UPPER BOUND FOR THE LOSS FACTOR OF ENERGY DETECTION OF RANDOM SIGNALS IN MULTIPATH FADING COGNITIVE RADIOS

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ABSTRACT

In this paper, the loss of energy detection compared with optimal sensing caused by neglecting signal correlation due to multipath fading is considered. The loss factor or relative performance of energy detection compared with optimal sensing is analyzed using Pitman's asymptotic relative efficiency (ARE) which is defined as the ratio of the required number of samples of one detector to that of the other to yield the same detection performance in large sample scheme. Under the assumption of L-tap finite impulse response (FIR) channel with zero-mean independent and identically distributed (i.i.d.) tap coefficients, it is shown that the loss factor of the energy detection relative to optimal sensing is no larger than 1/2 in large delay spread case (i.e., strong correlation); under the same signal power condition the required number of samples for energy detection neglecting the signal correlation is no more than twice of that required for optimal sensing exploiting the signal correlation fully.

Index Terms – Cognitive radio, multipath fading, energy detection, optimal sensing, asymptotic relative efficiency.

1. INTRODUCTION

With the congestion of radio spectrum cognitive radio communication has become an attractive solution [1]. In cognitive radio communications secondary users sense the primary user's signal in common radio band to access the channel opportunistically. Since the signal signature of the primary user is not known typically in this situation, the energy detection is widely considered and used for channel sensing [2, 3]. Under Gaussian signal and noise assumption, the energy detection is optimal for i.i.d. random signals. However, this is not the case in typical wireless channels. Due to multipath delay spread of wireless channel, the signal to sense is correlated and the energy detection is not optimal any more in most cases. In this paper, we analyze the loss of energy detection caused by neglecting the signal correlation, and provide a fundamental bound for the loss factor of energy detection compared with optimal sensing. Since the exact error performance for the detection of random signals with correlation is not available [4], we approach the problem using Pitman's ARE defined as the ratio of the required number of samples of one detector to that of the other to yield the same detection performance in large sample scheme. Under the assumption of L-tap FIR channel with zero-mean i.i.d. tap coefficients, e.g., L-tap Rayleigh fading, we show that the loss factor of the energy detection is no larger than 1/2 in large delay spread case; the required number of samples for energy detection neglecting the signal correlation is no larger than twice of that required for optimal sensing exploiting the signal correlation fully.

2. DATA MODEL AND OPTIMAL SENSING

We consider cognitive radio communications in which secondary users sense the signal of primary users for the opportunistic use of wireless channel. We assume that the primary user's signal is generated according to the general digital modulation procedure at the primary transmitter. That is, a symbol sequence $s[i]$ is filtered by a pulse shaping filter $p(t)$, and then the filtered signal is transmitted through an antenna[1]. We assume that the transmitted signal propagates through FIR channels to secondary users, which is the case in most wireless communications. At the receiver of a secondary user, the received signal is corrupted by additive white Gaussian noise (AWGN) and the noisy signal is filtered by $p^*(-t)$ and sampled at the symbol rate of the primary user signal which is assumed to be known to secondary users. Then, the discrete-time

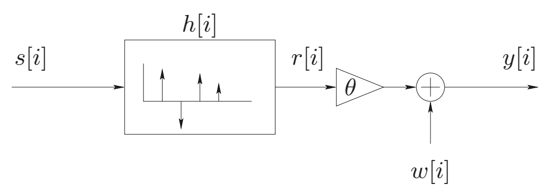

Fig. 1. Discrete-time system model

received signal $y[i]$ at the secondary user is well known, as shown in Fig. 1, and is given by

$$\begin{array}{lll} H_0: & y[i] = w[i], & i = 1, 2, \cdots, n, \\ H_1: & y[i] = \theta r[i] + w[i], & i = 1, 2, \cdots, n, \end{array} \quad (1)$$

where the null hypothesis H_0 represents the noise only case and the alternative hypothesis H_1 represents the case when the signal is present. Here, $w[i]$ are i.i.d. proper complex Gaussian noises drawn from $\mathcal{CN}(0, \sigma^2)$ with a known variance σ^2, which is independent of the signal part $r[i]$, and the signal part given by

$$r[i] = \sum_{k=1}^{L} h[k]s[i-k], \quad (2)$$

[1]The upconversion step is not important since we can adopt the baseband equivalent model.

