DOES CANONICAL CORRELATION ANALYSIS PROVIDE RELIABLE INFORMATION ON DATA CORRELATION IN ARRAY PROCESSING?

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ABSTRACT

This work provides analytical results on the canonical correlation analysis (CCA) of data sets from two spatially separated arrays of sensors. Our case studies cover both single source and multiple source signals in either white or colored noise fields for array signal processing. We derive analytical expressions of the canonical correlation for these examples and present a computer simulation analysis of empirical canonical correlations as a function of nominal correlation, signal-to-noise ratio (SNR), and sample support. Results obtained reveal an interesting fact that the canonical coefficients from CCA provide reliable information on the *spatial correlation* existing among data sets from two arrays only when the SNRs at *both arrays* are reasonably high. When sample correlation matrices (SCM) are used in the empirical CCA, reliable correlation can be estimated from CCA asymptotically (either at high SNRs from both arrays, or with a large number of snapshots in comparison with array dimensionality).

Index Terms— canonical correlation analysis (CCA), array signal processing,

1. INTRODUCTION

Canonical correlation analysis (CCA) [1, 2] is a standard statistical tool for analyzing and exploring the correlations among two multivariate data measurements. Our particular interest in studying the CCA for array applications is driven by the need to determine the signal spatial coherence across two spatially separated arrays (with inherent non-stationarity) and subsequent coherent signal processing for source localization and beamforming. Therefore, we put special emphasis on the impact of data quality, in terms of SNR, on the correlation revealing nature of the CCA. The non-stationary nature of our data motivates us to study the reliability aspect of the empirical CCA in comparison with the true statistic-based CCA when only a finite amount of data is available.

The CCA, which measures the cosines of principal angles between two random vectors, has been successfully applied for analyzing second-order filtering and communication systems over the Gaussian channel [3, 4]. They multiplicatively decompose concentration ellipses for second-order filtering and additively decompose information rate for the Gaussian channel. For array applications, the asymptotical property of CCA has been exploited by [5, 6] to design effective solutions to the estimation of sources' directions of arrival, to the determination of model order, and to the performance improvement of existing subspace DOA estimation methods. In this work, we focus on the application of CCA in array signal processing with emphasis on the impact of sources' SNRs at different receiving arrays. From several simplified but properly formulated nominal examples, we derive the analytical expressions for the canonical coefficients of one and two signal sources embedded in white and colored noise, and analyze the performance of empirical canonical correlations as a function of nominal correlation, SNR, and sample support.

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2. SINGLE SOURCE TWO ARRAYS

Given two sets of data vectors in the forms of,

$$\begin{array}{l}\mathbf{x}(t) = \mathbf{s}_x\alpha_x(t) + \mathbf{n}_x(t)\\ \mathbf{y}(t) = \mathbf{s}_y\alpha_y(t) + \mathbf{n}_y(t)\end{array},\quad t = 1, 2, \ldots, M. \tag{1}$$

At a fixed time index, the data vectors $\mathbf{x}(t)$ and $\mathbf{y}(t)$, of dimensions $N_x \times 1$ and $N_y \times 1$ respectively, can be treated as snapshots from two spatially separated arrays. We further assume that they are zero-mean complex Gaussian distributed with auto/cross-correlation matrices,

$$\begin{aligned}\mathbf{R}_{xx} &= E\{\mathbf{x}(t)\mathbf{x}^H(t)\} = \sigma_x^2\mathbf{s}_x\mathbf{s}_x^H + \mathbf{R}_{n_xn_x}\\ \mathbf{R}_{yy} &= E\{\mathbf{y}(t)\mathbf{y}^H(t)\} = \sigma_y^2\mathbf{s}_y\mathbf{s}_y^H + \mathbf{R}_{n_yn_y}\\ \mathbf{R}_{xy} &= E\{\mathbf{x}(t)\mathbf{y}^H(t)\} = \rho\sigma_x\sigma_y\mathbf{s}_x\mathbf{s}_y^H + \mathbf{R}_{n_xn_y}.\end{aligned}$$

Here, parameter $\rho = E\{\alpha_x(t)\alpha_y^*(t)\}/\sigma_x\sigma_y$ denotes the spatial correlation coefficient between two Gaussian processes $\alpha_x(t)$ and $\alpha_y(t)$, with zero means and prescribed correlation. When two arrays' spatial separation distance is *greater* than the spatial *coherence length* of the noise field, we can safely assume $\mathbf{R}_{n_xn_y} = \mathbf{0}$. The augmented Gaussian vector

$\mathbf{z}(t) = [\mathbf{x}^T(t)\ \mathbf{y}^T(t)]^T$ has a correlation matrix,

$$\mathbf{R}_{zz} = \begin{bmatrix} \mathbf{R}_{xx} & \mathbf{R}_{xy} \\ \mathbf{R}_{yx} & \mathbf{R}_{yy} \end{bmatrix} = \mathbf{R}_{ss} + \mathbf{R}_{nn},$$

with $\mathbf{R}_{ss} = \begin{bmatrix} \mathbf{s}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_y \end{bmatrix} \cdot \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho^*\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{s}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{s}_y \end{bmatrix}^H$, and $\mathbf{R}_{nn} = \begin{bmatrix} \mathbf{R}_{n_x n_x} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{n_y n_y} \end{bmatrix}$.

