Array Interpolation Based on Local Polynomial Approximation With Application to DOA Estimation Using Weighted MUSIC

M. Viberg* and A. Lundgren†
*Department of Signals and Systems, Chalmers University of Technology, Göteborg, Sweden
†Volvo Technology AB, Göteborg, Sweden

***Abstract*—The problem of Direction-of-Arrival (DOA) estimation using an array of sensors has received much attention for more than 3 decades. This is due to a rich interest from application areas such as radar, sonar and wireless communication channel characterization. However, high resolution DOA estimation requires an accurate model of the array response. This is usually achieved by measuring the response using sources at known positions (calibration). This paper considers interpolation of the calibration measurements using knowledge of a nominal response model. Standard linear interpolation is compared to an approach based on Local Polynomial Approximation (LPA). We also derive a weighted MUSIC estimator, which is applied using error estimates from the interpolation. Both LPA interpolation and weighted MUSIC are found to improve the performance, but not uniformly in all scenarios.**

***Index Terms*—Array signal processing, array calibration, doa estimation.**

I. Introduction

The general area of Sensor Array Signal Processing has received a tremendous attention during the last several decades. A large number of methods for Direction-of-Arrival (DOA) estimation and beamforming have been presented in the literature, see e.g. [1], [2]. In this paper we are primarily concerned with high-resolution DOA estimation using the MUSIC algorithm [3]. It is well-known that accurate DOA estimation requires a good mathematical model of the array response [4]. In practice, the array is usually subject to great uncertainty, for example due to perturbations in sensor positions, channel errors and mutual coupling. If a parameterized model of the uncertainty is available, one can apply so-called auto-calibration techniques. These estimate the array parameters and the DOAs simultaneously, see e.g. [5]. In general, this is an ill-conditioned problem, and further the model of the uncertainty may itself be incorrect. A possible remedy is to measure the response using sources at known positions. However, this can only be done at a limited set of DOAs, and the question arises of how to exploit these calibration data in the best way. If a parametric model is available, one can of course estimate the array parameters from the calibration data. The case we consider herein is that a nominal model of the array response is available, which is known to be subject to errors. Following [6], [7], the nominal model is used to aid the interpolation of the calibration data. In contrast to [7], where a local regression is performed, we apply the more general method of Local Polynomial Approximation [8] for the interpolation. In addition, we provide a new derivation of a Weighted MUSIC (W-MUSIC) estimator, which is a special case of the more general approach of [4]. The W-MUSIC method takes the errors in the individual sensor models into account, potentially leading to a reduced sensitivity to these errors. Finally, it is shown how error estimates from the interpolation schemes can be used to determine the weights in W-MUSIC. Simulation examples show that this can indeed lead to improved performance when the errors are not uniformly distributed over the different sensors. We also find that the LPA can lead to a significantly reduced mean-square error, but for more complicated functions the selection of the support in the local model is a limiting factor that needs further attention.

This work was supported in part by the Swedish Foundation for Strategic Research (SSF), within the Strategic Research Center Charmant.

II. Problem Formulation

This section presents the mathematical framework for the proposed methodology, as well as the underlying assumptions.

A. Direction-of-Arrival Estimation

Assume an array of m sensors receives a superposition of d narrowband signals from distant sources. The complex baseband representation of the array output is modeled by

$$\mathbf{x}(t) = \sum_{k=1}^{d} \mathbf{a}(\theta_k) s_k(t) + \mathbf{n}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}(t) + \mathbf{n}(t)\,, \qquad (1)$$

where $\mathbf{x}(t)$ is the m-vector of sensor outputs, $s_k(t)$ are the signal waveforms, $\mathbf{a}(\theta)$ is the array response to a signal from the DOA θ, and $\mathbf{n}(t)$ represents additive noise. The array output is sampled at N time instants, resulting in the data $\mathbf{x}(t),\ t = 1, 2, \ldots, N$. Based on these data, the problem is to estimate the DOAs $\theta_k,\ k = 1, \ldots, d$. The number of signals d is assumed to be known. Since no structural assumptions of the signal waveforms or the noise is done, the inference is usually based on the array sample covariance matrix:

