ACHIEVABLE THROUGHPUT APPROXIMATION FOR RBD PRECODING AT HIGH SNRS

Bin Song and Martin Haardt

Communications Research Laboratory
Ilmenau University of Technology
P. O. Box 100565, D-98694 Ilmenau, Germany, http://www.tu-ilmenau.de/crl
Email: {bin.song, martin.haardt}@tu-ilmenau.de

Abstract — In this paper, we study the achievable throughput for regularized block diagonalization (RBD) precoding in multi-user MIMO broadcast channels at high SNRs. By applying an analytical framework for a high SNR affine approximation to capacity, we derive the multiplexing gains and the power offsets for RBD in two cases separately. In the first case, we assume that the aggregate number of receive antennas is less than or equal to the number of transmit antennas. It is found that RBD can maintain the same multiplexing gain as dirty paper coding (DPC) and block diagonalization (BD) precoding at high SNRs and has a smaller power offset than BD. The sum rate differences relative to DPC and BD are analyzed and bounded as simple functions of the system parameters. In the second case, we assume that the aggregate number of receive antennas is larger than the number of transmit antennas. Although RBD can still be performed, the achievable throughput is degraded with an increasing number of receive antennas. The benefit of spatial multiplexing is completely lost due to a unit spatial multiplexing gain at high SNRs.

***Index Terms*—** Muli-user MIMO, dirty paper coding, linear precoding, capacity approximation

1. INTRODUCTION

Compared to point-to-point MIMO channels, multi-user MIMO broadcast channels can significantly increase the capacity and spectral efficiency by using space division multiple access (SDMA) to simultaneously transmit multiple data streams to a group of users. As an optimal SDMA strategy, dirty paper coding (DPC) has been shown to achieve the capacity region of the Gaussian MIMO broadcast channels in [1]. However, deploying DPC in real systems is very impractical due to the significant additional complexity at both transmitter and receiver.

An alternative sub-optimal SDMA strategy is linear precoding, which has a lower complexity and is able to transmit the same number of data streams as a DPC based system. Therefore linear precoding can achieve the same multiplexing gain as DPC, but does incur a sum rate loss compared to DPC. The authors of reference [2] have analyzed the ratio between the achievable sum rates of DPC and block diagonalization (BD) precoding [3]. In [4] the absolute rate and power offsets between these algorithms have been studied at high SNRs .

In this work we further consider a recently proposed linear precoding technique, regularized block diagonalization (RBD) [5], which has an improved sum rate and diversity order relative to BD. Furthermore, RBD has the advantage that it is not constrained by the dimensionality condition that the aggregate number of receive antennas is not larger than the number of transmit antennas.

The key analytical framework used in this paper is an affine approximation for the system capacity at high SNRs proposed in [6]. Thereby, we approximate the achievable throughput of an RBD based system and consider two cases separately. In the first case we assume that the aggregate number of receive antennas is less than or equal to the number of transmit antennas. Compared to DPC and BD based systems, the bounds of the average rate and power offsets among these strategies are derived as a function of the system parameters (e.g., the number of users and receive antennas). In the second case we assume that the aggregate number of receive antennas is larger than the number of transmit antennas. It is shown that the same multiplexing gain as for DPC cannot be maintained anymore. With an increase of the number of receive antennas, the achievable throughput is gradually degraded.

2. SYSTEM MODEL

We consider a multi-user MIMO system with a single base station (BS) and K users, where the BS is equipped with M_{T} transmit antennas and each user has M_r receive antennas. The aggregate number of receive antennas, denoted by M_{R}, is equal to $K \cdot M_r$. The propagation channel between the BS and each user is assumed to have a spatially uncorrelated Gaussian distribution. The received signal of the $i\mathrm{th}$ user is expressed as

$$\boldsymbol{y}_i = \boldsymbol{H}_i \boldsymbol{x} + \boldsymbol{n}_i \tag{1}$$

where $\boldsymbol{x} \in \mathbb{C}^{M_{\mathrm{T}} \times 1}$ is the transmit signal vector. Under an average total power limitation P_{T} at the BS, we require that $\mathrm{tr}(\mathrm{E}\left\{\boldsymbol{x}\boldsymbol{x}^H\right\}) \leq P_{\mathrm{T}}$. The matrix $\boldsymbol{H}_i \in \mathbb{C}^{M_r \times M_{\mathrm{T}}}$ is the channel gain matrix for user i and the vector $\boldsymbol{n}_i \in \mathbb{C}^{M_r \times 1}$ represents Additive White Gaussian Noise (AWGN) with unit variance. In this paper we assume that each user has perfect knowledge of its own channel and the BS has perfect knowledge of all users' channels. The signal-to-noise ratio (SNR) is defined as the ratio between P_{T} and the noise variance, i.e., $\mathrm{SNR} \hat{=} P_{\mathrm{T}}$.

