THROUGHPUT ANALYSIS OF COOPERATIVE WIRELESS MEDIUM ACCESS SCHEME EXPLOITING MULTI-BEAM ADAPTIVE ARRAYS

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ABSTRACT

Cooperative wireless network medium access schemes can achieve high throughput through collision resolution. By using a multi-beam adaptive array (MBAA) at a base station or access point, it can concurrently communicate with multiple nodes/users and thus the network performance can be further enhanced. In this paper, we provide an efficient packet resolution method and analyze the throughput of cooperative wireless medium access scheme exploiting MBAAs.

Keywords: antenna arrays, cooperative systems, information rates.

1. INTRODUCTION

Wireless networks such as cellular or wireless local area networks (WLANs) are continuously being developed to support higher network throughput to meet the increasing demands for the delivery of multimedia contents. The wireless network medium access schemes based on cooperation are able to improve network throughput and thus attract great research efforts. In the ALLIANCES (ALLow Improved Access in the Network via Cooperation and Energy Savings) scheme [1, 2], collided packets are buffered and a cooperative transmission epoch (CTE) is triggered once a collision occurs. In each CTE slot sequential to the collision slot, a relay node, which can be either a source node or a non-source node, is selected to transmit. A source node retransmits the packet, whereas a non-source node amplifies and forwards the collided packets it received in the collision slot. Thus, the scheme can form an equivalent MIMO problem to retrieve the original packets.

This scheme was extended to exploit multi-beam adaptive arrays (MBAAs) at the base station or access point (BS/AP) receiver so as to achieve higher data throughput [2, 3]. An MBAA may concurrently receive data packets from multiple users without collision. In the event of a collision, a reduced number of CTE slots suffices to collect enough independent equations involving the collided packets to recover the original packets. A CTE is triggered only when collisions are considered to happen.

Theoretically, the data received at an M-element array can be used to form up to M independent equations

in each time slot [4, 5, 6]. Collisions may still occur, e.g., when more than M users transmit packets simultaneously, and/or when the channels corresponding to different users are highly correlated. Thus, several consecutive CTE slots are likely to be needed. There are two fundamental questions associated with this approach: how to effectively resolve the collided packets and how high of the uplink throughput to be achieved. The purpose of this paper is to answer these questions. We provide an efficient packet resolution method and analyze the throughput performance.

This paper is organized as follows. Section 2 provides the cooperative medium access protocol. The throughput performance is analyzed in Section 3. Simulation results are provided in Section 4. Finally, we conclude this paper in Section 5.

2. COOPERATIVE MEDIUM ACCESS PROTOCOL WITH MBAA

2.1. General Description

Consider a BS/AP receiver (destination) which is able to concurrently receive signals from up to M nodes without collisions. Packet collision occurs and CTE slots are initialized when the number of active users exceeds the receiver capacity or the channels are highly correlated. In each CTE time slot, M_r nodes are permitted to simultaneously transmit as cooperative relays: the source relay nodes simply retransmit their own packets, and the non-source relay nodes amplify and forward the collided packets which are individually received in the collision time slot. The stacked channel matrix is updated in each CTE slot. The packets are jointly retrieved at the end of the final CTE slot.

The introduction of MBAA to the cooperative wireless medium access scheme has been shown to be effective to improve its throughput performance, since the spatial dimensionality can be utilized for the provision of robust and high throughput wireless links. For the underlying network system, similar to those introduced in [1], we make the following assumptions.

 Consider a slotted multiple access network system, where all the J nodes are synchronized to the destination. Both the link delay and decoding delay at the destination are ignored. All nodes operate in a half-duplex mode and transmit in the same carrier frequency channel.

- 2) Each node can hear the messages from the destination on a downlink control channel, and thus each knows when it should either transmit a source packet or forward a cooperative packet.
- 3) Each node in non-transmit state stays in receive state, and thus it can hear packets from other nodes. The destination is equipped with an *M*-element adaptive array, whereas only one antenna is used at each source and relay node. The uplink packet length is the same of one time slot.
- 4) Channel coefficients remain constant within a time slot. As such, array processing at the destination is feasible.

