SYNTHESIS OF PLANAR ARRAYS WITH ARBITRARY GEOMETRY FOR FLAT-TOP FOOTPRINT PATTERNS

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ABSTRACT

This paper presents a new synthesis algorithm to produce flat-top main beams with arbitrary footprint for array elements placed in an arbitrary planar geometry. The general framework of the paper would encompass the patterns produced by circular aperture distribution. It is shown that for patterns generated by circular apertures some efficient simplifying facts based on properties of Bessel functions can be applied. The method shows a high performance in generating optimally flat-top patterns with detailed geometry footprints.

Index Terms— Array pattern synthesis, flat-top pattern, shaped beam antennas

1. INTRODUCTION

Shaping the main beam of planar arrays is a problem of interest for many applications such as satellite communication and medical imaging where we require the main beam to specifically conform to a rather complex and not necessarily well geometrically defined region of interest. In such applications the main beam is usually preferred to be flat-top for a constant coverage over the region of interest. Most footprint pattern synthesis methods already proposed require the array elements to lie on a circular or rectangular lattice [1, 2]. They also require defining a desired pattern as a reference pattern, where obviously some values should be assigned to the reference function in the transient region. In addition to the assumptions made the patterns produced usually fail to have flat-top main beams [3]. In this paper using the formulation of continuous circular aperture and the properties of Bessel functions we change the synthesis problem to a finite series expansion problem where by imposing appropriate energy minimizations the problem ends up with a Rayleigh quotient problem, i.e. an eigenvalue-eigenvector problem. In FIR filter designs which are equivalent to array design problems, Rayleigh quotient mostly inspires the idea of Eigenfilters which also use some concept of energy minimization [4]. One of the problems with classic Eigenfilters is the need to a reference frequency point and normalization [5]. The method presented in this paper does not require a reference point or a reference pattern function and does not provide geometric restrictions to the lattice of array elements while it is capable of producing optimally flat-top and shaped main beams with low sidelobe levels.

2. DESCRIPTION OF THE METHOD

2.1. The Pattern Produced by a Circular Aperture

Consider a planar circular aperture of radius a centered at the origin possessing current distribution of $K(\rho, \beta)$, where ρ and β are the radial and angular coordinates in the aperture. As shown in [1] by Fourier series representation of

$$K(\rho, \beta) = \sum_{n=-\infty}^{\infty} K_n(\rho) e^{jn\beta}, \qquad (1)$$

the far-field pattern $F(\eta, \phi)$ radiated from this aperture can be formulated as:

$$F(\eta, \phi) = \sum_{n=-\infty}^{\infty} F_n(\eta) e^{jn\phi} \qquad (2)$$

where

$$F_n(\eta) = 2\pi (j)^n \int_0^a K_n(\rho) J_n(k\rho\eta) \rho d\rho, \qquad (3)$$

$\eta = \sin\theta$, θ and ϕ are the spherical coordinates and $k = 2\pi/\lambda$ the wavenumber. For a given $F(\eta, \phi)$, the quantities $F_n(\eta)$ are obtained by a Fourier series expansion as (2) and the problem then is to determine $K_n(\rho)$ from (3). As the current is limited within the aperture the unknown functions $K_n(\rho)$ should vanish for $\rho > a$ and to be able to apply inverse Hankel transform to (3) they should also be continuous at $\rho = a$. Assuming the existence of such functions the upper bound of integration in (3) can be converted to ∞ and by applying an inverse Hankel transform, hence, one may deduce

$$K_n(\rho) = \frac{k^2 (j)^{-n}}{2\pi} \int_0^{\infty} F_n(\eta) J_n(k\rho\eta) \eta d\eta. \qquad (4)$$

