INTERFERENCE ALIGNMENT VIA ALTERNATING MINIMIZATION

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ABSTRACT

Using interference alignment, it has been shown that the number of degrees of freedom in the interference channel scales linearly with the number of users. Unfortunately, closed-form solutions for interference alignment over constant-coefficient channels with more than 3 users are difficult to derive. This paper proposes an algorithm for interference alignment in the MIMO interference channel with an arbitrary number of users, antennas, or spatial streams. The algorithm is an alternating minimization over the precoding matrices at the transmitters and the interference subspaces at the receivers, and is proven to converge. Numerical results show how the algorithm is useful for simulation and can give insight into the limitations of interference alignment.

***Index Terms*—MIMO systems, Interference, Radio communication, Multiuser channels, Optimization methods**

I. INTRODUCTION

Interference alignment (IA) is a technique recently shown to achieve the maximum spatial degrees of freedom in the K-user interference channel [1]. By forcing interfering signals at each receiver into a reduced-dimensional subspace of the received space, the receivers can observe an interference-free desired signal if it lies outside of the interference subspace. If the multi-dimensional receive space corresponds to physical space (antennas) rather than frequency or time slots, then practical wireless multiple-input multiple-output (MIMO) techniques such as orthogonal precoding and zero-forcing receivers can be applied. Unfortunately, there appear to be no closed-form solutions for the precoders of such systems with more than 3 users. Further, MIMO interference alignment does not achieve as many degrees of freedom as when coding over an infinite number of independent time or frequency slots, so the limits of MIMO IA remain unclear.

This paper proposes an algorithm for MIMO IA that alternatively optimizes the precoders at the transmitters and the interference subspaces at the receivers. The precoders and interference subspaces are constrained to be orthonormal and, with the optimization used, will be shown to lie on the Grassmann manifold. The gradient of the objective function on this manifold has a closed-form solution so an alternating minimization approach can be applied. We establish convergence of the algorithm, although convergence to a global optimum requires additional work. The proposed algorithm gives insight into when MIMO interference alignment is feasible without any assumptions on number of users, method of obtaining CSI, reciprocity of the channel, antenna distribution, or stream allocation.

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Interference alignment was first studied in the MIMO X channel [6], where the achievability for fractional degrees of freedom spawned research into applying the same principal to the interference channel. In particular, [1] showed IA achieves the maximum degrees of freedom of the interference channel with time or frequency selectivity, though the degree of freedom frontier for constant channels is still unknown except with 3 users where it coincides with the time or frequency selective case. A distributed algorithm for interference alignment that requires reciprocal channels was proposed in [4]. Our algorithm removes this requirement and adds none, making it more general than [4]. It can be used in a distributed fashion as with [4], but imperfect or quantized CSI can be considered with the algorithm presented in this paper. Our formulation is also more conducive to mathematical and geometrical analysis and interpretation.

This paper uses the following notation: $\mathbf{A}$ is a matrix, $\mathbf{a}$ is a vector, and a is a scalar; $\mathbf{A}^*$ is the conjugate transpose of $\mathbf{A}$, and $\|\mathbf{A}\|_F$ is the Frobenius norm of $\mathbf{A}$; $\mathrm{tr}(\mathbf{A})$ is the trace of $\mathbf{A}$; $\mathbb{C}^N$ is N-dimensional complex space, and $\mathcal{N}(\mu, \mathbf{R})$ is the multivariate normal distribution with mean μ and covariance matrix $\mathbf{R}$.

II. INTERFERENCE ALIGNMENT

Consider the K-user MIMO interference channel of Figure 1. This model consists of $2K$ nodes, K of which are designated as transmitters while the other K are receivers. Each transmitter is paired with a single receiver in a 1-1 mapping. Finally, each transmitter interferes with all the