2.2. Collision Resolution by Array Processing

Consider that K packets are collided in the *n*th time slot (non-CTE slot) and the packet sent by the i_k th node consists of N symbols $\mathbf{x}_{i_k}(n) = [x_{i_k,0}(n), \ldots, x_{i_k,N-1}(n)]$. Let $S(n) = \{i_1, \ldots, i_K\}$ denote the set of active source nodes, $\overline{S}(n) = \{r_1, \ldots, r_{\overline{K}}\}$ be that of the $\overline{K} = J - K$ non-source nodes, and $D = \{d_1, \ldots, d_M\}$ contain the MBAA antennas. In the *n*th slot, the received signal at an MBAA antenna or a non-source node is given by the $1 \times N$ vector

$$\mathbf{y}_p(n) = \sum_{i_k \in S(n)} h_{pi_k}(n) \mathbf{x}_{i_k} + \mathbf{n}_p(n), \qquad (1)$$

where $p \in D \bigcup \overline{S}(n)$, h_{pi_k} is the channel coefficient from the i_k th source node to either the *p*th antenna of the BS/AP or the *p*th non-source node, and $\mathbf{n}_p(n)$ is the additive white Gaussian noise (AWGN) vector with mean zero and variance σ_n^2 . For convenience, the signal received by the MBAA in the *n*th slot (i.e., the collision slot) can be written in form of the $M \times N$ matrix

$$\mathbf{Y}(n) = [\mathbf{y}_{d_1}^T(n), \dots, \mathbf{y}_{d_M}^T(n)]^T = \mathbf{H}(n)\mathbf{X}(n) + \mathbf{n}(n), \qquad (2)$$

where $\mathbf{H}(n) = [\mathbf{h}_{Di_1}(n), \ldots, \mathbf{h}_{Di_K}(n)] \in \mathbb{C}^{M \times K}$ with $\mathbf{h}_{Di_k}(n) = [h_{d_1i_k}(n), \ldots, h_{d_Mi_k}(n)]^T$, $\mathbf{X}(n) = [\mathbf{x}_{i_1}^T(n), \ldots, \mathbf{x}_{i_K}^T(n)]^T \in \mathbb{C}^{K \times N}$, and $\mathbf{n}(n) \in \mathbb{C}^{M \times N}$ is the noise matrix. Furthermore, $(\cdot)^T$ denotes transpose. Each channel coefficient can be estimated, for example, through the orthogonal ID sequence that is embedded in the beginning of the packet [1]. Consequently, M linear equations are constructed.

On the one hand, when the channel matrix $\mathbf{H}(n)$ is fullcolumn rank, the array system at the destination receiver is able to resolve the packet collision, and thus there is no need to start a CTE slot. On the other hand, when the condition above does not hold, i.e., when $\mathbf{H}(n)$ becomes column rank-deficient, the K packets cannot be resolved. The CTE procedure is initiated so as to resolve all collided packets.

In a cooperation network, there are different ways to retrieve each collided packet. We adopt a simplified method here. Collided packets are retrieved at the end of the final CTE slot, i.e., the destination needs only to identify whether the channel matrix thus far is full-column rank; all the collided packets are jointly decoded at the end of the final CTE slot.