From infinite integration properties of Bessel functions we know that for integer orders of bessel functions

$$\int_0^\infty J_n(\alpha_1\eta)J_n(\alpha_2\eta)\frac{\eta d\eta}{\eta^2-\eta_0^2} = \begin{cases} \frac{1}{2}\pi j J_n(\alpha_2\eta_0)H_n^{(1)}(\alpha_1\eta_0) & \alpha_1 > \alpha_2 \\ \frac{1}{2}\pi j J_n(\alpha_1\eta_0)H_n^{(1)}(\alpha_2\eta_0) & \alpha_1 < \alpha_2 \end{cases} \tag{5}$$

where $H_n^{(1)}$ is the Hankel function of first kind and order n. Using (4) and (5) it is revealed that if $F_n(\eta)$ is expressed as

$$F_n(\eta) = \sum_{l=1}^{L_n} c_{n,l}\frac{J_n(ka\eta)}{\eta^2 - (\frac{\gamma_{n,l}}{ka})^2}, \tag{6}$$

where L_n is a positive integer and $\gamma_{n,l}$ is the l^{th} positive root of n^{th} order Bessel function, then

$$K_n(\rho) = \begin{cases} -\frac{k^2}{4}(j)^{-n}\sum\limits_{l=1}^{L_n} c_{n,l}Y_n(\gamma_{n,l})J_n(\frac{\rho\gamma_{n,l}}{a}) & 0 \le \rho < a \\ 0 & \rho \ge a, \end{cases} \tag{7}$$

denoting Y_n as the Bessel function of second kind and order n. For this form of $K_n(\rho)$, near $\rho = a$ all the terms in the summation vanish and produce a continuous function which satisfies the assumption already made about $K_n(\rho)$. Generally speaking if the following expansion is performed to express the desired pattern for $\eta \le a$:

$$F(\eta,\phi) = \sum_{n=-\infty}^{\infty}\sum_{l=1}^{L_n} c_{n,l}f_{n,l}(\eta,\phi) \tag{8}$$

where

$$f_{n,l}(\eta,\phi) = e^{jn\phi}\frac{J_n(ka\eta)}{\eta^2 - (\frac{\gamma_{n,l}}{ka})^2} \tag{9}$$

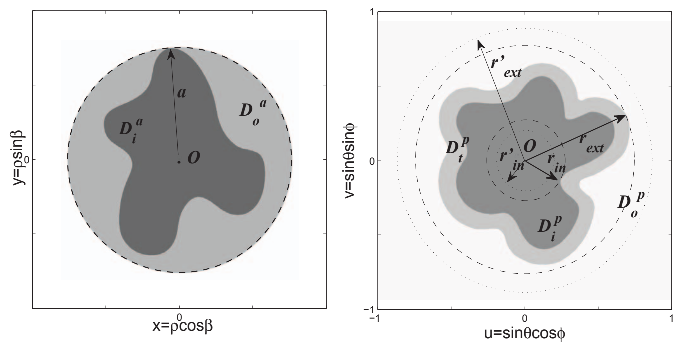
then the current distribution could be expressed as

$$K(\rho,\beta) = -\frac{k^2}{4}\sum_{n=-\infty}^{\infty}\sum_{l=1}^{L_n} c_{n,l}k_{n,l}(\rho,\beta) \tag{10}$$

where

$$k_{n,l}(\rho,\beta) = Y_n(\gamma_{n,l})e^{jn(\beta-\frac{\pi}{2})}J_n(\frac{\rho\gamma_{n,l}}{a}). \tag{11}$$

For the sake of simplicity in future notations we define two domains; the aperture domain with polar coordinates of ρ and β and corresponding rectangular coordinates of x and y Fig. 1.a, and the pattern domain with polar coordinates of η and ϕ and corresponding rectangular coordinates of u and v Fig. 1.b. As another convention throughout the paper for the functions already defined in polar coordinates of either domains, the same function defined for rectangular coordinates of that domain is denoted with an additional hat. e.g., $\hat{F}(u,v) \triangleq F(\eta,\phi)\Big|_{\substack{\eta=\sqrt{u^2+v^2} \\ \phi=\arctan\frac{v}{u}}}$ and $\hat{K}(x,y) \triangleq K(\rho,\beta)\Big|_{\substack{\rho=\sqrt{x^2+y^2} \\ \beta=\arctan\frac{y}{x}}}$.