where $\{h[i], i = 1, \cdots, L : \mathbb{E}\{\sum_i |h[i]|^2\} = 1\}$ are the (statistically) normalized FIR channel response from the primary user and the secondary user, and θ is the unknown amplitude parameter.[2] Here, we assume that the channel coefficient vector $\mathbf{h} \triangleq [h_1, \cdots, h_L]^T$ is a realization of proper zero-mean complex random vector satisfying $\mathbb{E}||\mathbf{h}||^2 = 1$ and does not change over one sensing interval. To simplify the sensing problem, we further assume that the primary symbol sequence $s[i]$ is an i.i.d. zero-mean (complex) Gaussian process with variance $\mathbb{E}\{|s[i]|^2\} = 1$, independent of the channel coefficients. Note that the received signal process $r[i]$ is wide-sense stationary but not i.i.d. because of the memory effect of the FIR channel even though the symbol sequence $s[i]$ at the primary transmitter is assumed to be i.i.d., which is valid for most coded transmissions. The autocorrelation function of $r[i]$ conditioned on the realization $\mathbf{h}$ is given by

$$\begin{aligned} \gamma_m &= \mathbb{E}\{r[i]r^*[i-m]\}, \\ &= \begin{cases} \sum_{k=m+1}^{L} h[k]h^*[k-m], & -L+1 \le m \le L-1, \\ 0 & \text{o.w.,} \end{cases} \end{aligned} \tag{3}$$

where $(\cdot)^*$ represents the complex conjugate and $\gamma_{-m} = \gamma_m^*$. The signal-to-noise ratio (SNR) under the alternative hypothesis in (1) is given by

$$\text{SNR} = \frac{\theta^2}{\sigma^2}, \tag{4}$$

since $\mathbb{E}\{r^2[i]\} = \mathbb{E}\{|s[i]|^2\}\mathbb{E}\{\sum_{k=1}^{L} |h[k]|^2\} = 1$.

2.1. Exploiting signal correlation and locally optimal sensing

Let us first assume that the channel realization $\{h[i]\}$ is known to the secondary user. When the signal amplitude θ is also known, the optimal sensing is given by a likelihood ratio detector given by

$$T_{lrt}(\mathbf{y}_n) = \mathbf{y}_n^H \Sigma_r (\sigma^{-2}\mathbf{I} + \theta^2 \mathbf{\Sigma}_r)^{-1} \mathbf{y}_n \underset{<_{H_0}}{\overset{\ge^{H_1}}{}} \tau_1, \tag{5}$$

where $\mathbf{y}_n = [y[1], y[2], \cdots, y[n]]^T$ and $\mathbf{\Sigma}_r$ is given by the $n \times n$ signal covariance matrix

$$\mathbf{\Sigma}_r = \begin{bmatrix} \gamma_0 & \gamma_{-1} & \cdots & \gamma_{-L+1} & 0 & \cdots & \cdots \\ \gamma_1 & \gamma_0 & \gamma_{-1} & \ddots & \gamma_{-L+1} & 0 & \cdots \\ \vdots & \gamma_1 & \gamma_0 & \gamma_{-1} & \ddots & \gamma_{-L+1} & 0 \\ \gamma_{L-1} & \ddots & \ddots & \ddots & \ddots & \ddots & \gamma_{-L+1} \\ 0 & \gamma_{L-1} & \ddots & \gamma_1 & \gamma_0 & \gamma_{-1} & \vdots \\ \vdots & 0 & \gamma_{L-1} & \ddots & \gamma_1 & \gamma_0 & \gamma_{-1} \\ \vdots & \vdots & 0 & \gamma_{L-1} & \cdots & \gamma_1 & \gamma_0 \end{bmatrix}.$$