Denote K-1 as the total number of required CTE time slots. Assume that in the *m*th $(1 \le m \le (\hat{K}-1))$ CTE time slot, there are *q* non-source relay nodes forwarding the collided packets which are individually received in the *n*th time slot, and there are $p=M_r-q$ source nodes retransmitting their own packets. The received signal at the MBAA can be written as

$$\mathbf{Y}(n+m) = [\mathbf{y}_{d_1}^T(n+m), \dots, \mathbf{y}_{d_M}^T(n+m)]^T$$
$$= [\mathbf{H}_{\mathbf{s}}(n+m) + \mathbf{H}_{\mathbf{r}}(n+m)] \mathbf{X}(n) + \mathbf{n}(n+m), \quad (3)$$

where

$$\mathbf{H}_{\mathbf{r}}(n+m) = \sum_{j=1}^{q} \gamma_{r_j} \mathbf{h}_{Dr_j}(n+m) \mathbf{h}_{r_j S}^{T}(n+m)$$
(4)

$$\mathbf{n}(n+m) = \sum_{j=1}^{q} \gamma_{r_j} \mathbf{h}_{Dr_j}(n+m) \mathbf{n}_{r_j}(n+m) + \mathbf{n}_D(n+m), \quad (5)$$

$$\begin{split} \mathbf{H_s}(n+m) \in \mathbb{C}^{M \times K} \text{ is the channel matrix denoting the uplink channel coefficients from the } p \text{ source relay nodes to the destination at the } mth CTE time slot; \\ \mathbf{H_r}(n+m) \in \mathbb{C}^{M \times K} \text{ is the channel matrix containing not only the channel matrix } \\ \mathbf{h}_{r_j S}(n+m) \in \mathbb{C}^{K \times 1}, \text{ representing channel coefficients from all source nodes to the } jth non-source relay node, but also the channel matrix \\ \mathbf{h}_{Dr_j}(n+m) \in \mathbb{C}^{M \times 1}, \text{ denoting channel coefficients from the } jth non-source relay node, but also the channel matrix \\ \mathbf{h}_{Dr_j}(n+m) \in \mathbb{C}^{M \times 1}, \text{ denoting channel coefficients from the } jth non-source relay node to the destination, \\ j=1,\ldots,q; \text{ and } \mathbf{n}(n+m) \in \mathbb{C}^{M \times N} \text{ is the noise matrix, consisting of both the AWGN } \mathbf{n}_D(n+m) \text{ in the destination and the relay noise from } \mathbf{n}_{r_j}, \text{ the noise at non-source relay nodes. The scaling factor is } \\ \gamma_{r_j} = \sqrt{\sigma_x^2/(K\sigma_h^2\sigma_x^2+\sigma_n^2)}, \text{ where } \sigma_x^2 \text{ is the average power of transmitted symbols and } \\ \sigma_h^2 \text{ is the power gain of channel among nodes.} \end{split}$$

Following the procedure above, when cooperative transmission proceeds up to the $(\hat{K}-1)$ th CTE time slot, the stacked signal at the destination can be expressed as

 $\mathbf{Y} = \mathbf{H}\mathbf{X}(n) + \mathbf{n},$

where

$$\mathbf{Y} = \left[\mathbf{Y}(n)^{T}, \mathbf{Y}(n+1)^{T}, \cdots, \mathbf{Y}(n+\widehat{K}-1)^{T}\right] \stackrel{T}{\in} \mathbb{C}^{M\widehat{K} \times N}, \qquad (7)$$

(6)

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}(n) \\ \mathbf{H}_{\mathbf{s}}(n+1) + \mathbf{H}_{\mathbf{r}}(n+1) \\ \vdots \\ \mathbf{H}_{\mathbf{s}}(n+\widehat{K}-1) + \mathbf{H}_{\mathbf{r}}(n+\widehat{K}-1) \end{bmatrix} \in \mathbb{C}^{M\widehat{K} \times K}, \quad (8)$$
$$\mathbf{n} = \begin{bmatrix} \mathbf{n}(n)^{T}, \mathbf{n}(n+1)^{T}, \cdots, \mathbf{n}(n+\widehat{K}-1)^{T} \end{bmatrix}^{T} \in \mathbb{C}^{M\widehat{K} \times N}. \quad (9)$$