2.1. Capacity Approximation Framework

The capacity approximation framework used in this paper has been developed in [6], where the channel capacity $C(P_{\mathrm{T}})$ is well approximated at high SNRs as

$$\begin{aligned} C(P_{\mathrm{T}}) &= \mathcal{S}_\infty \cdot (\log_2 P_{\mathrm{T}} - \mathcal{L}_\infty) + o(1)\,, && (2)\\ \text{where} \quad \mathcal{S}_\infty &= \lim_{P_{\mathrm{T}} \to \infty} \log_2 e \cdot P_{\mathrm{T}} \dot{C}(P_{\mathrm{T}}) && (3)\\ \text{and} \quad \mathcal{L}_\infty &= \lim_{P_{\mathrm{T}} \to \infty} (\log_2 P_{\mathrm{T}} - \frac{C(P_{\mathrm{T}})}{\mathcal{S}_\infty})\,. && (4) \end{aligned}$$

The authors gratefully acknowledge the partial support of the German Research Foundation (Deutsche Forschungsgemeinschaft, DFG) under contract no. HA 2239/1-2

Here $\mathcal{S}_\infty$ represents the multiplexing gain (i.e., the asymptotic slope of the spectral efficiency in bps/Hz per 3 dB) and $\mathcal{L}_\infty$ refers to the power offset in 3 dB units. $\dot{C}(P_{\mathrm{T}})$ denotes the first derivative of the capacity with respect to P_{T}. For either multi-user MIMO broadcast channels or point-to-point MIMO channels, the multiplexing gain $\mathcal{S}_\infty$ is found to be the minimum of the aggregate number of receive antennas and the number of transmit antennas under the assumption of uncorrelated Rayleigh fading. The rate offset $\mathcal{L}_\infty$ depends on the fading statistics and the signaling strategy.

3. SUM RATE APPROXIMATION ($M_{\mathrm{T}} \geq M_{\mathrm{R}}$)

In this section we compute the high SNR affine approximation to the RBD based sum rate under the condition that the aggregate number of receive antennas is less than or equal to the number of transmit antennas. Furthermore, compared to DPC and BD based sum rates, we derive the bounds for the average rate and power offsets among them.

3.1. Dirty Paper Coding

In [7] the DPC sum rate is shown to converge to the capacity of a point-to-point MIMO channel with the matrix $\boldsymbol{H} \in \mathbb{C}^{M_{\mathrm{T}} \times M_{\mathrm{R}}}$ under the condition $M_{\mathrm{T}} \geq M_{\mathrm{R}}$, thus

$$\lim_{P_{\mathrm{T}} \to \infty} \left(C_{\mathrm{DPC}}(P_{\mathrm{T}}) - \log_2 \left| \boldsymbol{I} + \frac{P_{\mathrm{T}}}{M_{\mathrm{R}}} \boldsymbol{H}^H \boldsymbol{H} \right| \right) = 0 \,. \tag{5}$$

Using this result we write the DPC sum rate at high SNRs as

$$C_{\mathrm{DPC}}(P_{\mathrm{T}}) \cong \log_2 \left| \boldsymbol{I} + \frac{P_{\mathrm{T}}}{M_{\mathrm{R}}} \boldsymbol{H}^H \boldsymbol{H} \right| \,, \tag{6}$$

which corresponds to the fact that we choose the covariance matrix $\boldsymbol{Q} = \frac{P_{\mathrm{T}}}{M_{\mathrm{R}}} \boldsymbol{I}_{M_{\mathrm{T}}}$ as an asymptotically optimal solution at high SNRs [4]. Then using the approximation framework of [6] we get

$$\begin{aligned} \mathcal{S}_\infty &= M_{\mathrm{R}} \,, & (7) \\ \mathcal{L}_\infty &= \log_2 M_{\mathrm{R}} - \frac{1}{\mathcal{S}_\infty} \log_2 \left| \boldsymbol{H}^H \boldsymbol{H} \right| \,, & (8) \end{aligned}$$

and the approximation of the DPC sum rate at high SNRs can be found as

$$C_{\mathrm{DPC}}(P_{\mathrm{T}}) \cong M_{\mathrm{R}} \log_2 P_{\mathrm{T}} - M_{\mathrm{R}} \log_2 M_{\mathrm{R}} + \log_2 \left| \boldsymbol{H}^H \boldsymbol{H} \right| \,. \tag{9}$$

3.2. Regularized Block Diagonalization Precoding

In a multi-user MIMO system RBD is deployed to suppress multi-user interference (MUI) while achieving a high sum rate. In this case the transmit signal vector $\boldsymbol{x}$ in equation (1) can be rewritten as

$$\boldsymbol{x} = \textstyle\sum_{i=1}^{K} \boldsymbol{F}_i \boldsymbol{s}_i \tag{10}$$

where $\boldsymbol{s}_i \in \mathbb{C}^{r_i \times 1}$ contains the data symbols for user i and r_i represents the number of data streams intended for user i. The matrix $\boldsymbol{F}_i \in \mathbb{C}^{M_{\mathrm{T}} \times r_i}$ denotes the RBD precoding matrix for user i. Then the received signal for user i is given by

$$\boldsymbol{y}_i = \boldsymbol{H}_i \boldsymbol{F}_i \boldsymbol{s}_i + \textstyle\sum_{j=1, j\neq i}^{K} \boldsymbol{H}_i \boldsymbol{F}_j \boldsymbol{s}_j + \boldsymbol{n}_i \,. \tag{11}$$