Here, the threshold τ_1 is determined to satisfy the size constraint, i.e., $\Pr\{T_{lrt}(\mathbf{y}_n) \ge \tau_1\} = P_F$, where P_F is the desired false alarm probability. In the sensing problem under cognitive radio context, however, it is difficult to know the channel gain $h[i]$ and the signal amplitude θ without explicit measuring of the signal part. (Note that we do not even know whether the signal is present or not.) With the unknown signal power, the sensing problem can be formulated as a composite hypothesis detection problem, and the null and alternative distributions are given by

$$\begin{cases} H_0 & : \quad \theta = 0, \\ H_1 & : \quad \theta > 0, \end{cases} \tag{6}$$

[2]Let $\mathbf{h}_t$ be the true propagation channel. Then, $\theta = \sqrt{\mathbb{E}||\mathbf{h}_t||^2}$ and $\mathbf{h} = \mathbf{h}_t/\sqrt{\mathbb{E}||\mathbf{h}_t||^2}$.

respectively. For the test of the hypothesis (6), no uniformly most powerful (UMP) detector exists [4]. Thus, the detection of a signal with unknown amplitude as in (6) is focused on the alternative hypothesis which is close to the null hypothesis $\theta = 0$ where the distributions of the null and alternative hypotheses are mutually contiguous. That is, the criterion focuses on the low signal-to-noise (SNR) range, especially, in the large sample size since the parameter θ represents the amplitude of the signal. This is especially suitable to cognitive radio communications in which the strength of the primary user's signal to sense is very weak. The locally most powerful (LMP) or locally optimal detection for (1, 6) focusing on the low SNR is given by the score test [4, 5]

$$T_{lo}(\mathbf{y}_n) = \frac{1}{n} \mathbf{y}_n^H \mathbf{\Sigma}_r \mathbf{y}_n \underset{<_{H_0}}{\overset{\ge^{H_1}}{}} \tau_2, \tag{7}$$

where the threshold τ_2 is determined to satisfy the size constraint. Since $\mathbf{\Sigma}_r$ is a banded matrix with bandwidth L, the computation of $T_{lo}(\mathbf{y}_n)$ can be easily done exploiting this sparsity, and is given by

$$\begin{aligned} T_{lo}(\mathbf{y}_n) &= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} y^*[i]y[j]\gamma_{i-j}, \\ &= \gamma_0 \hat{\gamma}_0 + 2 \sum_{k=1}^{L-1} \text{Re}\{\gamma_k \hat{\gamma}_k\}, \end{aligned} \tag{8}$$

where $\{\gamma_m, -L+1 \le m \le L+1\}$ are the true autocorrelation parameters, and $\hat{\gamma}_m$ is the sample autocorrelation given by

$$\hat{\gamma}_k = \frac{1}{n} \sum_{i=1}^{n-k} y^*[i]y[i+k], \quad k = 0, 1, \cdots, L-1. \tag{9}$$

Note that the computational complexity of the optimal sensing is $O(Ln)$, where the channel length L is fixed, and this complexity is in the same order $O(n)$ of the simple energy detector in the below.

3. SUBOPTIMAL ENERGY DETECTION AND ITS PERFORMANCE LIMIT

Note from (7 - 8) that the optimal LMP sensing requires the knowledge of the signal correlation $\{\gamma_m\}$, which in turn requires the knowledge of the FIR channel coefficients. (See (3).) The knowledge of the channel between the primary user and the secondary user is not available in most cognitive radio situations. Thus, the simple energy detector is widely used by ignoring the signal correlation, and is given by

$$\begin{aligned} T_{en}(\mathbf{y}_n) &= \frac{1}{n} \gamma_0 \mathbf{y}_n^H \mathbf{y}_n = \frac{1}{n} \gamma_0 \sum_{i=1}^{n} |y[i]|^2, \\ &= \gamma_0 \hat{\gamma}_0. \end{aligned} \tag{10}$$