(a) Aperture Domain (b) Pattern Domain

Fig. 1. Illustration of the synthesis domains, defined regions and parameters

2.2. Finite Aperture Radius Considerations

Most synthesis problems are concerned with having $F(\eta,\phi)$ a nonzero constant in a subset of u-v plane to be known as the coverage or *shaped region* and zero in the blockage or *unshaped region*. A typical case is shown in Fig. 1.b, where the dark region represents the shaped region denoted as $\mathcal{D}_i^p$, the lighter region represents the transient region denoted as $\mathcal{D}_t^p$ and the remaining region in $\eta \le 1$ which excludes $\mathcal{D}_i^p$ and $\mathcal{D}_t^p$, as the unshaped region. We also define $r_{in} = min\{\mathbf{d}(\mathrm{x})|\mathrm{x} \in \mathcal{D}_t^p\}$ and $r_{ext} = max\{\mathbf{d}(\mathrm{x})|\mathrm{x} \in \mathcal{D}_t^p\}$, where $\mathbf{d}(\cdot)$ is the distance function to the origin. In this section we illustrate that the summations in (8) and (10) are finite sums for finite values of aperture radius. Although the method works for appropriate multiply connected shaped regions, to easily explain the idea we consider the shaped region to be a simply connected region in u-v plane. With no loss of generality one may assume the origin of u-v plane belonging $\mathcal{D}_i^p$, as $\hat{F}(u+u_0, v+v_0)$ can be obtained through multiplying $\hat{K}(x,y)$ by $\exp(jk[xu_0+yv_0])$. In this paper, as already mentioned, we do not consider a reference function for $F(\eta,\phi)$, but to find the most impressive coefficients in (8) we assume it to be given. Therefore, from (2) we have

$$F_n(\eta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\eta,\phi)e^{-jn\phi}d\phi. \tag{12}$$

About the unknown coefficients $c_{n,l}$ in (6) the following lemma is true.

Lemma: In collocation (6), assume $F_n(\eta) = 0$ for some $\eta \in (\eta_1,\eta_2)$. If for some integer values of L, $\frac{\gamma_{n,L}}{ka} \in (\eta_1,\eta_2)$, then $c_{n,L} = 0$.

Proof: By taking η very close to $\gamma_{n,L}$, the left side of (6) must be zero according to the assumption. At the right side, because of the appearance of $J_n(ka\eta)$ in the numerator of all summation terms, all the terms tend to zero, except the corresponding term for $c_{n,L}$ which has a nonzero limit of $\frac{ka}{2\gamma_{n,L}} J_{n-1}(\gamma_{n,L})$. Equating both sides results in $c_{n,L} = 0$. ∎

Regarding our problem the roots of Bessel functions have the property that for an integer constant n, as l increases, $\gamma_{n,l}$ also increases. Also for $l = 1$, as $|n|$ increases, $\gamma_{n,1}$ increases. Such ordered structure for the roots and the lemma already mentioned can be used to change (8) into a finite sum. For $r'_{ext} \in (r_{ext}, 1]$ and $r'_{in} \in [0, r_{in})$ and reasonable values of a (i.e. $\frac{a}{\lambda} \gg 1$) the following simplifications are valid:

I. There always exists a positive n_{ext} that

$$\gamma_{(n_{ext}-1),1} < kar'_{ext} \leq \gamma_{n_{ext},1}. \tag{13}$$

As $F(\eta, \phi) = 0$ for $\eta > r'_{ext}$, one may deduce that $F_n(\eta) = 0$ for $\eta > r'_{ext}$. Therefore, using the lemma, $c_{n,l} = 0$ for all $|n| \geq n_{ext}$.