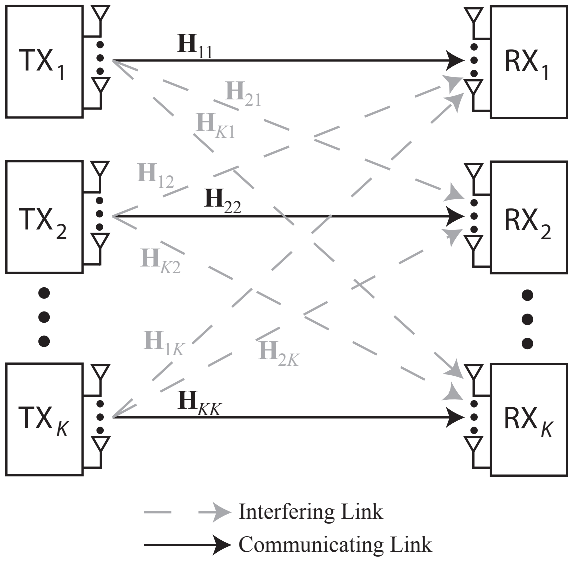

Fig. 1. The K-user MIMO interference channel. Transmitter k, with M_k antennas, has a message for receiver k with N_k antennas. All transmitters share a channel with all receivers.

receivers it is not paired with. Transmitter k is equipped with M_k transmit antennas while receiver k is equipped with N_k receive antennas. Without loss of generality, we assume transmitter k wishes to communicate with receiver k. For simplicity we assume a frequency flat channel.

The received signal at receiver k is given as

$$\mathbf{y}_k \ = \ \mathbf{H}_{k,k}\mathbf{F}_k\mathbf{s}_k \ + \ \sum_{\ell \neq k}\mathbf{H}_{k,\ell}\mathbf{F}_\ell\mathbf{s}_\ell \ + \ \mathbf{v}_k, \qquad (1)$$

where $\mathbf{H}_{k,\ell}$ is the narrowband matrix channel from transmitter ℓ to receiver k, $\mathbf{F}_\ell$ is the precoding matrix used by transmitter ℓ, $\mathbf{s}_k$ is the vector symbol transmitter k wishes to send to receiver k, and $\mathbf{v}_k$ is additive white Gaussian noise with distribution $\mathcal{N}(\mathbf{0}, N_0\mathbf{I})$.

The goal of interference alignment is to choose precoder matrices $\{\mathbf{F}_\ell\}_{\ell=1}^{K}$ such that each receiver can decode its own signal by forcing interfering users to share a reduced-dimensional subspace of the user's receive space. In particular, with S_k streams being transmitted by transmitter k, the interference at receiver k must lie in a linear subspace of dimension at most $N_k - S_k$ of $\mathbb{C}^{N_k}$ to detect the desired signal at each receiver with no interference. That is, if the interference at receiver k lies in a p-dimensional linear subspace $\mathcal{C}$ of N_k-dimensional (complex) space, then receiver k can remove all interference by projecting its received signal to the subspace orthogonal to $\mathcal{C}$. The leftover signal is only from $\mathbf{s}_k$. Of course, the projected signal is of dimension S_k, meaning receiver k can only resolve S_k-dimensional signals from transmitter k.

Aligning interfering signals is not altogether a new idea. The notion first arose in multiuser MIMO algorithms [7], and was more formally developed in [6] for the MIMO X channel. Cadambe and Jafar [1] showed that interference alignment achieves the maximum degrees of freedom in the K-user MIMO relay channel with frequency or time selectivity, at the same time proving (surprisingly) that the number of spatial degrees of freedom scales linearly with number of users in the interference channel. Interference alignment is thus optimal at high SNR, but likely suboptimal at low-to-moderate SNR, where the optimal strategy is unknown. For this proof, [1] codes an infinite block of symbols over an infinite number of independent frequency or time dimensions.

This paper focuses on the case of constant channel coefficients across time and frequency, in effect limiting block lengths to one. In this case, the maximum achievable spatial degrees of freedom are unknown [4], except for the 3-user case [1], where frequency or time selectivity do not increase the spatial degrees of freedom.