It is seen by comparing (10) with (7) that the simple energy detector is optimal if $\mathbf{\Sigma}_r = \gamma_0 \mathbf{I}$, i.e., the signal process $r[i]$ is i.i.d. However, this is not the case with multipath fading since multipath delay spread results in signal correlation. Since the exact error probability of the locally optimal detection and the energy detection under signal correlation is not available [4], we evaluate the performance degradation of the simple energy detection compared with the optimal LMP sensing using the Pitman ARE. The ARE of the suboptimal energy detector to the optimal LMP sensing is defined as the ratio of the number (n_{en}) of samples for the energy detector to that (n_{lo}) of the optimal LMP detector to achieve the same miss probability under the same size constraint as $n_{en} \to \infty$, i.e., [4, 5]

$$ARE_{en,lo} \triangleq \lim_{n_{en}\to\infty} \frac{n_{lo}}{n_{en}}. \quad (11)$$

Thus, the ARE can serve as a loss factor of the energy detector compared with the optimal LMP sensing. Under some regularity conditions the ARE is given by the asymptotic ratio of the generalized SNR, defined as [7]

$$ARE = \frac{S(T_{en,\infty})}{S(T_{lo,\infty})}, \quad (12)$$

where

$$\begin{aligned} S(T_{en,\infty}) &= \lim_{n\to\infty} \frac{S(T_{en}(\mathbf{y}_n))}{n}, & (13)\\ S(T_{lo,\infty}) &= \lim_{n\to\infty} \frac{S(T_{lo}(\mathbf{y}_n))}{n}, & (14)\end{aligned}$$

and

$$S(T_x(\mathbf{y}_n)) \triangleq \frac{(\mathbb{E}_1\{T_x(\mathbf{y}_n)\} - \mathbb{E}_0\{T_x(\mathbf{y}_n)\})^2}{\text{Var}_0\{T_x(\mathbf{y}_n)\}}. \quad (15)$$

Here, $\mathbb{E}_j\{\cdot\}$ represents expectation under H_j $(j = 0, 1)$, $\text{Var}_0\{\cdot\}$ denotes variance under H_0, and $T_x(\mathbf{y}_n)$ are given in (7) and (10). It is shown using the Toeplitz distribution theorem that the ARE of the energy detector to the optimal LMP sensing is given by [7]

$$ARE_{en,lo} = \frac{\left(\frac{1}{2\pi}\int_0^{2\pi} f_s(\omega)d\omega\right)^2}{\frac{1}{2\pi}\int_0^{2\pi} f_s^2(\omega)d\omega}, \quad (16)$$

where $f_s(\omega)$ is the spectrum of the signal process $s[i]$, given by

$$\begin{aligned} f_s(\omega) &= (2\pi)^{-1} \sum_{m=-\infty}^{\infty} \gamma_m e^{-jm\omega},\\ &= (2\pi)^{-1} \left(\gamma_0 + 2\sum_{m=1}^{L-1} \text{Re}\{\gamma_m e^{-jm\omega}\}\right). & (17)\end{aligned}$$

Note from (16) that $ARE_{en,lo} \le 1$ by the Cauchy-Schwarz inequality unless the signal spectrum is flat. That is, the energy detector always yields worse performance than the optimal LMP sensing with multipath fading (or delay spread). By direct computation we have

$$\begin{aligned} \int_0^{2\pi} f_s(\omega)d\omega &= \gamma_0, & (18)\\ \int_0^{2\pi} f_s^2(\omega)d\omega &= (2\pi)^{-1}\left(\gamma_0^2 + 2\sum_{m=1}^{L-1}|\gamma_m|^2\right), & (19)\end{aligned}$$

and the ARE is given by

$$ARE_{en,lo}(\gamma(\mathbf{h})) = \frac{\gamma_0^2}{\gamma_0^2 + 2\sum_{m=1}^{L-1}|\gamma_m|^2}. \quad (20)$$

Note that the ARE is a function of $\{\gamma_m\}$ and in turn a function of the channel coefficient vector $\mathbf{h}$ through (3). (This dependency is explicitly shown in the left-handed side of (20).)

Note that $ARE_{en,lo}$ is dependent on the channel realization $\mathbf{h}$ and thus random. To eliminate this instantaneous channel dependency and derive the loss factor depending only on the channel statistics, we define the average ARE as

$$\bar{ARE} \triangleq \mathbb{E}_{\mathbf{h}}\{ARE_{en,lo}(\gamma(\mathbf{h}))\}, \quad (21)$$

where the expectation is taken over the distribution of the channel. Now, we examine the impact of multipath fading on the relative performance of energy detectors to optimal LMP sensing by investigating the behavior of the average ARE as the channel length L increases.