Once the channel matrix **H** reaches full-column rank, the CTE procedure is stopped, and all the data packets originated from the K users can be retrieved through a multi-user detection. Optimally, the collided packets are decoded using a maximum likelihood (ML) decoder, minimizing the Frobenius norm with respect to $\mathbf{X}(n)$ under the set space Ω of all possible transmitted symbols, i.e.,

$$\widehat{\mathbf{X}}(n) = \underset{\mathbf{X}(n)\in\Omega}{\arg\min} \|\mathbf{Y} - \mathbf{H}\mathbf{X}(n)\|_{\mathrm{F}} .$$
(10)

The ML approach can fully exploit the cooperative diversity provided by the system and achieve optimal performance at the expense of high decoding complexity. An alternative suboptimal zero-forcing linear decoder with lower computation cost, expressed as

$$\widehat{\mathbf{X}}(n) = \mathbf{H}^{\dagger} \mathbf{Y}, \tag{11}$$

can also be used, where \mathbf{H}^{\dagger} is the pseudo-inverse of \mathbf{H} .

3. THROUGHPUT ANALYSIS

In this section, we analyze the throughput performance of the cooperative wireless medium access scheme proposed in Section 2. We define the uplink throughput G as the expected value of the simultaneous throughput (K/\hat{K}) , i.e., the number of nodes from which packets are successfully received in a time slot.

Assume that all J nodes are independently located in the network area with a uniform angular distribution around the destination and each node transmits a source packet with identical probability p_s . For convenience, we ignore queueing considerations at each node. Thus, the probability that, at the *n*th slot, K out of J active nodes simultaneously transmit source packets, is given by

$$P(K) = \binom{J}{K} p_s^K (1 - p_s)^{J - K}, \qquad (12)$$

where $\binom{J}{K}$ denotes the combination operation representing the number of different ways of selecting K out of J nodes.

Let $P(\hat{K}|K)$ be the conditional probability that \hat{K} slots are required by collision resolution under the condition of K active nodes in the *n*th time slot. Clearly, for a given K, the value of \hat{K} can be $1, 2, \dots, \infty$. Thus, the throughput in the presence of K active nodes can be written as

$$G_K = \frac{K}{\sum_{\hat{K}=1}^{\infty} \hat{K} P(\hat{K}|K)}.$$
(13)

Finally, the uplink throughput is given by

$$G = \sum_{K=1}^{J} G_K P(K).$$
 (14)

It can be seen that, in order to obtain the uplink throughput, the conditional probability $P(\hat{K}|K)$ needs to be known.

From the protocol described in Section 2, it is known that the event that the number of time slots required for collision resolution in the presence of K active nodes equals \hat{K} is equivalent to the event that the channel matrix given in (8) is full-column rank. So, their occurrence probabilities are

$$P(\vec{K}|K) = \Pr(\mathbf{H} \text{ is full-column rank}).$$
(15)

As is well known, the rank of a matrix equals the number of non-zero singular values. $\mathbf{H} \in \mathbb{C}^{M\widehat{K} \times K}$ has $\min(M\widehat{K}, K)$ singular values, so, for $M\widehat{K} \geq K$, **H** is full-column rank if only its smallest singular value σ_m is non-zero. Practically, the channel matrix **H** can be considered full-column rank if $\sigma_m \geq \sigma_{th}$, where σ_{th} is a preset threshold. Thus, (15) can be written as

$$P(\hat{K}|K) = \Pr(\sigma_m \ge \sigma_{th}), \tag{16}$$

which is relevant to the problem of probability distribution of the smallest singular value of a matrix [7]-[11].