III. INTERFERENCE ALIGNMENT VIA ALTERNATING MINIMIZATION

Suppose we have a p-dimensional linear subspace $\mathcal{U}$ of N dimensional complex space and a matrix $\mathbf{A} \in \mathbb{C}^{N\times q}$ ($p \geq q$). If $\mathbf{U}$ is an orthonormal basis of $\mathcal{U}$, then the orthogonal projection of $\mathbf{A}$ onto $\mathcal{U}$ is

$$\tilde{\mathbf{A}} = \mathbf{U}\mathbf{U}^*\mathbf{A}. \qquad (2)$$

Since the columns of $\tilde{\mathbf{A}}$ are the least squares solutions to the equations $\mathbf{a}_k = \mathbf{U}\mathbf{x}$ and have error $\|\mathbf{a}_k - \tilde{\mathbf{a}}_k\|$, where $\mathbf{a}_k$ is the kth column of $\mathbf{A}$, a natural way to measure the error between $\mathbf{A}$ and its closest point on $\mathcal{U}$ is

$$d(\mathbf{A}, \mathcal{U}) = \|\mathbf{A} - \tilde{\mathbf{A}}\|_F^2. \qquad (3)$$

More precisely, (3) is the sum of the squared Euclidean distances between the columns of $\mathbf{A}$ and their orthogonal projections onto $\mathcal{U}$. We now use this measure to derive a minimum squared error subspace to K matrices.

Lemma 1: Given K arbitrary matrices $\mathbf{A}_k \in \mathbb{C}^{N\times q}$, the p-dimensional subspace $\mathcal{U}$ with minimum overall Euclidean distance to the columns of all the $\mathbf{A}_k$ has orthonormal basis $\mathbf{U}$, where the columns of $\mathbf{U}$ are the eigenvectors associated with the p largest eigenvalues of $\sum_k \mathbf{A}_k\mathbf{A}_k^*$.

Proof: We formulate the problem by minimizing the sum of the squared errors in (3) over k, namely

$$\mathbf{U}_{opt} = \arg\min_{\mathbf{U}^*\mathbf{U}=\mathbf{I}} \sum_{k=1}^{K} \|\mathbf{A}_k - \mathbf{U}\mathbf{U}^*\mathbf{A}_k\|_F^2\,, \qquad (4)$$

where $\mathbf{U}$ is an orthonormal basis of $\mathcal{U}$. Note that this objective function is equivalent to forming a composite matrix of all the columns of the $\mathbf{A}_k$ matrices and minimizing its distance to $\mathcal{U}$. Using basic properties of linear algebra,

$$\mathbf{U}_{opt} = \arg \min_{\mathbf{U}^*\mathbf{U}=\mathbf{I}} \operatorname{tr}\left(\sum_{k=1}^{K} \mathbf{A}_k^*\mathbf{A}_k - \mathbf{A}_k^*\mathbf{U}\mathbf{U}^*\mathbf{A}_k\right) \quad (5)$$

$$= \arg \max_{\mathbf{U}^*\mathbf{U}=\mathbf{I}} \operatorname{tr}\left(\mathbf{U}^*\left(\sum_{k=1}^{K} \mathbf{A}_k\mathbf{A}_k^*\right)\mathbf{U}\right). \quad (6)$$

The solution to (6) for columns of $\mathbf{U}$ is known to be the p dominant eigenvectors of $\sum_k \mathbf{A}_k\mathbf{A}_k^*$ [5]. ∎

Lemma 1 can also be proved by noting that (4) is an optimization on the Grassmann manifold, taking the appropriate gradient with respect to $\mathbf{U}$ [3], and setting to zero.

The following lemma deals with finding the nearest matrix to K subspaces.

Lemma 2: Given K arbitrary p-dimensional subspaces $\mathcal{U}_k$ with respective orthonormal bases $\mathbf{U}_k$ and $M \times N$ matrix $\mathbf{B}$, the matrix $\mathbf{V}$ such that $\mathbf{A} = \mathbf{B}\mathbf{V}$, $\mathbf{V} \in \mathbb{C}^{N\times q}$, that minimizes the squared Euclidean distance from the columns of $\mathbf{A}$ to the subspaces has columns equal to the eigenvectors corresponding to the q minimum eigenvalues of $\sum_k \mathbf{B}^*(\mathbf{I} - \mathbf{U}_k\mathbf{U}_k^*)\mathbf{B}$.

Proof: The proof follows directly from (5), but with variable $\mathbf{A} = \mathbf{B}\mathbf{V}$ and fixed $\mathbf{U}_k$. ∎

Lemma 2 can be proven by noting that the optimization is on the Grassmann manifold and taking the appropriate gradient [3].