With RBD the channel $\boldsymbol{H}_i$ is converted to an equivalent channel $\bar{\boldsymbol{H}}_i = \boldsymbol{H}_i \boldsymbol{F}_i$ which has less overlap with the other users' channels. The achievable sum rate of RBD based system is expressed as

$$C_{\mathrm{RBD}}(P_{\mathrm{T}}) = \max_{\sum_i^K \mathrm{tr}(\boldsymbol{Q}_i) \leq P_{\mathrm{T}}} \textstyle\sum_{i=1}^{K} \log_2 \left| \boldsymbol{I} + \bar{\boldsymbol{H}}_i \boldsymbol{Q}_i \bar{\boldsymbol{H}}_i^H \right| \,, \tag{12}$$

where we set the covariance matrix $\boldsymbol{Q}_i = \mathrm{E}\left\{ \boldsymbol{s}_i \boldsymbol{s}_i^H \right\}$ to be $\frac{P_{\mathrm{T}}}{M_{\mathrm{R}}} \boldsymbol{I}_{r_i}$ for a fair comparison with DPC. Thus, $\mathcal{S}_\infty$ and $\mathcal{L}_\infty$ for RBD can be calculated from equations (3) and (4), and we get

$$\begin{aligned} \mathcal{S}_\infty &= M_{\mathrm{R}} \,, & (13) \\ \mathcal{L}_\infty &= \log_2 M_{\mathrm{R}} - \frac{1}{\mathcal{S}_\infty} \log_2 \textstyle\prod_{i=1}^{K} \left| \bar{\boldsymbol{H}}_i \bar{\boldsymbol{H}}_i^H \right| \,. & (14) \end{aligned}$$

It is found that RBD can maintain the same multiplexing gain as DPC, but has a different power offset.

Now let us further quantify the power offset for RBD. The RBD precoding matrix of user i is described as [5]

$$\boldsymbol{F}_i = \gamma \boldsymbol{F}_{a_i} \boldsymbol{F}_{b_i}, \quad \text{where } \boldsymbol{F}_{a_i} = \widetilde{\boldsymbol{V}}_i (\widetilde{\boldsymbol{\Sigma}}_i^T \widetilde{\boldsymbol{\Sigma}}_i + \alpha \boldsymbol{I}_{M_{\mathrm{T}}})^{-1/2} \,. \tag{15}$$

The matrix $\boldsymbol{F}_{a_i}$ is used to suppress MUI while balancing it with noise enhancement. The matrix $\boldsymbol{F}_{b_i}$ can further optimize the system performance by optimal power loading. In this work we assume that $\boldsymbol{F}_{b_i}$ is unitary. The parameter γ is chosen to set the total transmit power to P_{T}, and $\alpha = \left(\frac{P_{\mathrm{T}}}{M_{\mathrm{R}} \sigma_n^2} \right)^{-1}$. The matrices $\widetilde{\boldsymbol{V}}_i \in \mathbb{C}^{M_{\mathrm{T}} \times M_{\mathrm{T}}}$ and $\widetilde{\boldsymbol{\Sigma}}_i \in \mathbb{C}^{(M_{\mathrm{R}} - M_r) \times M_{\mathrm{T}}}$ are the right singular vectors and the diagonal matrix of the singular values of the combined channel matrix of all other users, respectively.

Note that in [5] equation (15) is derived under the condition $\mathrm{E}\left\{ \boldsymbol{s}_i \boldsymbol{s}_i^H \right\} = \boldsymbol{I}_{r_i}$. Taking into account $\mathrm{E}\left\{ \boldsymbol{s}_i \boldsymbol{s}_i^H \right\} = \frac{P_{\mathrm{T}}}{M_{\mathrm{R}}} \boldsymbol{I}_{r_i}$ in our system model, we rewrite the expressions (22) and (26) in [5] as

$$\boldsymbol{F}_a = \min_{\boldsymbol{F}_a} \mathrm{E}\Big\{ \textstyle\sum_{i=1}^{K} \left\| \widetilde{\boldsymbol{H}}_i \boldsymbol{F}_{a_i} \boldsymbol{s}_i \right\|_{\mathrm{F}}^2 + \frac{\|\boldsymbol{n}_i\|_{\mathrm{F}}^2}{\beta^2} \Big\} \quad \text{and} \tag{16}$$

$$\beta^2 = \frac{P_{\mathrm{T}}}{\sum_{i=1}^{K} \mathrm{tr}(\boldsymbol{F}_{a_i} \boldsymbol{s}_i \boldsymbol{s}_i^H \boldsymbol{F}_{a_i}^H)} = \frac{M_{\mathrm{R}}}{\sum_{i=1}^{K} \mathrm{tr}(\boldsymbol{F}_{a_i} \boldsymbol{F}_{a_i}^H)} \,. \tag{17}$$

Then using a similar derivation as in [5] we get a new expression for $\boldsymbol{F}_{a_i}$ as

$$\boldsymbol{F}_{a_i} = \widetilde{\boldsymbol{V}}_i \Big(\frac{P_{\mathrm{T}}}{M_{\mathrm{R}}} \widetilde{\boldsymbol{\Sigma}}_i^T \widetilde{\boldsymbol{\Sigma}}_i + \sigma_n^2 \boldsymbol{I}_{M_{\mathrm{T}}} \Big)^{-1/2} \,. \tag{18}$$