Using a square-root decomposition of a Hermitian matrix, $\mathbf{R}_{xx} = \mathbf{R}_{xx}^{1/2}\mathbf{R}_{xx}^{H/2}$ and $\mathbf{R}_{xx}^{-1} = \mathbf{R}_{xx}^{-H/2}\mathbf{R}_{xx}^{-1/2}$, we can find the analytical expressions for the canonical correlation coefficients, given the data model in eq. (1). In principle, the CCA involves two stages of data transformation. *The first stage* accomplishes a task of whitening each vector-component of the augmented vector $\mathbf{z}(t)$. That is, the block-whitened data $\boldsymbol{\zeta}(t)$ defined by,

$$\boldsymbol{\zeta}(t) \triangleq \begin{bmatrix} \boldsymbol{\mu}(t) \\ \boldsymbol{\nu}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{R}_{xx}^{-1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{yy}^{-1/2} \end{bmatrix}}_{\text{a block-whitening operator}} \mathbf{z}(t),$$

has correlation matrix of the following form,

$$\mathbf{R}_{\zeta\zeta} = E\{\boldsymbol{\zeta}(t)\boldsymbol{\zeta}^H(t)\} = \begin{bmatrix} \mathbf{I} & \mathbf{C} \\ \mathbf{C}^H & \mathbf{I} \end{bmatrix}.$$

Here $\mathbf{C} = \mathbf{R}_{xx}^{-1/2}\mathbf{R}_{xy}\mathbf{R}_{yy}^{-H/2}$ is the *coherence matrix*. *The second stage* in the CCA diagonalizes the coherence matrix $\mathbf{C}$ via orthogonal matrix transformation, or equivalently, using a transformation to produce decoupled canonical coordinates. When the background noise is white, we can derive the eigenstructures for the matrices $\mathbf{R}_{xx}$ and $\mathbf{R}_{yy}$. Specifically,

$$\mathbf{R}_{xx} = \sigma_x^2\mathbf{s}_x\mathbf{s}_x^H + \sigma_{n_x}^2\mathbf{I} = \sum_{i=1}^{N_x}\lambda_{x,i}\mathbf{v}_{x,i}\mathbf{v}_{x,i}^H,$$

with its eigen-values and eigen-vectors given by,

the principal pair: $\lambda_{x,\max} \triangleq \max\{\lambda_{x,i}\} = \sigma_x^2\|\mathbf{s}_x\|^2 + \sigma_{n_x}^2$

$$\mathbf{v}_{x,max} \triangleq \mathbf{v}_{x,1} = \mathbf{s}_x/\|\mathbf{s}_x\|,$$

the minor eigenpairs: $\lambda_{x,i} = \sigma_{n_x}^2$ *(repeated eigenvalues)*

$$\mathbf{v}_{x,i} \perp \mathbf{s}_x,\ i = 2, 3, \ldots, N_x$$

Hence, the spectral decomposition of $\mathbf{R}_{xx}^{-1/2}$ becomes,

$$\mathbf{R}_{xx}^{-1/2} = \frac{1}{\sqrt{\sigma_x^2\|\mathbf{s}_x\|^2 + \sigma_{n_x}^2}}\mathbf{P}_{s_x} + \frac{1}{\sigma_{n_x}}\mathbf{P}_{s_x}^{\perp}. \quad (2)$$

Here the matrices $\mathbf{P}_{s_x} = \mathbf{s}_x\mathbf{s}_x^H/\|\mathbf{s}_x\|^2$ and $\mathbf{P}_{s_x}^{\perp} = \mathbf{I} - \mathbf{P}_{s_x}$ are orthogonal projection operators onto the signal subspace $\langle\mathbf{s}_x\rangle$ and its orthogonal complement. Similarly, we have,

$$\mathbf{R}_{yy}^{-1/2} = \frac{1}{\sqrt{\sigma_y^2\|\mathbf{s}_y\|^2 + \sigma_{n_y}^2}}\mathbf{P}_{s_y} + \frac{1}{\sigma_{n_y}}\mathbf{P}_{s_y}^{\perp}. \quad (3)$$

Consequently, the analytical expression for the canonical coefficients can be derived from the singular value decomposition (SVD) of the coherence matrix $\mathbf{C}$, i.e.,

$$\mathbf{C} = \mathbf{R}_{xx}^{-1/2}\mathbf{R}_{xy}\mathbf{R}_{yy}^{-H/2} = \frac{\rho\sigma_x\sigma_y}{\sqrt{\sigma_x^2\|\mathbf{s}_x\|^2 + \sigma_{n_x}^2}\sqrt{\sigma_y^2\|\mathbf{s}_y\|^2 + \sigma_{n_y}^2}}\mathbf{s}_x\mathbf{s}_y^H. \quad (4)$$

