CAPACITY SCALING OF WIRELESS NETWORKS WITH COMPLEX FIELD NETWORK CODING*

Tairan Wang and Georgios B. Giannakis

Dept. of ECE, University of Minnesota, Minneapolis, MN 55455 USA
Emails: {wang0822, georgios}@umn.edu

ABSTRACT

Network coding in wired networks has been shown to achieve considerable throughput gains relative to traditional routing networks. While the ergodic capacity of wireless multihop networks is unknown, the scaling of capacity with the number of nodes (n) has recently received increasing attention. While existing works mainly focus on networks with n source-destination pairs, this paper deals with capacity scaling in any-to-any wireless links, where each node communicates with all other nodes. Complex field network coding (CFNC) is adopted at the physical layer to allow n nodes exchanging information with simultaneous transmissions from multiple sources. A hierarchical CFNC-based scheme is developed and shown to achieve asymptotically (as $n \to \infty$) optimal quadratic capacity scaling in a dense network, where the area is fixed and the density of nodes increases. This is possible by dividing the network into many clusters, with each cluster sub-divided into many sub-clusters, hierarchically.

Index Terms— Capacity scaling, hierarchical transmission, complex field network coding, multiple access channel (MAC), broadcast channel (BC).

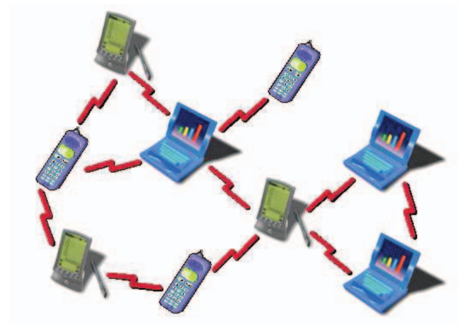

Fig. 1. An ad hoc wireless network.

1. INTRODUCTION

With the emergence of network science, capacity scaling laws in large ad-hoc wireless networks have attracted growing interest, since the exact ergodic capacity of wireless multihop networks is unknown. Gupta and Kumar first studied the scenario where n nodes are randomly located in the unit disk and each node communicates with a random destination at a rate $R(n)$ bits/second [1]. The problem was to assess how fast the total network capacity increases with n, i.e., the maximally achievable scaling of the total capacity $C(n) = nR(n)$. The results in [1] and [2] established that a multihop architecture with conventional single-user decoding and forwarding of packets can achieve $C(n)$ at most $\Theta(\sqrt{n})$, and the same scaling is achieved by a scheme using only nearest-neighbor communication.

Different from the *dense* network in [1], where the total area is fixed and the density of nodes increases, many subsequent works dealt with *extended* networks, whose size grows to cover an increasing area with the density of nodes remaining fixed. After successive refinements, the nearest-neighbor multihop scheme for a bounded transmit-power was shown to be order-optimal whenever the power path loss exponent α is greater than 3 [3–5]. Recently, a scheme based on hierarchical cooperation and distributed multi-input multi-output (MIMO) communication was developed to identify the scaling laws of random ad hoc networks for any path loss exponent $\alpha \geq 2$ [5]. For the dense networks considered in this paper, [5] established that the total capacity scales linearly with n.

The Gupta-Kumar model assumes that the signals received from nodes other than the source constitute interference that is regarded as noise degrading the communication link. Under this assumption, direct communication between source and destination pairs is not preferable, as the interference generated discourages most other nodes from communicating. Complex field network coding (CFNC), however, allows multiple users to transmit simultaneously to a destination after precoding, which turns destructive interference into a constructive signal [6]. This motivates the present paper's utilization of CFNC to achieve an improved capacity scaling law.

When traditional Galois field network coding (GFNC) is employed by random networks with n source-destination pairs, compared to the scheme in [1], there is only a constant (as opposed to a scaling) gain [7]. In contrast, this paper establishes that CFNC achieves asymptotically optimal capacity scaling in a wireless network, where each node transmits to all other nodes. This any-to-any connectivity appears often in both tactical and commercial ad hoc networks, as illustrated in Fig. 1.