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{t=1}^{N} \mathbf{x}(t)\mathbf{x}^*(t)\,, \qquad (2)$$

where $(\cdot)^*$ denotes complex conjugate transpose. The main concern in this paper is modeling errors, and therefore the infinite data case will be considered. We assume that $\mathbf{s}(t)$ and $\mathbf{n}(t)$ are independent stationary random processes with bounded moments of sufficient order. The noise is assumed to be zero-mean and spatially white, so that $\mathrm{E}[\mathbf{n}(t)\mathbf{n}^*(t)] = \sigma^2\mathbf{I}$. Under the stated assumptions, it holds that

$$\lim_{N\to\infty} \hat{\mathbf{R}} = \mathbf{R} = \mathrm{E}[\mathbf{x}(t)\mathbf{x}^*(t)] = \mathbf{APA}^* + \sigma^2\mathbf{I}\,, \qquad (3)$$

where $\mathbf{P} = \mathrm{E}[\mathbf{s}(t)\mathbf{s}^*(t)]$ is the signal covariance matrix. It is further assumed that $\mathbf{P}$ is positive definite. This property is necessary for the MUSIC algorithm, which exploits geometric properties of the array covariance matrix $\mathbf{R}$.

B. Array Models and Calibration

High-resolution DOA estimation requires that the functional form of $\mathbf{a}(\theta)$ is accurately known. This is often termed the *array manifold*. For an ideal Uniform Linear Array (ULA), the array manifold is given by

$$\mathbf{a}(\theta) = [1, e^{j\kappa\Delta\sin\theta}, \ldots, e^{j(m-1)\kappa\Delta\sin\theta}]^T\,, \qquad (4)$$

where $\kappa = \omega/c$ is the wavenumber, ω is the carrier frequency, c is the speed of propagation, Δ is the inter-element separation, and the DOA θ is defined relative the array broadside. In practice, the array response can never be perfectly modeled by (4) due to various imperfections. Firstly, all practical sensor arrays are subject to mutual coupling, which affects the array response in a way which is often difficult to predict accurately. Secondly, there is always manufacturing tolerances, resulting in sensor position errors, channel errors (gain and phase), etc. This leads inevitably to that any explicit model for the array manifold is subject to errors. It will be assumed here that a *nominal model* $\mathbf{a}_0(\theta)$ is available, which is a reasonable approximation of the true response $\mathbf{a}(\theta)$. To improve on the performance which is achieved using the nominal model, a set of *calibration data* is collected, using sources at known positions. This results in calibration vectors $\hat{\mathbf{a}}_l,\ l = 1, \ldots, L$, that are (good) estimates of the true array response vectors at the calibration DOAs $\theta_l,\ l = 1, \ldots, L$. More precisely we assume that $\mathrm{E}[\hat{\mathbf{a}}_l] = \mathbf{a}(\theta_l)$ (coherent calibration) and that $\mathrm{E}[(\hat{\mathbf{a}}_l - \mathbf{a}(\theta_l))(\hat{\mathbf{a}}_l - \mathbf{a}(\theta_l))^*] = \mathbf{C}_a$ and $\mathrm{E}[(\hat{\mathbf{a}}_l - \mathbf{a}(\theta_l))(\hat{\mathbf{a}}_l - \mathbf{a}(\theta_l))^T] = 0$, where $\mathbf{C}_a$ is known.

The DOA estimation problem using calibration data is now formulated as follows: given data $\hat{\mathbf{R}}$ ($\hat{\mathbf{R}} = \mathbf{R}$ in the infinite data case), a nominal model $\mathbf{a}_0(\theta)$ and calibration vectors $\{\hat{\mathbf{a}}_l\}_{l=1}^L$, estimate the DOAs $\theta_k,\ k = 1, \ldots, d$. It is of course interesting to make the most efficient use of the calibration data, so that the calibration grid $\{\theta_l\}_{l=1}^L$ can be kept as sparse as possible. This is particularly important in the multi-parameter case, where the calibration is done over a vector-valued parameter θ, which can include azimuth, elevation, frequency as well as polarization parameters.