Lemmas 1 and 2 show that our intuitive measure (3) between a matrix and a subspace have nice optimization solutions and is thus amenable to an iterative algorithm. Using this metric, we can pose our problem as follows:

$$\min_{\substack{\mathbf{F}_\ell^*\mathbf{F}_\ell=\mathbf{I},\forall\ell \\ \mathbf{C}_k^*\mathbf{C}_k=\mathbf{I},\forall k}} \sum_{k=1}^{K}\sum_{\substack{\ell=1\\ \ell\neq k}}^{K} \left\|\mathbf{H}_{k,\ell}\mathbf{F}_\ell - \mathbf{C}_k\mathbf{C}_k^*\mathbf{H}_{k,\ell}\mathbf{F}_\ell\right\|_F^2. \quad (7)$$

Here, the matrix $\mathbf{C}_k$ is an orthonormal basis for received interference subspace $\mathcal{C}_k$. The linear receiver would then be formed by $\mathbf{W}_k = \mathbf{I}_{N_k} - \mathbf{C}_k\mathbf{C}_k^*$.

We have now formulated the interference alignment problem as an optimization over $2K$ variables that can be solved via an alternating minimization [2]. If $2K-1$ of the variables are temporarily fixed, we can optimize the objective function for the remaining variable, alternating between which variables are held fixed and which are optimized. Via inspection we can see that to solve for $\mathbf{F}_\ell$, we need only hold the $\mathbf{C}_k$ fixed, and vice versa. Thus our alternating minimization takes the following form:

1) Fix $\mathbf{F}_\ell$ arbitrarily for all ℓ
2) Let the columns of $\mathbf{C}_k$ be the $N_k - S_k$ dominant eigenvectors of $\sum_{\ell\neq k} \mathbf{H}_{k,\ell}\mathbf{F}_\ell\mathbf{F}_\ell^*\mathbf{H}_{k,\ell}^*$ $\forall k$
3) Let the columns of $\mathbf{F}_\ell$ be the S_ℓ least dominant eigenvectors of $\sum_{k\neq\ell} \mathbf{H}_{k,\ell}^*(\mathbf{I}_{N_k} - \mathbf{C}_k\mathbf{C}_k^*)\mathbf{H}_{k,\ell}$ $\forall\ell$
4) Repeat steps 2,3 until convergence

Steps 2 and 3 are given by Lemmas 1 and 2, respectively.

Since the objective function is minimized at 2 and 3, an iteration will never increase it. Also, the objective function is nonnegative. These two properties alone prove convergence to a solution. As an alternating minimization, however, one may be able to prove or disprove convergence to the optimal solution [2]. This is left for future work.

IV. DISCUSSION

It is important to distinguish the above algorithm with that of [4]. Our algorithm iteratively updates the precoders for each transmitter and the receive intereference subspaces at each receiver. Alternatively, [4] updates the projection matrix at each receiver, and, assuming reciprocity in the channel, treats this as the precoder for the reciprocal channel. Since the receive interference subspace is orthogonal to the space spanned by the projection matrix, the two algorithms give identical results if reciprocity can be assumed. Our algorithm makes no assumptions on the reciprocity of the channel, the distribution of antennas or streams, or on how information is passed between the two iterative steps. Because of this, it is better suited for studying how imperfect CSI or quantized precoders affect the interference-aligned solution.

V. SIMULATIONS

The algorithm presented in Section III produces precoding filters that will approximately minimize a measure of global interference assuming the receivers use a zero-forcing equalizer. Even when a perfectly aligned solution exists, the objective function will likely not reach zero because of numerical rounding errors, and in fact may have a practical nonzero lower bound. In fact, for a given finite iteration, there is no upper bound on the objective function even if the algorithm will converge to an optimal solution (although the authors have observed numerically that the algorithm finds a low-objective solution within a few iterations).

Further, there is no lower bound on the objective function when a perfectly aligned solution does not exist. That is, the objective function may theoretically flatten out at 10^{-10}, making it look like perfect interference alignment is feasible when in fact it is not because the objective is not approaching zero asymptotically. Thus, this algorithm cannot prove whether interference alignment is feasible for a particular antenna/stream allocation, but it can still gain significant insight into the problem. This feasibility question is an open problem whose exact mathematical solution is elusive [4], so the insight the algorithm can give is novel.