Considering $\sigma_n^2 = 1$ in this work, we approximate $\left| \bar{\boldsymbol{H}}_i \bar{\boldsymbol{H}}_i^H \right|$ with the above expression as follows

$$\begin{aligned}
&\left| \bar{\boldsymbol{H}}_i \bar{\boldsymbol{H}}_i^H \right| \\
&= \left| \boldsymbol{H}_i \widetilde{\boldsymbol{V}}_i \Big(\frac{P_{\mathrm{T}}}{M_{\mathrm{R}}} \widetilde{\boldsymbol{\Sigma}}_i^T \widetilde{\boldsymbol{\Sigma}}_i + \boldsymbol{I}_{M_{\mathrm{T}}} \Big)^{-1} \widetilde{\boldsymbol{V}}_i^H \boldsymbol{H}_i^H \right| \\
&\overset{(1)}{=} \left| \boldsymbol{H}_i \Big[\widetilde{\boldsymbol{V}}_i^{(1)} \; \widetilde{\boldsymbol{V}}_i^{(0)} \Big] \Big(\frac{P_{\mathrm{T}}}{M_{\mathrm{R}}} \widetilde{\boldsymbol{\Sigma}}_i^T \widetilde{\boldsymbol{\Sigma}}_i + \boldsymbol{I}_{M_{\mathrm{T}}} \Big)^{-1} \Big[\widetilde{\boldsymbol{V}}_i^{(1)} \; \widetilde{\boldsymbol{V}}_i^{(0)} \Big]^H \boldsymbol{H}_i^H \right| \\
&\overset{(2)}{=} \Big| \boldsymbol{H}_i \widetilde{\boldsymbol{V}}_i^{(1)} \Big(\frac{P_{\mathrm{T}}}{M_{\mathrm{R}}} \widetilde{\boldsymbol{\Lambda}}_i^2 + \boldsymbol{I}_{(M_{\mathrm{R}} - M_r)} \Big)^{-1} \widetilde{\boldsymbol{V}}_i^{(1)H} \boldsymbol{H}_i^H \\
&\qquad\qquad + \boldsymbol{H}_i \widetilde{\boldsymbol{V}}_i^{(0)} \widetilde{\boldsymbol{V}}_i^{(0)H} \boldsymbol{H}_i^H \Big| \\
&\overset{(3)}{\approx} \left| \boldsymbol{H}_i \widetilde{\boldsymbol{V}}_i^{(1)} \Big(\frac{P_{\mathrm{T}}}{M_{\mathrm{R}}} \widetilde{\boldsymbol{\Lambda}}_i^2 \Big)^{-1} \widetilde{\boldsymbol{V}}_i^{(1)H} \boldsymbol{H}_i^H + \boldsymbol{H}_i \widetilde{\boldsymbol{V}}_i^{(0)} \widetilde{\boldsymbol{V}}_i^{(0)H} \boldsymbol{H}_i^H \right| \\
&\overset{(4)}{=} \left| \frac{M_{\mathrm{R}}}{P_T} \boldsymbol{G}_i^{(1)} \boldsymbol{G}_i^{(1)H} + \boldsymbol{G}_i^{(0)} \boldsymbol{G}_i^{(0)H} \right| \\
&\overset{(5)}{\approx} \left| \boldsymbol{G}_i^{(0)} \boldsymbol{G}_i^{(0)H} \right| \Big(1 + \frac{M_{\mathrm{R}}}{P_T} \underbrace{\mathrm{tr} \Big[(\boldsymbol{G}_i^{(0)} \boldsymbol{G}_i^{(0)H})^{-1} \boldsymbol{G}_i^{(1)} \boldsymbol{G}_i^{(1)H} \Big]}_{\mu_i} \Big) \\
&= \left| \boldsymbol{G}_i^{(0)} \boldsymbol{G}_i^{(0)H} \right| \Big(1 + \frac{\mu_i M_{\mathrm{R}}}{P_T} \Big) \,.
\end{aligned} \tag{19}$$

At step (1) we separate $\widetilde{\boldsymbol{V}}_i$ into $\widetilde{\boldsymbol{V}}_i^{(1)} \in \mathbb{C}^{M_{\mathrm{T}} \times (M_{\mathrm{R}} - M_r)}$ and $\widetilde{\boldsymbol{V}}_i^{(0)} \in \mathbb{C}^{M_{\mathrm{T}} \times (M_{\mathrm{T}} - M_{\mathrm{R}} + M_r)}$, which refer to the right singular vectors corresponding to non-zero singular values and the right singular vectors corresponding to zero singular values, respectively. At step (2) we use $\widetilde{\boldsymbol{\Lambda}}_i \in \mathbb{C}^{(M_{\mathrm{R}}-M_r)\times(M_{\mathrm{R}}-M_r)}$ to represent the economy-size version of $\widetilde{\boldsymbol{\Sigma}}_i$. For large P_{T}, we neglect $\boldsymbol{I}_{(M_{\mathrm{R}}-M_r)}$ and reach step (3). Then we replace $\boldsymbol{H}_i\widetilde{\boldsymbol{V}}_i^{(1)}\widetilde{\boldsymbol{\Lambda}}_i^{-1}$ and $\boldsymbol{H}_i\widetilde{\boldsymbol{V}}_i^{(0)}$ by $\boldsymbol{G}_i^{(1)}$ and $\boldsymbol{G}_i^{(0)}$, respectively. At step (4) we use the following property of matrix determinants

$$\det(\boldsymbol{A}+\epsilon\boldsymbol{X}) = \det(\boldsymbol{A})(1+\mathrm{tr}(\boldsymbol{A}^{-1}\boldsymbol{X})\epsilon)+o(\epsilon^2), \qquad (20)$$

where $\boldsymbol{A}$ and $\boldsymbol{X}$ are square matrices and ϵ is very small number, which leads to step (5).