III. Interpolation and Local Polynomial Approximation

Since we are given calibration data $\{\hat{\mathbf{a}}_l\}_{l=1}^L$, a very natural idea is to interpolate these to obtain the array response at some given direction θ. In general, this approach does not result in satisfactory DOA estimates, since the dependency of $\mathbf{a}(\theta)$ on θ is not smooth. The remedy is to exploit the nominal model $\mathbf{a}_0(\theta)$ in some way. The most direct approach is to apply a correction to the nominal model, i.e.,

$$\mathbf{a}(\theta) = \mathbf{Q}\mathbf{a}_0(\theta)\,, \qquad (5)$$

where $\mathbf{Q}$ is the correction matrix. The correction can be *global*, implying that the same $\mathbf{Q}$ is used for all θs, or it can be *local*, which means that $\mathbf{Q}$ is a function of θ. Global calibration, using either a diagonal or a full matrix $\mathbf{Q}$ is discussed e.g. in [9]. This calibration is efficient for direction-independent errors, such as channel errors and mutual coupling. However, in the presence of θ-dependent errors, also $\mathbf{Q}$ needs to be a function of θ. It is easy to see that it is enough to consider a diagonal matrix $\mathbf{Q} = \mathrm{diag}(\mathbf{q})$, where $\mathbf{q} = \mathbf{q}(\theta)$ depends on the DOA. We call this *local array interpolation*, and this is the case considered herein. Since $\mathbf{q}(\theta)$ is potentially a much smoother function of θ than is $\mathbf{a}(\theta)$, it is natural to try a simple interpolation scheme, such as linear interpolation. The interpolation is usually applied to the m components of $\mathbf{q}(\theta)$ independently, and also treating the real and imaginary parts separately. It is of course possible to interpolate gain and phase instead, but this gives essentially the same results. In linear interpolation, only the nearest neighbors are used to calculated a sought value. This is optimal if the calibration data are perfect, and the sought function is rapidly varying. However, when the calibration measurements are noisy and the true function $q_k(\theta)$ is smooth, it is beneficial to exploit several neighboring data points in the interpolation. In [7], a local modeling approach is proposed, where the sought value $q_k(\theta)$ (real or imaginary part) is determined as a linear combination of the calibration data $\{q_k(\theta_l)\}_{l=1}^L$. The terms are weighted according to their distance $|\theta_l - \theta|$ to the DOA in question. Here we propose a more general approach, based on Local Polynomial Approximation [8]. The calibration data are modeled as linear combinations of a set of basis functions $\{\phi_p(x)\}_{p=0}^P$, usually taken as monomials $\phi_p(x) = x^p$. The model is expressed as

$$q_k(\theta_l) = \sum_{p=0}^{P} \alpha_p \phi_p(\theta - \theta_l)\,, \qquad (6)$$

where, with some abuse of notation, we assume let $q_k(\cdot)$ represent either the real or the imaginary part. The coefficients α_p are determined by solving a Weighted Least-Squares (WLS) problem:

$$\hat{\boldsymbol{\alpha}} = \arg\min_{\boldsymbol{\alpha}} \sum_{l=1}^{L} w(|\theta_l - \theta|) \left[\hat{q}_k(\theta_l) - \sum_{p=0}^{P} \alpha_p \phi_p(\theta - \theta_l)\right]^2, \qquad (7)$$

where the $\hat{q}_k(\theta_l) = \hat{a}_{l,k}/a_{0,k}(\theta_l)$ are computed from the calibration data. Any weighting function can be used, but in our examples we used a Gaussian window, $w(x) = (1/\sqrt{2\pi h^2})\exp(-0.5x^2/h^2)$, where the "bandwidth" h controls the amount of smoothing. If polynomial basis functions are used, the first coefficient $\hat{\alpha}_0 = \hat{q}_k(\theta)$ is the sought interpolator of $q_k(\theta)$. If desired, the second coefficient $\hat{\alpha}_1$ gives an estimate of the derivative $q'_k(\theta)$.