As the distributed MIMO scheme of [5] does not attain a desirable capacity scaling in a network with n^2 source-destination pairs, the CFNC-based hierarchical scheme here divides the network (or sub-network) into multiple clusters in each layer of the hierarchy. Each layer includes five transmission phases, which entail MIMO multiple access (MAC) and MIMO broadcast (BC) channels [8]. CFNC is used to overcome the interference during simultaneous transmissions in MIMO-MAC and MIMO-BC. The number of clus-

* Work in this paper was prepared through collaborative participation in the Communications and Networks Consortium sponsored by the U. S. Army Research Laboratory under the Collaborative Technology Alliance Program, Cooperative Agreement DAAD19-01-2-0011. The U. S. Government is authorized to reproduce and distribute reprints for Government purposes notwithstanding any copyright notation thereon.

ters M per layer is critical to the total capacity scaling, which is now defined as $C(n) = n^2 R(n)$ for the n^2 pairing network. With M increasing slowly as n increases, the 5-phase scheme achieves a capacity scaling of order $\Theta(n^{2-\epsilon})$ in dense networks, for any $\epsilon > 0$. As the capacity scaling is upper-bounded by $\Theta(n^2 \log n)$, this scheme is nearly optimal. Moreover, the associated capacity scaling exponent approaches the upper bound as n grows large, which justifies the asymptotic optimality claim.

Notation: Upper and lower case bold symbols denote matrices and column vectors, respectively; $(\cdot)^T$ denotes transpose; $\mathcal{CN}(0, \sigma^2)$ the circular symmetric complex Gaussian distribution with zero mean and variance σ^2; for a random variable γ, $E[\gamma]$ denotes its mean; $C(n) = \Theta(n^t)$ means that $\lim_{n\to\infty} C(n)/n^t = K$, for some bounded constant $K > 0$.

2. MODELING

Consider n nodes uniformly and independently distributed in a square of unit area in dense networks. Any node can be the source of information to all other nodes, and at the same time, any node can be the destination of all source nodes. Hence, there can be $n(n-1)$ possible source-destination pairs in total. Suppose that each source has the same traffic rate to send to its destination node and a common average transmit power budget of P Joules per symbol. The overall network throughput is $C(n) = n(n-1)R(n)$, where $R(n)$ is the achievable rate per source-destination pair. For simplicity in exposition, suppose that every node is also the destination for itself, that is $C(n) = n^2 R(n)$ from now on.

We assume that wireless communication takes place over a flat channel of bandwidth W Hertz around a carrier frequency f_c with $f_c \gg W$. The complex baseband-equivalent channel gain between node i and node k at time slot m is given by

$$H_{ik}[m] = \sqrt{G} r_{ik}^{-\alpha/2} \exp(j\theta_{ik}[m]) \quad (1)$$

where r_{ik} is the distance between nodes, $\theta_{ik}[m]$ denotes random phase at time m, uniformly distributed in $[0, 2\pi]$ and $\{\theta_{ik}[m]\}_{i,k=1}^{n}$ is a collection of independent and identically distributed (i.i.d.) random processes. Variables $\theta_{ik}[m]$ and r_{ik} are also assumed independent, while the gain G and the path-loss exponent $\alpha \geq 2$ are assumed constant.

Note that the channel is random and depends on the location of nodes and the channel phases. The locations are assumed to be fixed, while the phases are allowed to vary in a stationary ergodic manner (fast fading). All channel gains are assumed available to all nodes. The signal received by node i at time m is

$$Y_i[m] = \sum_{k=1}^{n} H_{ik}[m] X_k[m] + Z_i[m] \quad (2)$$

where $X_k[m]$ stands for the symbol sent by node k at time m and $Z_i[m] \sim \mathcal{CN}(0, \sigma^2)$.

3. CAPACITY SCALING

Capacity scaling quantifies how fast the information-theoretic capacity increases with the network size n. The pertinent metric is provided by the scaling exponent $e(n)$, which is defined as

$$e(n) := \lim_{n\to\infty} \frac{\log C(n)}{\log n}. \quad (3)$$

In networks for which the exact capacity expression is not available, capacity scaling reveals how much throughput gain one can expect as the network size grows. This in turn delineates the tradeoff between throughput gain and deployment cost, which is critical for the network design.