Due to the fact that

$$0 \le \mathrm{tr}(\boldsymbol{AB})^n \le \mathrm{tr}(\boldsymbol{A})^n\mathrm{tr}(\boldsymbol{B})^n \qquad (21)$$

if $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive semi-definite matrices of the same order, we can bound μ_i defined in equation (19) as

$$0 \le \mu_i \le \mathrm{tr}\Big((\boldsymbol{G}_i^{(0)}\boldsymbol{G}_i^{(0)H})^{-1}\Big)\mathrm{tr}\Big(\boldsymbol{G}_i^{(1)}\boldsymbol{G}_i^{(1)H}\Big). \qquad (22)$$

Substituting equation (19) into equation (14), $\mathcal{L}_\infty$ can be rewritten as

$$\mathcal{L}_\infty = \log_2 M_{\mathrm{R}} - \frac{1}{M_{\mathrm{R}}}\log_2\prod_{i=1}^K\left|\boldsymbol{G}_i^{(0)}\boldsymbol{G}_i^{(0)H}\right| - \frac{1}{M_{\mathrm{R}}}\log_2\prod_{i=1}^K\left(1+\tfrac{\mu_i M_{\mathrm{R}}}{P_T}\right). \quad (23)$$

As a result, we approximate the RBD sum rate at high SNR as

$$C_{\mathrm{RBD}}(P_{\mathrm{T}}) \cong M_{\mathrm{R}}\log_2 P_{\mathrm{T}} - M_{\mathrm{R}}\log_2 M_{\mathrm{R}} + \log_2\prod_{i=1}^K\left|\boldsymbol{G}_i^{(0)}\boldsymbol{G}_i^{(0)H}\right| + \log_2\prod_{i=1}^K\left(1+\tfrac{\mu_i M_{\mathrm{R}}}{P_T}\right). \qquad (24)$$

3.3. Block Diagonalization Precoding

BD is an extension of zero-forcing precoding for the case that the users have multiple receive antennas [3]. With BD, the ith user's precoding matrix $\boldsymbol{F}_i$ lies in the null space of all other users' channels, i.e., $\widetilde{\boldsymbol{V}}_i^{(0)} \in \mathbb{C}^{M_{\mathrm{T}}\times(M_{\mathrm{T}}-M_{\mathrm{R}}+M_r)}$. Thus the system is converted into K parallel MIMO channels with effective channel matrices $\boldsymbol{G}_i^{(0)} = \boldsymbol{H}_i\widetilde{\boldsymbol{V}}_i^{(0)}, i=1,\cdots,K$. There is no MUI at each user. The sum rate approximation for BD at high SNRs is given by [4]

$$C_{\mathrm{BD}}(P_{\mathrm{T}}) \cong M_{\mathrm{R}}\log_2 P_{\mathrm{T}} - M_{\mathrm{R}}\log_2 M_{\mathrm{R}} + \log_2\prod_{i=1}^K\left|\boldsymbol{G}_i^{(0)}\boldsymbol{G}_i^{(0)H}\right|, \qquad (25)$$

which has the same multiplexing gain as DPC and RBD whenever $M_{\mathrm{T}} \ge M_{\mathrm{R}}$, but a different power offset.

3.4. Average Rate and Power Offset

In this subsection we derive bounds of the average rate and power offsets among the sum rates of DPC, RBD, and BD at high SNRs.

3.4.1. RDB vs. BD

We define the rate offset between the sum rates of RBD and BD at high SNRs as

$$\Delta_{\mathrm{RBD-BD}} = \lim_{P_{\mathrm{T}}\to\infty}\left[C_{\mathrm{RBD}}(P_{\mathrm{T}}) - C_{\mathrm{BD}}(P_{\mathrm{T}})\right], \qquad (26)$$

which is for one channel realization. By averaging over the fading distribution, we can get the average rate offset as

$$\bar{\Delta}_{\mathrm{RBD-BD}} = \mathrm{E}\left\{\Delta_{\mathrm{RBD-BD}}\right\}. \qquad (27)$$

Theorem 1. The average rate offset between RBD and BD at high SNRs is upper bounded by

$$\bar{\Delta}_{\mathrm{RBD-BD}} \le K\log_2\left(1+\frac{\mu M_{\mathrm{R}}}{P_T}\right),$$

where $\begin{cases} 0 \le \mu \le \frac{M_r^2(M_{\mathrm{R}}-M_r)}{M_{\mathrm{T}}-M_{\mathrm{R}}} & \text{, for } M_{\mathrm{T}} > M_{\mathrm{R}} \\ 0 \le \mu \le \frac{(M_r^2/\xi+M_r-\xi)(M_{\mathrm{R}}-M_r)}{2-\xi/M_r} & \text{, for } M_{\mathrm{T}} = M_{\mathrm{R}} \end{cases}$

and ξ is a small positive number which should be less than or equal to the smallest eigenvalue of $\boldsymbol{G}_i^{(0)}\boldsymbol{G}_i^{(0)H}$. (28)