IV. A WEIGHTED MUSIC ESTIMATOR

Of the various suboptimal approaches to DOA estimation, the MUSIC algorithm has gained much attention due to its simplicity and elegance. It is based on geometric properties of the array covariance matrix $\mathbf{R} = \mathbf{APA}^* + \sigma^2\mathbf{I}$. Let

$$\mathbf{R} = \sum_{k=1}^{m} \lambda_k \mathbf{e}_k \mathbf{e}_k^* = \mathbf{E}_s \mathbf{\Lambda}_s \mathbf{E}_s^* + \mathbf{E}_n \mathbf{\Lambda}_n \mathbf{E}_n^* \tag{8}$$

be the eigendecomposition of $\mathbf{R}$, where the signal subspace matrix $\mathbf{E}_s = [\mathbf{e}_1, \ldots, \mathbf{e}_d]$ contains the d principal eigenvectors. Then $\lambda_1 \geq \cdots \geq \lambda_d > \lambda_{d+1} = \cdots = \lambda_m = \sigma^2$, and $\mathbf{E}_n^*\mathbf{A} = 0$. This last relation is exploited by the MUSIC algorithm. An alternative derivation of the method is to express it as an *Inverse Subspace Fitting* (ISF) problem (see also [10]):

$$P(\theta) = \min_{\mathbf{t}} \|\mathbf{a}(\theta) - \mathbf{E}_s\mathbf{t}\|^2, \tag{9}$$

where $\mathbf{t}$ is a $d \times 1$ vector and $P(\theta)$ is the inverse MUSIC spectrum (or "null spectrum"), which is obtained as

$$P(\theta) = \mathbf{a}^*(\theta)\mathbf{a}(\theta) - \mathbf{a}^*(\theta)\mathbf{E}_s{\mathbf{E}_s}^*\mathbf{a}(\theta) = \mathbf{a}^*(\theta)\mathbf{E}_n\mathbf{E}_n^*\mathbf{a}(\theta)\,. \tag{10}$$

The MUSIC DOA estimates $\hat{\theta}_k$ are the arguments of the d smallest minima of $P(\theta)$. The name inverse subspace fitting stems from the relation to the subspace fitting formulation of [11]. If $\|\mathbf{a}(\theta)\|$ depends on θ, which is likely for perturbed arrays, it is beneficial to normalize (10) by $\|\mathbf{a}(\theta)\|^2$.

Assume now that $\mathbf{E}_s$ is known perfectly (infinite sample case), whereas only an estimate $\hat{\mathbf{a}}(\theta)$ of the array manifold is available. It is then natural to modify the ISF to include a weighting:

$$P_w(\theta) = \min_{\mathbf{t}} (\hat{\mathbf{a}}(\theta) - \mathbf{E}_s\mathbf{t})^*\mathbf{W}(\hat{\mathbf{a}}(\theta) - \mathbf{E}_s\mathbf{t})\,, \tag{11}$$

where $\mathbf{W} = \mathbf{W}^* > 0$ is the weighting matrix. From the general theory of WLS estimators, the weighting matrix can be chosen to minimize the estimation error variance of the resulting DOA estimates. If $\mathrm{E}[(\hat{\mathbf{a}}(\theta) - \mathbf{a}(\theta))(\hat{\mathbf{a}}(\theta) - \mathbf{a}(\theta))^*] = \mathbf{C}_a$ and $\mathrm{E}[(\hat{\mathbf{a}}(\theta) - \mathbf{a}(\theta))(\hat{\mathbf{a}}(\theta) - \mathbf{a}(\theta))^T] = 0$, then $\mathbf{W} = \mathbf{C}_a^{-1}$ results in minimum variance estimates to first order in $\|\mathbf{C}_a\|$. It is straightforward to extend the result to the case where circular symmetry does not hold, but the details are omitted here. Solving (11) w.r.t. $\mathbf{t}$, the inverse weighted MUSIC spectrum is expressed as

$$P_w(\theta) = \mathbf{a}^*(\theta)\mathbf{W}\mathbf{a}(\theta) - \mathbf{a}^*(\theta)\mathbf{W}\mathbf{E}_s({\mathbf{E}_s}^*\mathbf{W}\mathbf{E}_s)^{-1}{\mathbf{E}_s}^*\mathbf{W}\mathbf{a}(\theta)\,. \tag{12}$$

It is noted that the above can be interpreted as a special case of the weighted MUSIC version proposed in [4], although our derivation is much simpler and more direct.

V. SIMULATIONS

In this section we present some simulation results to compare the performance of the different interpolation schemes. It will also be shown how interpolation with error estimation can be used together with the weighted MUSIC estimator.