3.1. Upper Bound

This section provides an information-theoretic upper bound on the achievable scaling law for the aggregate throughput in the network model of Section 2. Before pursuing practical communication strategies, the following theorem establishes the best one can hope for.[1]

Theorem 1 *The aggregate throughput in the network of Section 2 is bounded above by*

$$C(n) \leq K' n^2 \log n \quad (4)$$

with high probability (i.e., with probability going to 1 as n grows) for some constant K' independent of the number of nodes n.

Now let us consider an unrealistic example which achieves this upper bound by capitalizing on standard properties of wireless communications, namely: $(p1)$ omnidirectional transmissions, $(p2)$ interference due to simultaneous transmissions from different sources, and $(p3)$ the half duplex constraint, which disallows simultaneous packet transmission and reception by any node (due to the constraint that nodes are equipped with a single transceiver). If one could bypass constraints $p2$ and $p3$, then all n nodes in the network would be allowed to broadcast together, while at the same time, each node would receive the messages from all other nodes. One can easily verify that each source-destination pair in such a scheme freed from $p2$ and $p3$, achieves capacity scaling $R(n) = \Theta(1)$, implying a total capacity scaling of $C(n) = \Theta(n^2)$, with scaling exponent $e(n) = 2$. We will term this kind of scheme asymptotically optimal, as the scaling exponent difference from the upper bound of Theorem 1 is just $\epsilon(n) = \log_n(\log n)$, which disappears as n increases to infinity.

3.2. Lower bound

Having envisioned an asymptotically optimal scheme that is too ideal to be true, one is motivated to look also for lower bounds on the capacity scaling. To this end, notice first that any realistic scheme obviously yields an achievable rate scaling, which at the same time provides a lower bound on the capacity scaling of the wireless network. Furthermore, the hierarchical cooperation scheme introduced by [5], which achieves the asymptotically optimal capacity scaling in a network of n pairs, does not lead to an asymptotically optimal capacity scaling in the network of n^2 pairs considered here. Actually, when the hierarchical cooperation scheme in [5] is modified to apply in the network of n^2 pairs, the capacity scaling is still linear: $C(n) = \Theta(n)$ [10].

For the lower bound, the main result of this paper can be summarized as follows:

Theorem 2 *With $\alpha \geq 2$ and for any $\epsilon > 0$, there exists a constant $K_\epsilon > 0$ independent of n such that with high probability, the aggregate throughput*

$$C(n) \geq K_\epsilon n^{2-\epsilon} \quad (5)$$

is achievable by the network model of Section 2 with n^2 source-destination pairs.

[1] Omitted due to space limitations, the proof of this and other results stated in this paper can be found in [9].

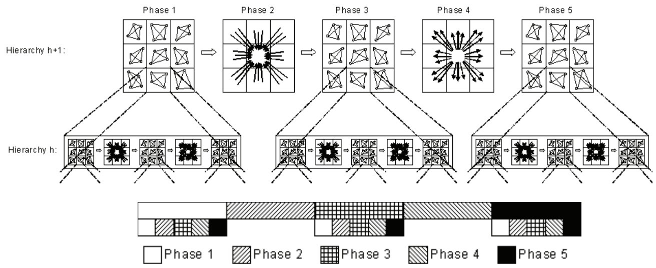

Fig. 2. Hierarchical structure and time division of the 5-phase scheme.

Theorem 2 asserts that the achievable capacity scaling can come arbitrarily close to the upper bound of Theorem 1, i.e., one can devise an asymptotically optimal scheme in the wireless network with n^2 pairs. Instrumental to proving Theorem 2 is to show that the interference property $p2$ can be mitigated with cooperation among nodes using the complex field network coding (CFNC) approach introduced in [6] to achieve high throughput and the maximum diversity gain provided by the wireless network.

