on f. Consequently, there is a constant C_6 such that, for all $N \ge N_1$ and $t \in [-T, T]$, we have

$$|f(t) - (A_N^\phi f)(t)| \le C_6\|f\|_{\mathcal{PW}_\pi^1}. \tag{15}$$

Let $\epsilon > 0$ be arbitrary but fixed. There exists a function $f_\epsilon \in \mathcal{PW}_\pi^2$ such that $\|f - f_\epsilon\|_{\mathcal{PW}_\pi^1} < \epsilon$. As a consequence, we have

$$\begin{aligned}
&|f(t) - (A_N^\phi f)(t)| \\
&\le |f(t) - f_\epsilon(t) - (A_N^\phi(f - f_\epsilon))(t)| + |f_\epsilon(t) - (A_N^\phi f_\epsilon)(t)| \\
&\le C_6\|f - f_\epsilon\|_{\mathcal{PW}_\pi^1} + |f_\epsilon(t) - (A_N^\phi f_\epsilon)(t)| \\
&\le C_6\epsilon + |f_\epsilon(t) - (A_N^\phi f_\epsilon)(t)|
\end{aligned} \tag{16}$$

for all $N \ge N_1$ and $t \in [-T, T]$. Furthermore, since $f_\epsilon \in \mathcal{PW}_\pi^2$, we can use (9), i.e., the uniform convergence of the series. It follows that there exists a $N_2(\epsilon) \ge N_1$ such that $\max_{t\in[-T,T]}|f(t) - (A_N^\phi f)(t)| \le (C_6 + 1)\,\epsilon$ for all $N \ge N_2(\epsilon)$, which completes the proof. □

3. GLOBAL CONVERGENCE BEHAVIOR

3.1. Global Convergence Behavior with Oversampling

In [4] it has been shown that the Shannon sampling series with equidistant samples is uniformly convergent on whole of $\mathbb{R}$ for all $f \in \mathcal{PW}_\pi^1$ if oversampling is used. In Theorem 2 we will see that this result can be extended to non-equidistant sampling.

Theorem 2. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered according to* (3)*. Furthermore, let ϕ_k be defined as in* (7)*. Then, for all $0 < \beta < 1$ and all $f \in \mathcal{PW}_{\beta\pi}^1$, we have*

$$\lim_{N\to\infty}\max_{t\in\mathbb{R}}\left|f(t) - \sum_{k=-N}^{N} f(t_k)\phi_k(t)\right| = 0.$$

Proof. The proof of Theorem 2 is in the spirit of the proof of Theorem 1, but omitted due to space constraints. □

3.2. Global Convergence Behavior without Oversampling

In the sections 2 and 3.1 we gave two positive results for non-equidistant sampling, namely the local uniform convergence of (2) in the case where no oversampling is used, and the global uniform convergence of (2) in the case where oversampling is used. However, so far we made no statement about the global convergence behavior of (2) when no oversampling is used. In this section we analyze this remaining question.

Previous investigations [5] have shown for the space $\mathcal{PW}_\pi^1$ and a large class of reconstruction processes, that a globally bounded signal approximation is impossible if the samples are taken equidistantly at Nyquist rate. By using non-equidistant sampling, an additional degree of freedom is created, which may help to improve the convergence behavior. However, we suspect that non-equidistant sampling is not capable to improve the global convergence behavior.

In this section we restrict our analysis to entire functions ϕ with separated real zeros $\{t_k\}_{k\in\mathbb{Z}}$ that have a representation as Fourier-Stieltjes integral in the form

$$\phi(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i\omega t}\ \mathrm{d}\mu(\omega), \tag{17}$$

where $\mu(\omega)$ is a real function of bounded variation on the interval $[-\pi, \pi]$ and has a jump discontinuity at each endpoint. It can be shown that all functions ϕ with the representation (17) satisfy part (ii) of Definition 1 [1, p. 143], and hence the class of functions ϕ that we consider here is a subclass of the functions of sine type. In section 4.2, where we present a simple method to construct such functions, we will see that this subclass is still very large.

For this subclass the global convergence behavior is not improved by using non-equidistant sampling, because there exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\limsup_{N\to\infty}\max_{t\in\mathbb{R}}\left|f_1(t) - \sum_{k=-N}^{N} f_1(t_k)\phi_k(t)\right| = \infty.$$

4. APPLICATIONS

4.1. Stochastic Processes

In this section we analyze the mean square sense convergence behavior of (2) for bandlimited stochastic processes X. We restrict our analysis to wide-sense stationary processes. Furthermore, we assume that X is mean square continuous, which implies the correlation function R_X has a representation $R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{i\omega\tau} \ \mathrm{d}\mu(\omega)$, with a positive and finite measure μ. For details and further facts we refer to the standard literature [6]. We additionally assume that the measure μ is absolutely continuous with respect to the Lebesgue measure λ, which implies that there exists a function $S_X \in L^1(\mathbb{R})$ with $\mathrm{d}\mu = S_X \ \mathrm{d}\lambda$. S_X is called power spectral density.