In the simulations we consider the following scenario: The nominal array is an $m = 10$ element ULA, with half wavelength element separation. Two signals arrive from DOAs θ_1 and θ_2, where θ_2 is fixed at 13° relative array broadside, whereas θ_1 is varied from 8° to 12°. The signal waveforms are i.i.d., so the signal covariance matrix is proportional to the identity. The estimates are formed from the exact array covariance matrix $\mathbf{R}$, thus the presence of noise is irrelevant.

The sensor positions are perturbed by i.i.d. Gaussian random variables in two dimensions, each with zero mean and standard deviation $\lambda/20$. In addition, each array response vector is premultiplied by a random matrix $\mathbf{G}$, which is either diagonal (channel errors) or a full matrix (mutual coupling). To correct for these errors, calibration data are collected on a uniform grid within the interval $\theta_l \in \{-40^\circ, 40^\circ\}$, where the calibration grid is varied. Each calibration measurement $\hat{\mathbf{a}}_l$ is perturbed by a $\mathcal{N}(0, 0.01^2\mathbf{I})$ random vector, i.e. $\mathbf{C}_a = 10^{-4}\mathbf{I}$.

In each experiment we apply the MUSIC or Weighted MUSIC algorithm, using different interpolation schemes. For each DOA separation, 400 independent perturbed arrays are generated and the corresponding $\mathbf{R} = \mathbf{APA}^* + \sigma^2\mathbf{I}$ are calculated. Statistics for the different methods are then generated and compared. The following interpolation schemes are compared:

- Linear interpolation of $\mathrm{Re}(\hat{q}_k(\theta))$ and $\mathrm{Im}(\hat{q}_k(\theta))$.
- LPA-based interpolation with $P = 1$ (locally linear model). A Gaussian window is used, where the window parameter h is optimized using Cross Validation (CV) as follows: first, an estimation data set is formed from calibration points $\{\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_3, \ldots, \hat{\mathbf{a}}_{L-1}\}$ (assuming L even), and the remaining calibration vectors constitute the validation set. The validation points are then interpolated using the estimation data, and the MSE is computed. The roles of the estimation data and the validation data are then reversed, and the procedure is repeated. The h that minimizes the average MSE is found using Matlab's `fminbnd` routine.

Since the errors are relatively large, the MUSIC method (or any other DOA estimator) performs very poorly without calibration, and it is therefore not included in the comparison.

The Weighted MUSIC estimator is implemented as follows: The average (over θ) variance of the estimated $\mathrm{Re}(\hat{q}_k(\theta))$ and $\mathrm{Im}(\hat{q}_k(\theta))$ are first estimated using CV. Any correlation between the real and imaginary parts, or between the different $\hat{q}_k$s is ignored. The resulting covariance matrix of the interpolated $\hat{\mathbf{a}}(\theta)$, thus assumed diagonal, is determined and added to the error covariance $\mathbf{C}_a$ of the calibration vectors to give the total covariance matrix of the estimated array model. The inverse covariance is then used in the Weighted MUSIC method

In the first example we use channel errors only, so $\mathbf{G} = \mathbf{I} + \mathrm{diag}(\tilde{\mathbf{g}})$, where the elements of $\tilde{\mathbf{g}}$ are $\mathcal{N}(0, 0.1^2)$. The calibration grid is chosen as 5°. The empirical RMS errors for the source at $\theta = 13^\circ$ are shown in Figure 1. It can be seen that the LPA interpolation outperforms the standard linear interpolator in this scenario, where only diagonal θ-dependent errors are used. The weighting has an insignificant effect on