4. HIERARCHICAL TRANSMISSIONS WITH CFNC

The goal of this section is to prove Theorem 2 by constructing a realistic scheme based on hierarchical clustering and CFNC transmissions among clusters. To achieve an asymptotically optimal capacity scaling, the network is split into multiple (M) subnetworks or clusters, each covering a smaller square of area $A = 1/M$. Since there are n nodes uniformly distributed in the network, there will be on average $nA = n/M$ nodes inside each cluster, and each cluster will contain order n/M nodes with probability higher than $1 - Me^{-\Lambda(\delta)n/M}$, where $\Lambda(\delta)$ is independent of n and satisfies $\Lambda(\delta) > 0$ when $\delta > 0$ [5]. While n increases, each cluster should be divided again into another M clusters, each containing n/M^2 nodes. This kind of hierarchical sub-division can be successively performed until each cluster contains less than or equal to M nodes, which results in a total of $\log_M(n)$ layers in the hierarchical clustering.[2]

Focusing on the transmission taking place in a particular layer $h+1$ of the hierarchy, consider that layer h has transmission rate R_h, $h = 1, \ldots, \log_M(n)$. The last layer $h = 1$ corresponds to the bottom layer of the hierarchy, while $h = \log_M(n)$ denotes the top layer which includes the entire network of size n. In layer $h+1$, each of the M clusters operates at rate R_h and the entire transmission proceeds in five phases:

Phase 1. Information Exchange within Each Cluster: As illustrated in Fig. 2, clusters start communicating in parallel. Within a cluster, each node distributes B bits to each of the other nodes, so that at the end of this phase, each node has B bits from each of the other nodes in the same cluster. This requires transmitting B bits for each source-destination pair. As each node in the cluster is also the destination of other nodes in the cluster, there is no extra traffic demand introduced by this clustering operation. With this per-cluster transaction occupying T_h time slots, the throughput in Phase 1 is B/T_h, where h denotes the layer in the hierarchy.

Phase 2. MIMO-MAC using CFNC: In this phase, MIMO-MAC transmissions from $M-1$ clusters are directed to the single designated relay cluster. The remaining $M-1$ clusters will be henceforth termed source clusters. The relay cluster is chosen to minimize the total transmit power in Phases 2 and 4. During the MAC transmissions, the bits from the $M-1$ source clusters are transmitted using CFNC and arrive simultaneously at the nodes in the relay cluster. Letting r_{S_iR} denote the distance between the midpoints of the source cluster S_i and the relay cluster R, the average transmit power per node is $P(r_{S_iR})^\alpha/M^h$ at layer h. As in the previous section, precoding and symbol synchronization precede each CFNC transmission. The nodes in cluster R quantize and accumulate the signals without decoding the information symbols in this phase. From [6], it is clear that this phase requires B time slots, one per bit transmitted from the nodes in each source cluster.

The per-cluster area at layer h is $A_h = 1/M^{\log_M(n)-h}$, and the per-node power is assumed upper bounded by $P(A_h)^{\alpha/2}/M^h$. For the parallel operation to be reliable, it is necessary to further bound the inter-cluster interference as in the following lemma.

Lemma 1 *For a network of size n, consider clusters of size M^h and area A_h operating as in the 5-phase scheme. Let each node have an average power $P(A_h)^{\alpha/2}/M^h$. For $\alpha > 2$, the interference power received by a node from other simultaneously operating clusters is upper-bounded by MK_{I_1} with a constant K_{I_1} independent of n. For $\alpha = 2$, the interference power is upper-bounded by $MK_{I_2}\log n$ with a constant K_{I_2} independent of n. And the interference signals received by different nodes in the cluster are zero-mean and uncorrelated.*

Phase 3. Joint Decoding in the Relay Cluster: Since nodes inside the relay cluster form a distributed receive antenna array, each node receives B MIMO-MAC transmissions during Phase 2. Thus, each node in the cluster receives B observations, one from each MIMO-MAC transmission, and each observation is to be conveyed to all other nodes for decoding. Since these observations are real numbers, nodes in the relay cluster quantize each observation to Q bits; hence, there are now a total of at most QB bits to exchange inside the relay cluster. Using exactly the same scheme as in Phase 1, it is clear that this phase requires QT_h time slots.

Phase 4. MIMO-BC using CFNC: This phase entails MIMO-BC transmissions from the relay cluster to the source clusters, as depicted in Fig. 2. CFNC is used again as in the previous section, and by analogy it follows that this phase is completed in B time slots.

Phase 5. Joint Decoding in Source Clusters: Since each source cluster receives B MIMO-BC transmissions in Phase 4, each node in the source clusters quantizes and exchanges each observation similar to the relay nodes during Phase 3 using a total of QT_h time slots.