This expression can be proved by substituting equations (24) and (25) into equation (27), and we get

$$\begin{aligned}\bar{\Delta}_{\mathrm{RBD-BD}} &= K\mathrm{E}\left\{\log_2\left(1+\frac{\mu_i M_{\mathrm{R}}}{P_{\mathrm{T}}}\right)\right\} \\ &\le K\log_2\left(1+\frac{M_{\mathrm{R}}}{P_{\mathrm{T}}}\mathrm{E}\left\{\mu_i\right\}\right),\end{aligned} \qquad (29)$$

where

$$\begin{aligned} 0 \le \mathrm{E}\left\{\mu_i\right\} &\le \overbrace{\mathrm{E}\left\{\mathrm{tr}\Big((\boldsymbol{G}_i^{(0)}\boldsymbol{G}_i^{(0)H})^{-1}\Big)\right\}}^{(a)}\overbrace{\mathrm{E}\left\{\mathrm{tr}\Big(\boldsymbol{G}_i^{(1)}\boldsymbol{G}_i^{(1)H}\Big)\right\}}^{(b)} \\ &= \frac{M_r^2(M_{\mathrm{R}}-M_r)}{M_{\mathrm{T}}-M_{\mathrm{R}}}. \end{aligned} \qquad (30)$$

According to the properties of Wishart matrices in [8], we calculcate the term (b) and the term (a) as $M_r(M_{\mathrm{R}}-M_r)$ and $\frac{M_r}{M_{\mathrm{T}}-M_{\mathrm{R}}}$, respectively. For the case that $M_{\mathrm{T}} = M_{\mathrm{R}}$, the term (a) can be evaluated by equation (9) in [9].

It can been seen that with increasing SNR the average rate offset between RBD and BD converges to zero. In other words, using RBD, at very high SNR each user transmits only in the null space of all other users as in BD precoding.

The average rate offset can easily be translated into an average power offset. We represent the average power offset by $\Delta\bar{P}_{\mathrm{RBD-BD}}$ and get

$$\Delta\bar{P}_{\mathrm{RBD-BD}} = \frac{3\bar{\Delta}_{\mathrm{RBD-BD}}}{\mathcal{S}_\infty} = \frac{3\bar{\Delta}_{\mathrm{RBD-BD}}}{M_{\mathrm{R}}}. \qquad (31)$$

3.4.2. DPC vs. RBD

The rate offset between the sum rates of DPC and RBD at high SNRs is also defined as

$$\Delta_{\mathrm{DPC-RBD}} = \lim_{P_{\mathrm{T}}\to\infty}\left[C_{\mathrm{DPC}}(P_{\mathrm{T}}) - C_{\mathrm{RBD}}(P_{\mathrm{T}})\right]. \qquad (32)$$

Averaging over the fading distribution, we calculate the average rate offset by

$$\bar{\Delta}_{\mathrm{DPC-RBD}} = \mathrm{E}\left\{\Delta_{\mathrm{DPC-RBD}}\right\}. \qquad (33)$$

Theorem 2. The average rate offset between DPC and RBD at high SNRs is low bounded by

$$\bar{\Delta}_{\mathrm{DPC-RBD}} \ge \log_2 e\sum_{m=0}^{M_{\mathrm{R}}-1}\varphi(M_{\mathrm{T}}-m) - K\log_2(1+\tfrac{\mu M_{\mathrm{R}}}{P_T}) - K\cdot\log_2 e\sum_{n=0}^{M_r-1}\varphi(M_{\mathrm{T}}-M_{\mathrm{R}}+M_r-n), \qquad (34)$$

where $\varphi(\cdot)$ denotes the digamma function.

This expression can be proved by substituting equations (9) and (24) into equation (33), and we get

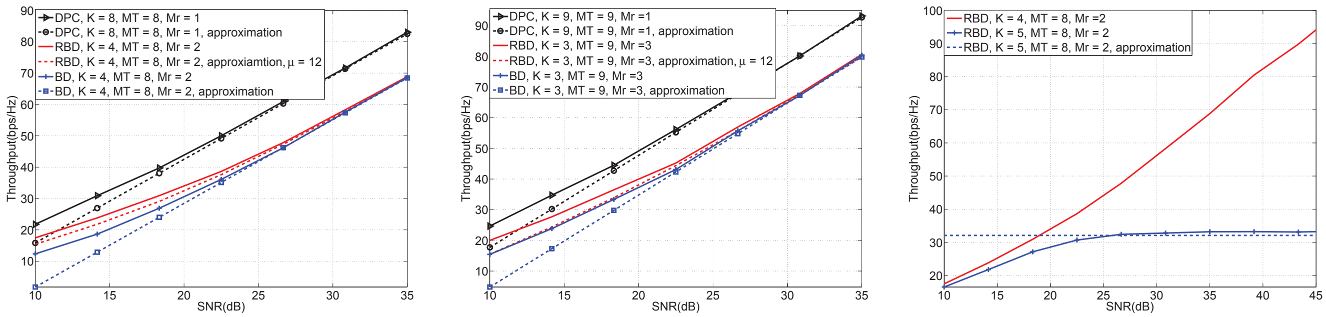
$$\bar{\Delta}_{\mathrm{DPC-RBD}} \ge \mathrm{E}\left\{\log_2\left|\boldsymbol{H}^H\boldsymbol{H}\right|\right\} - K\log_2\left(1+\frac{M_{\mathrm{R}}}{P_{\mathrm{T}}}\mathrm{E}\left\{\mu_i\right\}\right) - K\mathrm{E}\left\{\log_2\left|\boldsymbol{G}_i^{(0)}\boldsymbol{G}_i^{(0)H}\right|\right\}. \qquad (35)$$

Note that $\boldsymbol{H}^H\boldsymbol{H}$ is Wishart distributed with M_{T} degrees of freedom and $\boldsymbol{G}_i^{(0)}\boldsymbol{G}_i^{(0)H}$ is Wishart distributed with $M_{\mathrm{T}}-M_{\mathrm{R}}+M_r$ degrees of freedom. According to the property of Wishart matrices in equation (2.12) of [8], we can reach to Theorem 2. The upper bound of $\bar{\Delta}_{\mathrm{DPC-RBD}}$ is the average rate offset between DPC and BD, i.e., $\bar{\Delta}_{\mathrm{DPC-BD}}$ which has been calculated in [4].