Lemma 1 *Suppose that the channel coefficients $h_1, \dots, h_L$ are i.i.d. with zero mean and variance $1/L$. Then, we have*

$$f(L) \triangleq \mathbb{E}_{\mathbf{h}}\left\{\sum_{m=1}^{L-1}|\gamma_m|^2\right\} = \frac{L(L-1)}{2L^2} \le \frac{1}{2} \quad (22)$$

for all $L \ge 1$, and $f(L)$ is a monotone increasing function of L and converges to 1/2 as L tends to infinity.

Proof: By definition we have

$$\begin{aligned} \gamma_m &= \mathbb{E}\{r[i]r^*[i-m]|\mathbf{h}\},\\ &= \begin{cases} \sum_{k=m+1}^{L} h[k]h^*[k-m], & -L+1 \le m \le L-1,\\ 0 & \text{o.w.}, \end{cases}\end{aligned}$$

and thus we have

$$\begin{aligned} \mathbb{E}_{\mathbf{h}}|\gamma_m|^2 &= \mathbb{E}_{\mathbf{h}}\{\gamma_m\gamma_m^*\},\\ &= \mathbb{E}_{\mathbf{h}}\left\{\sum_{k=m+1}^{L} h[k]h^*[k-m]\left(\sum_{i=m+1}^{L} h[i]h^*[i-m]\right)^*\right\},\\ &= \mathbb{E}_{\mathbf{h}}\left\{\sum_{k=m+1}^{L}\sum_{j=m+1}^{L} h[k]h^*[k-m]h^*[i]h[i-m]\right\},\\ &= \sum_{k=m+1}^{L}\sum_{i=m+1}^{L}\mathbb{E}_{\mathbf{h}}\left(h[k]h^*[k-m]h^*[i]h[i-m]\right),\\ &= \sum_{k=m+1}^{L}\sum_{i=m+1}^{L}\delta_{ki}/L^2,\\ &= (L-m)/L^2.\end{aligned}$$

Summing all $\mathbb{E}_{\mathbf{h}}|\gamma_m|^2$, $m = 1, \cdots, L-1$, we have

$$\mathbb{E}_{\mathbf{h}}\left\{\sum_{m=1}^{L-1}|\gamma_m|^2\right\} = \sum_{m=1}^{L-1}\frac{L-m}{L^2} = \frac{1}{L^2}\frac{L(L-1)}{2} = \frac{1}{2}\left(1-\frac{1}{L}\right).$$

Since $\frac{1}{L}$ decreases monotonically to zero as L increases unboundedly. Therefore, $\mathbb{E}_{\mathbf{h}}\left\{\sum_{m=1}^{L-1}|\gamma_m|^2\right\}$ is monotonically increasing and converges to 1/2 as L increases. ■

Theorem 1 *Under the same conditions as in Lemma 1, for sufficiently large L, the average ARE is lower-bounded by 1/2, i.e.,*

$$\bar{ARE} \ge LB(L) \ge \frac{1}{2}, \quad (23)$$

and $LB(L)$ converges to 1/2 with rate of $O\left(\frac{1}{L}\right)$ as L increases unboundedly.

Proof: Note that

$$\gamma_0 = \sum_{i=1}^{L} |h_i|^2,$$

where h_i, $i = 1, \cdots, L$, are i.i.d. with zero-mean and variance $1/L$. By the strong law of large numbers (SLLN), we have for sufficiently large L

$$|\gamma_0 - 1| \le \epsilon,$$

almost surely since $\mathbb{E}\{\gamma_0\} = 1$. Thus, we have for sufficiently large L

$$1 - \epsilon' \le (1-\epsilon)^2 \le \gamma_0^2 \le (1+\epsilon)^2 \le 1 + \epsilon' \qquad (24)$$

almost surely for any $\epsilon' > 0$.