Definition 3. We call a bandlimited wide-sense stationary process X I-processes if its correlation function R_X has the representation

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(\omega)\,\mathrm{e}^{i\omega\tau} \ \mathrm{d}\omega, \tag{18}$$

for some non-negative $S_X \in L^1[-\pi,\pi]$.

It is well known that for all I-processes X and $T > 0$ we have

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \mathbb{E}\left| X(t) - \sum_{k=-N}^{N} X(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|^2 = 0,$$

i.e., the variance of the reconstruction error is bounded on all compact subsets of $\mathbb{R}$ and converges to zero for $N \to \infty$ [7]. Using the findings from the previous sections we can extend this result to non-equidistant sampling, and furthermore, we are even able to make statements about the global convergence behavior. Due to space constraints we omit the proofs.

Theorem 3. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered according to* (3). *Then, for all I-processes X and all $T > 0$, we have*

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \mathbb{E}\left| X(t) - \sum_{k=-N}^{N} X(t_k)\phi_k(t)\right|^2 = 0.$$

Theorem 3 shows that we have a good local convergence behavior, however for practical applications it is important to upper bound the mean-square approximation error over whole of $\mathbb{R}$. This is not possible in general, because there exist I-processes such that the global mean-square approximation error increases unboundedly.

Theorem 4. *Let ϕ be a function of sine type that has the representation* (17)*, and whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered according to* (3). *Then there exists an I-process X_1 such that*

$$\limsup_{N\to\infty} \sup_{t\in\mathbb{R}} \mathbb{E}\left| X_1(t) - \sum_{k=-N}^{N} X_1(t_k)\phi_k(t)\right|^2 = \infty.$$

The next theorem shows, similar to the deterministic case in section 3.1, that oversampling improves the convergence behavior of (2) for I-processes.

Theorem 5. *Let ϕ be a function of sine type, whose zeros $\{t_k\}_{k\in\mathbb{Z}}$ are all real and ordered according to* (3). *Then, for all $0 < \beta < 1$ and all I-processes X, whose power spectral density $S_X(\omega)$ is supported in $[-\beta\pi, \beta\pi]$, we have*

$$\sup_{N\in\mathbb{N}} \sup_{t\in\mathbb{R}} \mathbb{E}\left| \sum_{k=-N}^{N} X(t_k)\phi_k(t)\right|^2 < \infty.$$

4.2. Construction of Possible Sampling Patterns

Next, consider for an arbitrary real-valued signal $g \in \mathcal{PW}_\pi^1$, with $\|g\|_{\mathcal{PW}_\pi^1} < 1$, the function

$$\phi_g(t) = g(t) - \cos(\pi t). \tag{19}$$

Functions of this kind were analyzed for example in [8]. Since

$$\begin{aligned}\phi_g(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{g}(\omega)\,\mathrm{e}^{i\omega t} \ \mathrm{d}\omega - \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i\omega t} \ \mathrm{d}\mu_1(\omega)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{i\omega t} \ \mathrm{d}\mu_2(\omega)\end{aligned}$$

with $\mu_2(\omega) = -\mu_1(\omega) + \int_{-\pi}^{\omega} \hat{g}(\omega_1) \ \mathrm{d}\omega_1$, we see that ϕ_g is a function of sine type. The zeros $\{t_k\}_{k\in\mathbb{Z}}$ of ϕ_g are all real, because we assumed that g is real-valued and $\|g\|_{\mathcal{PW}_\pi^1} \leq 1$. Thus, by equation (19) we have a method to construct arbitrarily many functions of sine type ϕ_g and hence arbitrarily many sampling patterns $\{t_k\}_{k\in\mathbb{Z}}$ for which the theorems in this paper are valid. The sampling points $\{t_k\}_{k\in\mathbb{Z}}$ are nothing but the crossings of some bandlimited signal $g \in \mathcal{PW}_\pi^1$, $\|g\|_{\mathcal{PW}_\pi^1} \leq 1$, with the cosine function. It follows by Theorem 2 that

$$\sum_{k=-N}^{N} f(t_k)\phi_{g,k}(t), \tag{20}$$

where $\phi_{g,k}(t) = \phi_g(t)/(\phi_g'(t_k)(t-t_k))$, is globally uniformly convergent for all $f \in \mathcal{PW}_{\beta\pi}^1$, $0 < \beta < 1$, and in particular for $f = g$. Furthermore, we know by section 3.2 that, in the case without oversampling, i.e., $f \in \mathcal{PW}_\pi^1$, (20) is only locally uniformly convergent and not globally uniformly convergent in general. However, this does not answer the question whether $\sum_{k=-N}^{N} g(t_k)\phi_{g,k}(t)$, i.e., the sampling series with matched reconstruction function, is uniformly convergent on whole of $\mathbb{R}$ for all $g \in \mathcal{PW}_\pi^1$. We conjecture that even in this case the series is not globally uniformly convergent in general.

5. REFERENCES

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