Using a similar expression as in equation (31), we calculate the average power offset between DPC and RBD by

$$\Delta\bar{P}_{\mathrm{DPC-RBD}} = \frac{3\bar{\Delta}_{\mathrm{DPC-RBD}}}{\mathcal{S}_\infty} = \frac{3\bar{\Delta}_{\mathrm{DPC-RBD}}}{M_{\mathrm{R}}}. \qquad (36)$$

In Figure 1(a) and 1(b) we compare the approximation results with simulation results, which correspond to dashed lines and solid lines, respectively. The sum rate of DPC is simulated by applying an algorithm proposed in [10], and the sum rates of RBD and BD precoding are simulated by using $C_i = \mathrm{E}\left\{\log_2(1+\mathrm{SINR}_i)\right\}$ for each user with Gaussian inputs. From the figures we can see that even at moderate SNRs the approximation results still can provide an accurate characterization.



(a) (b) (c)

Fig. 1. Performance comparisons for simulation results and approximation results

4. SUM RATE APPROXIMATION ($M_{\mathrm{T}} < M_{\mathrm{R}}$)

Under the condition $M_{\mathrm{T}} < M_{\mathrm{R}}$ each user's precoding matrix cannot lie in the null space of all other users' channels anymore, since there is no null space left. As a result, zero-forcing precoding techniques (e.g., BD) cannot be performed in this case. However, the RBD precoding matrix $\boldsymbol{F}_i$ ($i = 1, \ldots, K$) does not only lie in the null space of all other users' channels (i.e., $\widetilde{\boldsymbol{V}}_i^{(0)}$), but also the space $\widetilde{\boldsymbol{V}}_i^{(1)}$ with a power that is inversely proportional to the singular values of the all other users' channel. In this section we study the sum rate of RBD precoding for this case and approximate this sum rate at high SNRs.

First let us use some results we have derived in Section 3.2, but rewrite the expression for $\left|\bar{\boldsymbol{H}}_i\bar{\boldsymbol{H}}_i^H\right|$ as

$$\begin{aligned}\left|\bar{\boldsymbol{H}}_i\bar{\boldsymbol{H}}_i^H\right| &= \left|\boldsymbol{H}_i\widetilde{\boldsymbol{V}}_i\Big(\frac{P_{\mathrm{T}}}{M_{\mathrm{R}}}\widetilde{\boldsymbol{\Sigma}}_i^T\widetilde{\boldsymbol{\Sigma}}_i + \boldsymbol{I}_{M_{\mathrm{T}}}\Big)^{-1}\widetilde{\boldsymbol{V}}_i^H\boldsymbol{H}_i^H\right| \\ &\overset{(1)}{\approx} \left|\frac{M_{\mathrm{R}}}{P_{\mathrm{T}}}\boldsymbol{H}_i\widetilde{\boldsymbol{V}}_i\Big(\widetilde{\boldsymbol{\Sigma}}_i^T\widetilde{\boldsymbol{\Sigma}}_i\Big)^{-1}\widetilde{\boldsymbol{V}}_i^H\boldsymbol{H}_i^H\right| \\ &\overset{(2)}{=} \left|\frac{M_{\mathrm{R}}}{P_{\mathrm{T}}}\boldsymbol{H}_i\Big(\widetilde{\boldsymbol{H}}_i^H\widetilde{\boldsymbol{H}}_i\Big)^{-1}\boldsymbol{H}_i^H\right|. \qquad (37)\end{aligned}$$

In step (1) we neglect $\boldsymbol{I}_{M_{\mathrm{T}}}$ by considering the high SNR regime. We represent the all users' channels except for user i by $\widetilde{\boldsymbol{H}}_i \in \mathbb{C}^{(K-1)M_r\times M_{\mathrm{T}}}$. Using its SVD $\widetilde{\boldsymbol{H}}_i = \widetilde{\boldsymbol{U}}_i\widetilde{\boldsymbol{\Sigma}}_i\widetilde{\boldsymbol{V}}_i^H$, leads to step (2).