We first consider the following special case. For $\mathbf{H} \sim \mathcal{CN}(\mathbf{0}, a^2 \mathbf{I}_K)$, following the derivation in [7], the probability distribution of the smallest singular value of \mathbf{H} can be expressed as

$$\Pr(\sigma_m \ge \sigma_{th}) = \left| \frac{\det(\mathbf{A}_{K,K_d,\sigma_{th}^2/a^2})}{\det(\mathbf{A}_{K,K_d,0})} \right|, \quad (17)$$

where $K_d = M\hat{K} - K$,

$$\mathbf{A}_{K,K_{d},b} = \begin{bmatrix} \widetilde{\Gamma}_{b}(K_{d}+1) & \cdots & \widetilde{\Gamma}_{b}(K_{d}+K) \\ \widetilde{\Gamma}_{b}(K_{d}+2) & \cdots & \widetilde{\Gamma}_{b}(K_{d}+K+1) \\ \vdots & & \vdots \\ \widetilde{\Gamma}_{b}(K_{d}+K) & \cdots & \widetilde{\Gamma}_{b}(K_{d}+2K-1) \end{bmatrix}, \quad (18)$$

is a $K \times K$ matrix,

$$\widetilde{\Gamma}_b(x) = \int_b^\infty t^{x-1} e^{-t} dt = \Gamma(x) \left[1 - \Gamma_b(x)\right], \qquad (19)$$

 $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \text{ and } \Gamma_b(x) = \frac{1}{\Gamma(x)} \int_0^b t^{x-1} e^{-t} dt.$ In general, the covariance matrix of **H**, $\Sigma \in \mathbb{C}^{K \times K}$, given

In general, the covariance matrix of $\mathbf{H}, \boldsymbol{\Sigma} \in \mathbb{C}^{\times n}$, given in (8), is not an identity matrix. In this case, we consider $\mathbf{H} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{\Sigma})$, and thus $\mathbf{W} = \mathbf{H}^H \mathbf{H}$ is the complex central Wishart matrix with K degrees of freedom [7]–[11]. From the distribution of the smallest eigenvalue λ_m of \mathbf{W} , the distribution of the smallest singular value of \mathbf{H} is derived as [9]

$$\Pr(\sigma_m \ge \sigma_{th}) = \Pr(\lambda_m \ge \sigma_{th}^2)$$

= etr($-\sigma_{th}^2 \Sigma^{-1}$) $\sum_{k=0}^{K(M\hat{K}-K)} \sum_{\mathcal{K}} \frac{\mathcal{C}_{\mathcal{K}}(\sigma_{th}^2 \Sigma^{-1})}{k!}$, (20)

where $\operatorname{etr}(\cdot)$ denotes the exponential of the trace, $\sum_{\mathcal{K}} \operatorname{denotes}$ the summation over the partitions¹ $\mathcal{K} = (k_1, \cdots, k_K)$ of k with $k_1 \leq (M\hat{K} - K)$, and $\mathcal{C}_{\mathcal{K}}(\mathbf{B})$ is the complex zonal polynomial of the complex matrix $\mathbf{B} \in \mathbb{C}^{K \times K}$, $\mathbf{B} \neq \mathbf{I}_K$, defined as [9, 10]

 $\mathcal{C}_{\mathcal{K}}(\mathbf{B}) = \chi_{[\mathcal{K}]}(1) \cdot \chi_{[\mathcal{K}]}(\mathbf{B}),$

and

$$\prod_{i< j}^{K} (k_i - k_j - i + j)$$

$$\chi_{[\mathcal{K}]}(1) = k! \frac{\prod_{i=1}^{K} (N_i - N_j - i + j)}{\prod_{i=1}^{K} (k_i + K - i)!},$$
(22)

$$\chi_{[\mathcal{K}]}(\mathbf{X}) = \frac{\det(\mathbf{A}_n)}{\det(\mathbf{A}_d)}.$$
(23)

(21)

In the above expression, both \mathbf{A}_n and \mathbf{A}_d have the same size of $K \times K$, and their *ij*th entries are respectively expressed

¹For the partition $\mathcal{K} = (k_1, \dots, k_K)$ of the integer k, two conditions hold: (1) $k_1 \geq \dots \geq k_K \geq 0$, and (2) $k = k_1 + \dots + k_K$.

as $(\mathbf{A}_n)_{ij} = \lambda_i^{k_j + K - j}$ and $(\mathbf{A}_d)_{ij} = \lambda_i^{K - j}$, with $\lambda_1, \dots, \lambda_K$ denoting the K eigenvalues of **B**. Particularly, $\mathcal{C}_{\mathcal{K}}(x) = x^k$ for K = 1 [10], and $\mathcal{C}_{\mathcal{K}}(\alpha \mathbf{B}) = \alpha^k \mathcal{C}_{\mathcal{K}}(\mathbf{B})$ [8].