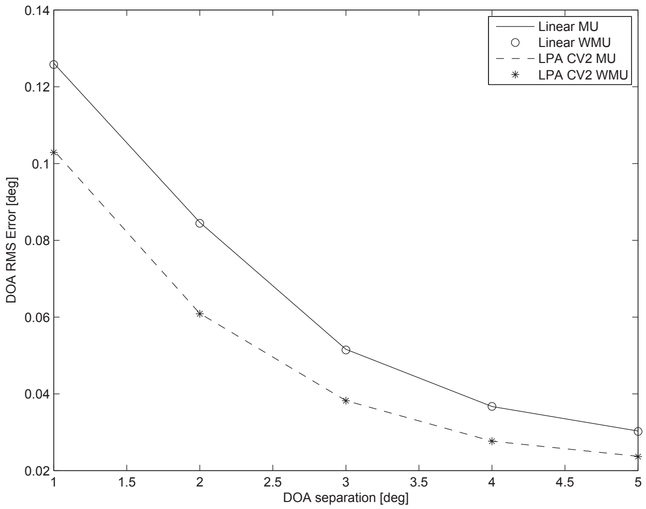

Fig. 1. Empirical RMS errors for the MUSIC and W-MUSIC algorithms using different interpolation schemes versus the DOA separation. Channel and position errors only.

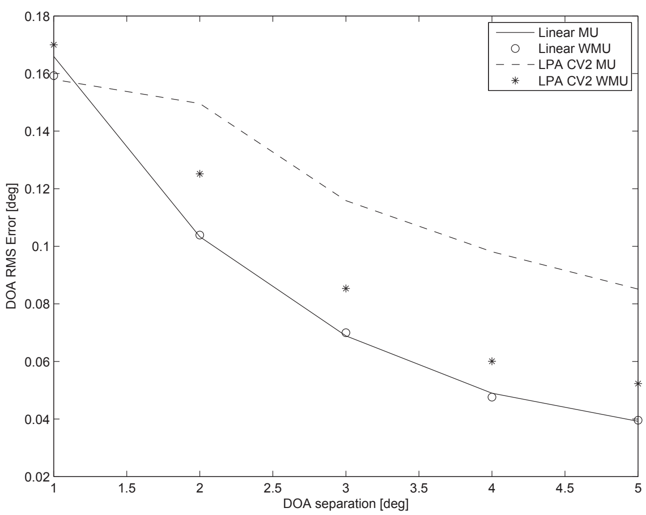

Fig. 2. Empirical RMS errors for the MUSIC and W-MUSIC algorithms using different interpolation schemes versus the DOA separation. Mutual coupling and position errors.

the estimates in this case, due to the uniform nature of the errors.

In the second experiment, all elements of $\mathbf{G}$ are perturbed, which resembles the case of an incorrectly specified mutual coupling model. Here, $\mathbf{G} = \mathbf{I} + \tilde{\mathbf{G}}$, where the elements of $\tilde{\mathbf{G}}$ are zero-mean i.i.d. $\mathcal{N}(0, 0.1^2/m)$. Since this is a more difficult case, the calibration grid is reduced to $2.5°$. The different methods are applied as above, and the results for the source at $13°$ are shown in Figure 2. In this case we see that the simple linear interpolation scheme performs the best, but its performance is still not improved by using W-MUSIC. The superiority of linear interpolation is due to the complicated nature of the errors, which means that the $q_k(\theta)$s are generally rapidly varying. The truly optimal h in the LPA approach is then very small, leading essentially to linear interpolation. However, when h is determined from data, and a fixed value is used for the whole field-of-view, larger values are sometimes selected leading to worse interpolation performance. Using weighted MUSIC reduces the influence of these poor interpolations, and the resulting performance is then close to that of the "optimal" linear interpolation.

VI. Conclusion

We have presented a new method for array calibration based on Local Polynomial Approximation (LPA). The method exploits knowledge of a nominal model as well as calibration data using sources at known positions. The interpolation yields a correction function for each sensor separately, which is then used to update the nominal model. We have also presented a simplified derivation of a Weighted MUSIC (W-MUSIC) estimator, which can lead to improved performance when the perturbations on the different sensors are imbalanced. It was shown how the weighting can be determined from calibration data. The different methods were tested on simulated data, and both LPA and W-MUSIC showed a potential for improving the DOA estimation performance. However, for LPA it is crucial to determine the optimal bandwidth of the window used in the local model. A simple scheme based on cross-validation is proposed here, but it does not always lead to improved performance compared to standard linear interpolation. In our future work we plan to investigate alternative schemes for bandwidth selection, including locally adapted (θ-varying) methods. Another interesting option is to use a non-diagonal correction matrix $\mathbf{Q}(\theta)$, in order to achieve a smoother θ-dependence.

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