Substituting equation (37) into equation (14) we get

$$\mathcal{L}_\infty = \log_2 M_{\mathrm{R}} - \frac{1}{\mathcal{S}_\infty}\log_2\prod_{i=1}^K\left|\frac{M_{\mathrm{R}}}{P_{\mathrm{T}}}\boldsymbol{H}_i(\widetilde{\boldsymbol{H}}_i^H\widetilde{\boldsymbol{H}}_i)^{-1}\boldsymbol{H}_i^H\right|. \quad (38)$$

Since the average power offset is more interesting than a power offset per realization, we average the above power offset over the fading distribution and get

$$\begin{aligned}\bar{\mathcal{L}}_\infty &= \mathrm{E}\{\mathcal{L}_\infty\} \\ &= \log_2 M_{\mathrm{R}} - \frac{K}{\mathcal{S}_\infty}\mathrm{E}\left\{\log_2\left|\frac{M_{\mathrm{R}}}{P_{\mathrm{T}}}\boldsymbol{H}_i(\widetilde{\boldsymbol{H}}_i^H\widetilde{\boldsymbol{H}}_i)^{-1}\boldsymbol{H}_i^H\right|\right\} \\ &= \log_2 P_{\mathrm{T}} - \frac{K}{\mathcal{S}_\infty}\mathrm{E}\left\{\log_2\left|\boldsymbol{H}_i(\widetilde{\boldsymbol{H}}_i^H\widetilde{\boldsymbol{H}}_i)^{-1}\boldsymbol{H}_i^H\right|\right\} \\ &\overset{(1)}{=} \log_2 P_{\mathrm{T}} - \frac{K\log_2 e}{\mathcal{S}_\infty}\sum_{n=0}^{Mr-1}\Big(\varphi(M_{\mathrm{T}}-n) - \varphi(M_{\mathrm{R}}-M_{\mathrm{T}}-n)\Big).\end{aligned}$$

At step (1) we rewrite $\mathrm{E}\Big\{\log_2|\boldsymbol{H}_i(\widetilde{\boldsymbol{H}}_i^H\widetilde{\boldsymbol{H}}_i)^{-1}\boldsymbol{H}_i^H|\Big\}$ by applying Theorem 2.12 in [8].

Then we can approximate the RBD sum rate at high SNRs as

$$C_{\mathrm{RBD}}(P_{\mathrm{T}}) \cong K\log_2 e\sum_{n=0}^{Mr-1}\Big(\varphi(M_{\mathrm{T}}-n) - \varphi(M_{\mathrm{R}}-M_{\mathrm{T}}-n)\Big).$$

From this equation and Figure 1(c) we can see that under the condition $M_{\mathrm{T}} < M_{\mathrm{R}}$, the RBD sum rate stays constant for high SNRs, which show that the benefit of spatial multiplexing is completely lost and $\mathcal{S}_\infty = 1$. In this case ($M_{\mathrm{T}} < M_{\mathrm{R}}$), RBD should be performed only for low SNRs.

5. CONCLUSION

In this paper we have studied the sum rates approximations for DPC, RBD, and BD based multi-user MIMO systems at high SNRs. First, under the condition that the aggregate number of receive antennas is not larger than the number of transmit antennas, it is found that DPC, RBD, and BD can maintain the same spatial multiplexing gain, but have different power offsets. We quantify the average rate and power offsets bounds among them as simple functions of the system parameters. Then under the condition that the aggregate number of receive antennas is larger than the number of transmit antennas, the approximated sum rate of RBD is calculated. Although RBD still can be performed in this case, the benefit of spatial multiplexing is completely lost due to a unit spatial multiplexing gain at high SNRs. In this case, we suggest to use RBD only for low to medium SNRs.

6. REFERENCES

[1] H. Weingarten, Y. Steinberg, and S. Shamai, "The capacity region of the Gaussian MIMO broadcast channel," in *Proc. IEEE Int. Symposium on Information Theory*, June 2004.

[2] Z. Shen, R. Chen, J. G. Andrews, R. W. Heath, and B. L. Evans, "Sum capacity of multiuser MIMO broadcast channels with block diagonalization," in *Proc. IEEE Int. Symp. on Inform. Theory (ISIT)*, Seattle, WA, July 2006, pp. 886–890.

[3] Q. H. Spencer, A. L. Swindlehurst, and M. Haardt, "Zero-forcing methods for downlink spatial multiplexing in multi-user MIMO channels," *IEEE Trans. Signal Processing*, vol. 52, pp. 461–471, Feb. 2004.

[4] J. Lee and N. Jindal, "High SNR analysis for MIMO broadcast channels: Dirty paper coding vs. linear precoding," *IEEE Trans. Information Theory*, vol. 53, pp. 4787–4792, Dec. 2007.

[5] V. Stankovic and M. Haardt, "Generalized design of multi-user MIMO precoding matrices," *IEEE Trans. on Wireless Communications*, vol. 7, pp. 953–961, 2008.

[6] S. Shamai and S. Verdú, "The impact of frequency-flat fading on the spectral efficiency of CDMA," *IEEE Trans. Information Theory*, vol. 47, May 2001.

[7] G. Gaire and S. Shamai, "On the achievable throughput of a multiantenna Gaussian broadcast channel," *IEEE Trans. Information Theory*, vol. 49, pp. 1691–1706, July 2003.

[8] A. M. Tulino and S. Verdú, *Random Matrix Theory and wireless communications*, now Publishers, Hanover, MA, 2004.

[9] Z. Bai and G. H. Golub, "Bounds for the trace of the inverse and the determiant of symmetric positive definite matrices," *Baltzer Journals*, Apr. 1996.

[10] N. Jindal, W. Rhee, S. Vishwanath, S. Jafar, and A. Goldsmith, "Sum power iterative water-filling for multi-antenna Gaussian broadcast channels," *IEEE Trans. on Information Theory*, vol. 51, Apr. 2005.