$$\begin{aligned} A\bar{R}E &= \mathbb{E}\left\{ \frac{\gamma_0^2}{\gamma_0^2 + 2\sum_{m=1}^{L-1} |\gamma_m|^2} \right\}, \\ &= \mathbb{E}\left\{ \frac{1}{1 + 2\sum_{m=1}^{L-1} |\gamma_m|^2/\gamma_0^2} \right\}, \\ &\overset{(a)}{\ge} \frac{1}{1 + 2\mathbb{E}\{\sum_{m=1}^{L-1} |\gamma_m|^2/\gamma_0^2\}}, \\ &\overset{(b)}{\ge} \frac{1}{1 + 2\mathbb{E}\{\sum_{m=1}^{L-1} |\gamma_m|^2\}/(1-\epsilon')}, \\ &\overset{(c)}{\ge} \frac{1}{1 + 2f(L)/(1-\epsilon')}, \qquad (25) \\ &\overset{(d)}{\ge} \frac{1}{2} \quad \text{almost surely.} \end{aligned}$$

Here, (a) is by Jenssen's inequality, (b) is by applying (24), (c) is by the definition of $f(L)$ in Lemma 1, and (d) is because ϵ' is arbitrary and $f(L) \le 1/2$ by Lemma 1. The second claim is by substituting $f(L) = (1/2)(1 - 1/L)$ from Lemma 1 in (25). ■

Theorem 1 provides a fundamental limit for the loss of the energy detection compared with the optimal LMP sensing caused by neglecting the signal correlation due to multipath fading for large L, i.e., very strong correlation. *The loss cannot be bigger than 50 % compared with the optimal sensing under the same signal power condition!* For the same detection performance the energy detection requires no more than twice of the number of samples that is required for the optimal sensing even in the case of very strong correlation caused by large delay spread.

Figure 2 shows the average ARE of the energy detection to the optimal LMP sensing for L-tap equal power Rayleigh fading. As proven in Theorem 1, the average ARE is lower bounded by 1/2 for large L. It is trivial to see that the average ARE is equal to one for $L = 1$, i.e., in flat fading. In the intermediate values of L it is seen in the figure that the average ARE is monotonically decreasing and converges to the lower bound, 1/2, as the channel L increases. (This is consistent with our intuition.) Note that the performance of the energy detector degrades quickly as L increases initially from $L = 1$. At $L = 5$, the performance degradation is already almost 40 % compared with the optimal LMP sensing, i.e., the average ARE = 0.6. Thus, we can improve the sensing performance by exploiting the signal correlation almost twice compared with the simple energy detector even for small values of L.

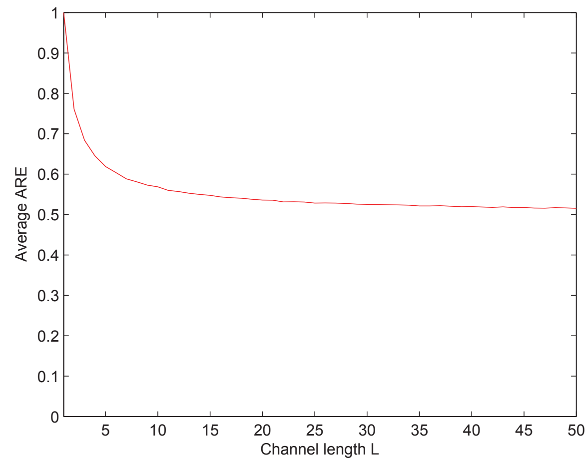

Fig. 2. ARE of the energy detector to the optimal LMP sensing for L-tap FIR Rayleigh channel with equal tap power averaged over 10000 channel realizations.

4. CONCLUSION AND DISCUSSION

We have considered the loss of energy detection compared with optimal sensing caused by neglecting the signal correlation induced by multipath delay spread which is common in wireless channels. We have investigated the loss using Pitman's ARE, and have shown that under the L-tap FIR channel model with equal power i.i.d. tap coefficients the loss of energy detection is no more than 50 % compared with optimal sensing exploiting the signal correlation fully. We have seen that the performance of energy detection degrades quickly as the channel length increases initially and the sensing performance can be improved almost by a factor of two by exploiting the signal correlation even for short channel lengths. Future works include the development of such algorithms.

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