Phases 1, 3, and 5 contain further MIMO-MAC and MIMO-BC transmissions at lower hierarchies, as illustrated in Fig. 2. Therefore, all transmissions in this 5-phase scheme take place during Phases 2 and 4 in each layer of the hierarchy. The inter-cluster interference power received at each node in Phase 4 also follows Lemma 1.

With each destination node capable of decoding the source bits from the quantized signals it collects by the end of Phase 5, the total number of time slots used in layer $h+1$ is

$$T_{h+1} = (2Q+1)T_h + 2B, \quad h = 1, 2, \ldots, \log_M(n) - 1 \qquad (6)$$

where $T_1 = 2B$. It then follows readily that

$$T_h = B\frac{(2Q+1)^h - 1}{Q}, \quad h = 1, 2, \ldots, \log_M(n) \qquad (7)$$

[2]Unless stated otherwise, it is assumed for simplicity that n is an integer power of M.

and the total number of time slots used in this 5-phase scheme is

$$T_{total} = T_{\log_M(n)} = B\frac{(2Q+1)^{\log_M(n)} - 1}{Q}. \tag{8}$$

Before returning to the capacity scaling issue, it is useful to clarify several definitions of the achievable rate. Following the conventional definition in n pairing networks, the total rate is B/T_{total}. While there are n^2 source-destination pairs in the network here, we will focus on the achievable rate for each source-destination pair. The following lemma quantifies the capacity scaling of this 5-phase scheme.

Lemma 2 *When $\alpha > 2$, the sum mutual information achieved by the MIMO-MAC from M nodes to one node, each equipped with N antennas, grows at least linearly with N. The other way around, same scaling for the sum mutual information can be achieved in a MIMO-BC transmission. When $\alpha = 2$, the sum mutual information in both MIMO-MAC and MIMO-BC grows at least on the order of $\Theta(N/\log n)$ for a network of size n.*

Consider first the case of $\alpha > 2$. Recall that all transmissions in this CFNC scheme are either MAC or BC. While joint encoding and decoding is employed in other phases, during MAC and BC transmissions, each cluster is treated as a single node with multiple antennas. At layer $h+1$, each node in the MAC from $M-1$ nodes to one node, has $N = M^h$ antennas. From Lemma 2, this leads to a capacity scaling of $\Theta(N)$ for the MAC. While considering each transmission pair, the achievable capacity scaling per pair is $R(n) = \Theta(N/(MN)) = \Theta(1/M)$. As the rate during the BC transmission is $R(n) = \Theta(1/M)$, the achievable capacity scaling per pair in the 5-phase scheme suffers a penalty of M relative to the conventional definition. Thus, the aggregate capacity scaling per pair in the 5-phase scheme is

$$R(n) = \frac{B}{MT_{total}} = \frac{Q}{M}\frac{1}{(2Q+1)^{\log_M(n)} - 1} \tag{9}$$

and as a result, the capacity scaling of the entire network is

$$C(n) = n^2 R(n) = \frac{Q}{M}\frac{n^2}{(2Q+1)^{\log_M(n)} - 1}. \tag{10}$$

Using (10), the following lemma yields the capacity scaling asserted in Theorem 2.

Lemma 3 *There exists a strategy to encode the observations at a fixed rate of Q bits per observation and arrive at a sum mutual information growth rate of $\Theta(N)$ (when $\alpha > 2$) or $\Theta(N/\log n)$ (when $\alpha = 2$) for the resultant quantized MIMO-MAC and MIMO-BC channels.*

Having fixed Q, let us turn our attention to M. If M is also fixed, the capacity scaling from (10) is

$$C(n) = \Theta(n^{2-\log_M(2Q+1)}) \tag{11}$$

which is not as high as asserted by Theorem 2.