II. For every $n < n_{ext}$ there exists $\gamma_{n,l^{(n)}_{ext}}$, a positive root of order n Bessel function, which

$$\gamma_{n,(l^{(n)}_{ext}-1)} < kar'_{ext} \leq \gamma_{n,l^{(n)}_{ext}}, \tag{14}$$

therefore, again using the lemma $c_{n,l} = 0$ for every $l > l^{(n)}_{ext}$.

III. For some $n < n_{ext}$ there can exist $l^{(n)}_{in}$, a positive root of order n Bessel function, which

$$\gamma_{n,l^{(n)}_{in}} \leq kar'_{in}. \tag{15}$$

As $F(\eta, \phi)$ has a constant value in the shaped region, then using (12), $F_n(\eta) = 0$ for $\eta < r'_{in}$ and $n \neq 0$. Therefore if $l^{(n)}_{in}$ exists, $c_{n,l} = 0$ for all $l < l^{(n)}_{in}$ and and $n \neq 0$. It should be noted that if $l^{(n)}_{in}$ does not exist for an n, it means that $\gamma_{n,l} > kar'_{in}$ and therefore $kar'_{in} < \gamma_{n,l} < \gamma_{n+1,l} < \cdots < \gamma_{n_{ext},l}$, which means $l^{(n)}_{in}$ does not also exist for the next values of n.

Applying aforementioned simplifications and ignoring the indices for which $c_{n,l} = 0$ in (8), only a finite number of impressive functions $f_{n,l}(\eta, \phi)$ or $\hat{f}_{n,l}(u, v)$ remain, in terms of which $F(\eta, \phi)$ or $\hat{F}(u, v)$ should be expressed. Assuming M to be the total number of such functions, we can switch to summation over a single index as

$$\hat{F}(u, v) = \sum_{m=1}^{M} c_m \hat{f}_m(u, v) \tag{16}$$

where $m \Leftrightarrow (n, l)$.

2.3. Flat-top Shaped Beams through Arbitrary Geometry of Aperture

This section utilizes the concepts already mentioned to propose a method of obtaining flat-top shaped beams through an arbitrary planar aperture.

Back to Fig. 1.a, it shows a typical arbitrary shaped aperture in dark color in $x - y$ plane. The original aperture is denoted as $\mathcal{D}^a_i$. We can always consider $\mathcal{D}^a_i$ as a subset of a circular aperture, radius of which is defined as $a = max\{\mathbf{d}(\mathrm{x}) | \mathrm{x} \in \mathcal{D}^a_i\}$. The portion of the circle which excludes $\mathcal{D}^a_i$ is denoted as $\mathcal{D}^a_o$.

Imagining the synthesis problem as a minimization problem of finding appropriate coefficients in (16), a cost function can be defined as