When either $\Sigma = a^2 \mathbf{I}$ or $\mathbf{B} = \alpha \mathbf{I}$ holds, (21) becomes invalid. Instead, (17) is applicable in this case, or the following expression derived from [9, 10] can be used

$$\mathcal{C}_{\mathcal{K}}(\alpha \mathbf{I}_{K}) = (4\alpha)^{k} k! \left[\frac{K}{2}\right] \frac{\prod_{i < j}^{r} (2k_{i} - 2k_{j} - i + j)}{\kappa \prod_{i = 1}^{r} (2k_{i} + r - i)!}, \quad (24)$$

where

$$\left[\frac{K}{2}\right]_{\mathcal{K}} = \prod_{i=1}^{r} \frac{\Gamma\left[(K-i+1)/2 + k_i\right]}{\Gamma\left[(K-i+1)/2\right]},$$
(25)

and r is the number of non-zero parts in the partition \mathcal{K} of k.

The above discussions provide the distribution of the smallest singular value of the channel matrix \mathbf{H} and the probability of \mathbf{H} being full-column rank , provided that the channel matrix \mathbf{H} is a Gaussian matrix with zero-mean entries. As a result, the following expression of the uplink throughput is ready for numerical evaluations:

$$G = \sum_{K=1}^{J} \frac{KP(K)}{\sum_{\hat{K}=1}^{\infty} \hat{K} \Pr(\lambda_m \ge \sigma_{th}^2)}.$$
 (26)

4. SIMULATION RESULTS

In this section, simulation results are provided to demonstrate the performance of the throughput analysis. we establish a simulation environment similar to [2]. In the simulations, a Bernoulli model is used for the generation of active nodes under a given traffic load λ . Assume that there are J=32 nodes in a network and they are statistically independent. Each node transmits a packet with probability $p_s = \lambda/J$ at each non-CTE slot. Perfect detection of active users is assumed. The ID sequences of the active nodes are selected from a Hadamard matrix of order J. QPSK modulation with symbol rate of 6 MSPS is used, and the packet length is N = 424 bits. The zero-forcing decoding method is considered. Channel coefficients from nodes to the destination are assumed to have unit variance and are simulated using the Jakes' model, and the input signal-to-noise ratio (SNR) at each receive antenna is 20 dB. The carrier frequency is 5.2 GHz, and the maximum Doppler frequency is assumed to be $f_d = 52$ Hz. The channel coefficients among nodes are free of fading. Consider that the destination is equipped with a uniform linear array (ULA) consisting of M = 4 elements with a half-wavelength inter-element spacing. The channel matrix is considered full-column rank if the minimum singular value is not below 0.07. The packets with bit error rate (BER) higher than 0.02 are considered corrupted. Simulation results $(M_r = M)$ are plotted in Figure 1, which shows that an improved throughput consistent with analytical results is achieved. When the traffic load is very high, the throughput is still sufficiently high due to the application of the cooperative medium access scheme.



Figure 1: Uplink throughput versus traffic load.

5. CONCLUSIONS

In this paper, we have presented an efficient packet resolution method and throughput performance analysis of cooperative wireless network medium access scheme. Using the probability distribution of the smallest singular value of the stacked channel matrix, we have analyzed the throughput performance of the cooperative scheme in an analytical framework. The analytical and simulation results showed good agreements and verified that the presented cooperative wireless network medium access scheme achieves high throughput with the use of multi-beam adaptive array.

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