We therefore declare that an antenna/stream allocation for the K user interference channel is *numerically feasible* if the objective function falls below 10^{-4} by the 5000th iteration for each of 100 randomly generated channel realizations or if the slope of the objective function is visibly decreasing at the 5000th iteration. Although this definition may seem arbitrary, or even strict, we have observed that the algorithm

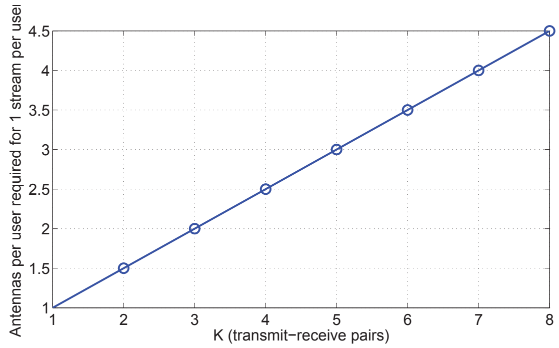

Fig. 2. The number of antennas per node required to achieve 1 interference-free spatial stream per user plotted versus number of transmit/receive pairs in the interference channel.

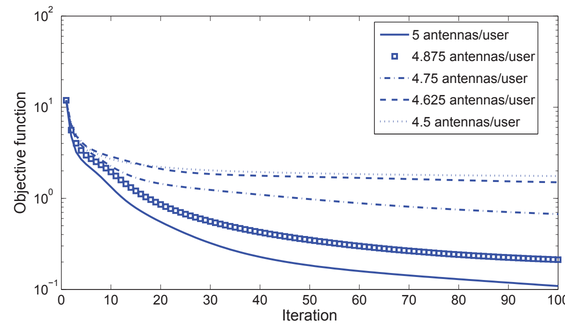

Fig. 3. The objective function for iterative algorithm applied to the 4-user MIMO interference channel with 2 streams per user. For $M_k = N_k = 5, \forall k$, the objective approaches zero. Antennas/user is $(\sum_{k=1}^{K} M_k + N_k)/(2K)$.

may converge on an objective of around 10^{-3} in cases where perfect interference alignment appears to be not quite feasible. Some configurations take longer to converge and will be above 10^{-4} at this iteration, so a visual inspection of these cases can confirm whether the objective function has leveled off or not. Although numerical feasibility does not imply mathematical feasibility and vice versa, there is evidence of significant overlap of the two regions.

Figure 2 plots the number of antennas per user required for numerical feasibility to achieve one stream per user for variable K. For instance, in the 3-user case it is known that two antennas per user are required to achieve one degree of freedom per user [1]. This corresponds to the point $K = 3$.

Note that the relation is a line with slope 1/2. For intuition consider the following scenario. We have a K-user interference channel with each transmit/receive pair transmitting a single stream. Then there are $(K + 1)/2$ antennas at each node. Now we want to add a transmit/receive pair with its own ability to communicate a single stream without disrupting the same ability of the others. We must then add a single antenna to half the nodes. Although our simulations achieved this by first adding antennas to the receivers and then to the transmitters, we have found this is an arbitrary ordering as long as they are evenly distributed with odd K.

Figure 3 plots the objective function for a fixed 4-user interference channel over 100 iterations with two streams per user and different antenna configurations. Antennas per user is computed by $(\sum_{k=1}^{K} M_k + N_k)/(2K)$ and assumes the antennas are as evenly distribted as is possible. The objective function flattens out relatively quickly, before the 100th iteration, for 4.5 antennas per user. As antennas are placed in the system, the objective function decreases and flattens out at later iterations with an objective function well below 1. At 5 antennas per user, the objective function approaches zero and thus will not become flat. This suggests that 5 antennas and 2 streams per user are achievable in the 4-user MIMO interference channel.

VI. CONCLUSION

We have proposed an alternating minimization approach to interference alignment with an arbitrary number of users or distribution of antennas and spatial streams. The algorithm, which is useful for simulating IA in any interference channel, is proven to converge and possibly converges to an optimal solution, the proof or disproof of which is left for future work. Insight into the performance limits of interference alignment has been gained via the convergence properties of the algorithm.

VII. REFERENCES

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