To achieve an asymptotically optimal capacity scaling promised by Theorem 2, consider $M = \log n$, which implies that the size of each layer in the hierarchy M increases sufficiently slowly with the network size n. Furthermore, it is prudent to seek an optimal M to maximize the capacity scaling exponent. As M increases with n, the capacity scaling from (10) is

$$C(n) = \Theta\left(n^{2-\log_n(M)-\log_M(2Q+1)}\right). \tag{12}$$

To maximize the capacity scaling exponent is equivalent to:

$$\min_M \left\{\log_n(M) + \log_M(2Q+1)\right\}. \tag{13}$$

The optimal solution is $2/\sqrt{\log_{2Q+1}(n)}$, when M is chosen as $\log_{2Q+1}(M) = \sqrt{\log_{2Q+1}(n)}$. As a consequence, the capacity scaling of $C(n) = \Theta\left(n^{2-2/\sqrt{\log_{2Q+1}(n)}}\right)$ is achievable, which proves Theorem 2 for $\alpha > 2$.

When $\alpha = 2$, the per node capacity scaling incurs a penalty of $M\log n$ compared to the conventional case given in Lemma 2. Moreover, the number of transmissions in Phases 2 and 4 will scale as $\log n$, which makes the number of observations at each receiver node also scale as $\log n$. Hence, instead of QT_h, we will have $QT_h \log n$ time slots in Phases 3 and 5. After incorporating these modifications, the overall capacity scaling for $\alpha = 2$ is [cf. (9) and (10)]

$$C(n) = n^2 R(n) = \frac{Q}{M}\frac{n^2}{(2Q\log n+1)^{\log_M(n)} - 1}. \tag{14}$$

Although it is cumbersome to obtain the optimal M maximizing this capacity scaling, we are ready to complete the proof of Theorem 2. To achieve a capacity scaling of $\Theta(n^{2-\epsilon})$, it suffices to choose $M = (\log n)^{\log(\log n)}$, which yields a capacity scaling exponent

$$e(n) \geq 2 - \frac{[\log(\log n)]^2}{\log n} - \frac{2}{\log(\log n)}. \tag{15}$$

With $e(n)$ in (15) vanishing as n goes to infinity, this completes the proof of Theorem 2.

5. REFERENCES

[1] P. Gupta and P. Kumar, "The capacity of wireless networks," *IEEE Trans. on Info. Theory*, vol. 46, no. 2, pp. 388–404, Mar. 2000.

[2] M. Franceschetti, O. Dousse, D. N. C. Tse, and P. Thiran, "Closing the gap in the capacity of wireless networks via percolation theory," *IEEE Trans. on Info. Theory*, vol. 53, no. 3, pp. 1009–1018, Mar. 2007.

[3] S. Ahmad, A. Jovicic, and P. Viswanath, "Outer bounds to the capacity region of wireless networks," *IEEE Trans. on Info. Theory*, vol. 52, no. 6, pp. 2770–2776, June 2006.

[4] L.-L. Xie and P. R. Kumar, "On the path-loss attenuation regime for positive cost and linear scaling of transport capacity in wireless networks," *IEEE Trans. on Info. Theory*, vol. 52, no. 6, pp. 2313–2328, June 2006.

[5] A. Ozgur, O. Leveque, and D. Tse, "Hierarchical cooperation achieves optimal capacity scaling in ad hoc networks," *IEEE Trans. on Info. Theory*, vol. 53, no. 10, pp. 3549–3572, October 2007.

[6] T. Wang and G. B. Giannakis, "Complex field network coding for multiuser cooperative communications," *IEEE Journal on Selected Areas in Communications*, vol. 26, no. 3, pp. 561–571, Apr. 2008.

[7] J. Liu, D. Goeckel, and D. Towsley, "Bounds on the gain of network coding and broadcasting in wireless networks," in *Proc. of INFOCOM Conf.*, Anchorage, Alaska, USA, May 6-12, 2007.

[8] A. Goldsmith, S. A. Jafar, N. Jindal, and S. Vishwanath, "Capacity limits of MIMO channels," *IEEE Journal on Selected Areas in Communications*, vol. 21, no. 5, pp. 684–702, June 2003.

[9] T. Wang and G. B. Giannakis, "Capacity scaling of wireless networks with complex field network coding," *IEEE Journal on Selected Areas in Communications*, 2008; downloadable from http://www.ece.umn.edu/users/twang/CSJSAC.pdf.

[10] U. Niesen, P. Gupta, and D. Shah, "On capacity scaling in arbitrary wireless networks," *IEEE Trans. on Info. Theory*, 2007; downloadable from http://arxiv.org/PS_cache/arxiv/pdf/0711/0711.2745v1.pdf.