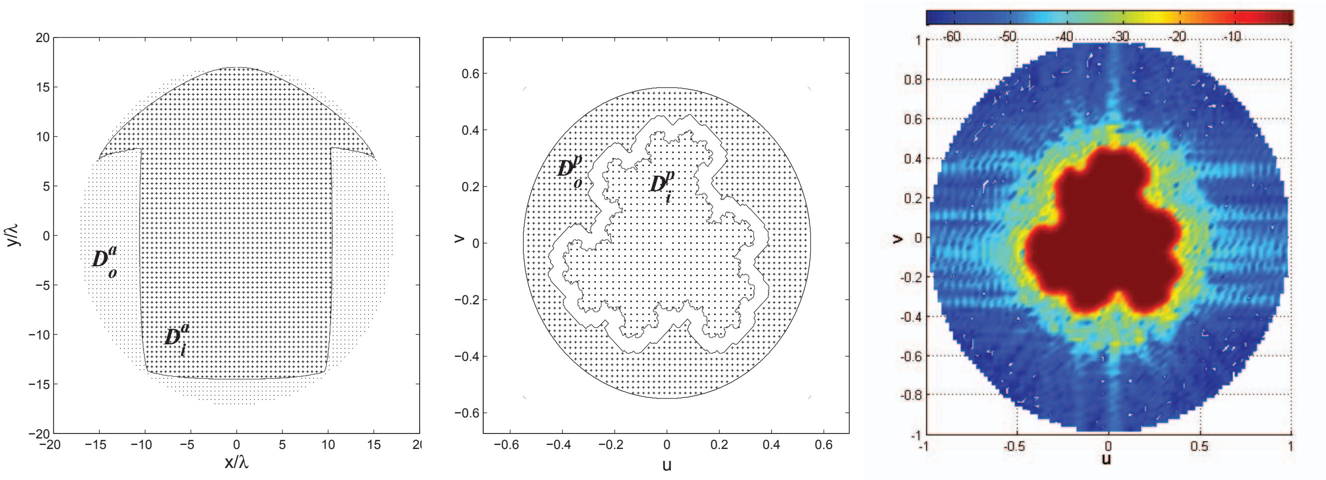
$$J_L(c_1, c_2, \ldots c_M) = \iint\limits_{\mathcal{D}^p_i} \|\nabla \hat{F}(u, v)\|^2 du dv$$
$$+ \lambda_1 \iint\limits_{\mathcal{D}^p_o} \|\hat{F}(u, v)\|^2 du dv + \lambda_2 \iint\limits_{\mathcal{D}^a_o} \|G(x, y)\|^2 dx dy, \tag{17}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are two regularization parameters, $G(x, y) = \frac{-4}{k^2} \hat{K}(x, y)$ is a frequency independent version of current distribution $\hat{K}(x, y)$, and $\mathcal{D}^p_o$ as shown in Fig. 1.b denotes the circle of radius r'_{ext} excluding $\mathcal{D}^p_i \bigcup \mathcal{D}^p_t$.

In fact the first term in (17) forces the pattern to be flat-top, the second term suppresses the sidelobes and the last term forces the current to be restricted to $\mathcal{D}^a_i$.

It should be noted that constructing a collocation problem as (16) with basis functions of $\hat{f}_m(u, v)$, made two major simplifications in formation of (17). First, the sidelobes should be suppressed within $\mathcal{D}^p_o$ only and not over the whole unit circle excluding $\mathcal{D}^p_i \bigcup \mathcal{D}^p_t$. This is valid because by forming (16) we have ignored the coefficients in (8) that cause significant values taking place in $\eta > r'_{ext}$. By ignoring such coefficients we only get some very small and negligible tails coming from the functions $\hat{f}_m(u, v)$ into the region $\eta > r'_{ext}$. Second, because of the property implied in (7), whatever the quantities c_m are, no current exists in $\rho > a$. Therefore, the current suppression of our minimization problem should not take place in the whole region of $x - y$ plane excluding $\mathcal{D}^a_i$, and only $\mathcal{D}^a_o$ is enough. In order to write (17) in a matrix form, we can exploit various numerical integration strategies, i.e. Monte-Carlo,..., quadrature, as $\iint_{\mathcal{D}} P(\mu) d\mu = \sum_i w_i P(\mu_i)$, where quantities μ_i are samples in the region $\mathcal{D}$ and w_i their corresponding weight. Therefore by choosing N_1 sample points in $\mathcal{D}^p_i$, N_2 in $\mathcal{D}^p_o$ and N_3 in $\mathcal{D}^a_o$, one may rewrite (17) as $\mathbf{J}(\mathbf{c}) = \mathbf{c}^H \mathbf{Q}^H \mathbf{w} \mathbf{Q} \mathbf{c}$ where ^{H} denotes the conjugate transpose operator, $\mathbf{c}$ is a vertical vector containing the unknowns c_m, $\mathbf{Q}$ is a matrix consisting four blocks of matrices as the following format

$$\mathbf{Q} = \begin{bmatrix} \begin{bmatrix} \frac{\partial}{\partial u}\widehat{\boldsymbol{f}}_1 & \frac{\partial}{\partial u}\widehat{\boldsymbol{f}}_2 & \cdots & \frac{\partial}{\partial u}\widehat{\boldsymbol{f}}_M \end{bmatrix}_{N_1 \times M} \\ \begin{bmatrix} \frac{\partial}{\partial v}\widehat{\boldsymbol{f}}_1 & \frac{\partial}{\partial v}\widehat{\boldsymbol{f}}_2 & \cdots & \frac{\partial}{\partial v}\widehat{\boldsymbol{f}}_M \end{bmatrix}_{N_1 \times M} \\ \sqrt{\lambda_1} \begin{bmatrix} \widehat{\boldsymbol{f}}_1 & \widehat{\boldsymbol{f}}_2 & \ldots & \widehat{\boldsymbol{f}}_M \end{bmatrix}_{N_2 \times M} \\ \sqrt{\lambda_2} \begin{bmatrix} \widehat{\boldsymbol{k}}_1 & \widehat{\boldsymbol{k}}_2 & \ldots & \widehat{\boldsymbol{k}}_M \end{bmatrix}_{N_3 \times M} \end{bmatrix} \tag{18}$$



(a) (b) (c)

Fig. 2. (a) The lattice of array elements (b) The desired coverage region (c) Synthesis results

and $\mathbf{w}$ a diagonal matrix containing corresponding integration weights. It is obvious that $\mathbf{c} = \mathbf{o}$ is the trivial minimizer for $\mathbf{J}$ and to avoid that we can assume $\mathbf{c}^{\mathbf{H}}\mathbf{c} = 1$, which converts the problem to a Rayleigh quotient problem, where basically optimal weights can be found by determining the eigenvector corresponding to the smallest nonzero eigenvalue of $\mathbf{Q}^H\mathbf{w}\mathbf{Q}$.

3. SIMULATION RESULTS

To show the performance of the method we assume array elements with $\lambda/2$ spacing placed on an aperture shown in Fig. 2.a (the house shaped region). There are 2545 array elements shown inside $\mathcal{D}_i^a$. The surrounding circle has a radius of $a = 17\lambda$. Our aim is to obtain a detailed fractal coverage zone as shown in Fig. 2.b. We choose $r'_{in} = 0.12$, $r'_{ext} = 0.55$ as $r_{in} = 0.17$ and $r_{ext} = 0.47$. By applying finite aperture considerations, the number of efficient functions in (16) comes out to be $M = 788$. The integration scheme we used was the simplest scheme of Monte-Carlo. Integrating samples are taken from regular grids in each integrating domain with $N_1 = 605$, $N_2 = 1110$ and $N_3 = 1491$. The places of sample points are shown in all regions by dots. The regularization values are taken to be $\lambda_1 = 2$ and $\lambda_2 = 10^3$. After applying the method and sampling the continuous current of aperture at element positions, a pattern shown in Fig. 2.c is resulted which demonstrates to be very close to the desired pattern. The ripple in the main beam is less than 0.6 dB and all the sidelobes in the unshaped region are below -25.3 dB. It is evident that if this problem was going to be solved by writing a general pattern form for the elements, there were 2545 unknowns to be considered in the minimization, while using this method reduced the number of unknowns to $M = 788$.

4. CONCLUSION

A synthesis technique to obtain optimally flat-top patterns with arbitrary footprint is presented. Apposing to the most already available methods that require the elements to be on circular or rectangular lattice, our method has the capability of accepting arbitrary geometry as the lattice structure. The simulation results depicted in this paper show that using this method and finding the most impressive basis functions, not only reduces the size of integration regions in minimization, but also significantly reduces the number of unknowns.

5. REFERENCES

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