This indicates that the only non-zero canonical correlation coefficient is given by the principal singular value of $\mathbf{C}$,

$$k(1) = |\rho|\frac{\sigma_x\sigma_y\|\mathbf{s}_x\|\cdot\|\mathbf{s}_y\|}{\sqrt{\sigma_x^2\|\mathbf{s}_x\|^2 + \sigma_{n_x}^2}\sqrt{\sigma_y^2\|\mathbf{s}_y\|^2 + \sigma_{n_y}^2}} = |\rho|\cdot\sqrt{\frac{\eta_x}{1+\eta_x}}\cdot\sqrt{\frac{\eta_y}{1+\eta_y}}, \quad (5)$$

with $\eta_x \triangleq \sigma_x^2\|\mathbf{s}_x\|^2/\sigma_{n_x}^2$ and $\eta_y \triangleq \sigma_y^2\|\mathbf{s}_y\|^2/\sigma_{n_y}^2$ representing the output SNR of array-x and array-y, respectively. Besides the known fact that the number of sources impinging on the arrays is indicated by the number of non-zero canonical correlation coefficients [2], our result in eq. (5) brings up the important correlation revealing property of the CCA. That is, only when reasonably high SNRs are observed *in both data sets*, i.e. $\eta_x,\ \eta_y \gg 1$, hence $k(1) \to |\rho|$, can the canonical correlation coefficients truthfully reveal the spatial coherence existing in data sets from two spatially separated arrays.

3. EXTENSION TO COLORED NOISE FIELDS

Quite often, colored noise fields are present at each array due to the compact design of arrays. In such applications, the noise correlation matrices $\mathbf{R}_{n_x}$ and $\mathbf{R}_{n_y}$ are no longer diagonal. We first introduce a preliminary transformation matrix on the augmented data vector $\mathbf{z}(t)$ as follows,

$$\boldsymbol{\zeta}(t) \triangleq \begin{bmatrix} \boldsymbol{\mu}(t) \\ \boldsymbol{\nu}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{R}_{n_x}^{-1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{n_y}^{-1/2} \end{bmatrix}}_{\text{a noise whitening operator}} \mathbf{z}(t),$$

so that the correlation matrix of data $\boldsymbol{\zeta}(t)$ becomes,

$$\mathbf{R}_{\zeta\zeta} = E\{\boldsymbol{\zeta}(t)\boldsymbol{\zeta}^H(t)\} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{12}^H & \mathbf{R}_{22} \end{bmatrix},$$

with,

$$\mathbf{R}_{11} = \mathbf{R}_{n_x}^{-1/2}\cdot\sigma_x^2\mathbf{s}_x\mathbf{s}_x^H\cdot\mathbf{R}_{n_x}^{-H/2} + \mathbf{I}$$
$$\mathbf{R}_{22} = \mathbf{R}_{n_y}^{-1/2}\cdot\sigma_y^2\mathbf{s}_y\mathbf{s}_y^H\cdot\mathbf{R}_{n_y}^{-H/2} + \mathbf{I}$$
$$\mathbf{R}_{12} = \mathbf{R}_{n_x}^{-1/2}\cdot\rho\sigma_x\sigma_y\mathbf{s}_x\mathbf{s}_y^H\cdot\mathbf{R}_{n_x}^{-H/2}.$$

We then adopt the standard block-whitening procedure in the CCA to the vector-components in data $\boldsymbol{\zeta}(t)$, resulting in the block-whitened data vector

$$\boldsymbol{\xi}(t) \triangleq \underbrace{\begin{bmatrix} \mathbf{R}_{11}^{-1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{22}^{-1/2} \end{bmatrix}}_{\text{a block-whitening operator}} \boldsymbol{\zeta}(t),$$

with $\mathbf{R}_{11}^{-1/2}$ and $\mathbf{R}_{22}^{-1/2}$ given as,

$$\begin{aligned}\mathbf{R}_{11}^{-1/2} &= \frac{1}{\sqrt{\sigma_x^2\|\mathbf{R}_{n_x}^{-1/2}\mathbf{s}_x\|^2+1}}\mathbf{P}_{\mathbf{R}_{n_x}^{-1/2}\mathbf{s}_x}+\mathbf{P}^{\perp}_{\mathbf{R}_{n_x}^{-1/2}\mathbf{s}_x}\\ \mathbf{R}_{22}^{-1/2} &= \frac{1}{\sqrt{\sigma_y^2\|\mathbf{R}_{yn_y}^{-1/2}\mathbf{s}_y\|^2+1}}\mathbf{P}_{\mathbf{R}_{n_y}^{-1/2}\mathbf{s}_y}+\mathbf{P}^{\perp}_{\mathbf{R}_{n_y}^{-1/2}\mathbf{s}_y}\end{aligned} \tag{6}$$

This results in a correlation matrix

$$\mathbf{R}_{\xi\xi} = E\{\boldsymbol{\xi}(t)\boldsymbol{\xi}^H(t)\} = \begin{bmatrix} \mathbf{I} & \mathbf{C} \\ \mathbf{C}^H & \mathbf{I}\end{bmatrix}, \tag{7}$$

with coherence matrix

$$\begin{aligned}\mathbf{C} &= \mathbf{R}_{11}^{-1/2}\mathbf{R}_{12}\mathbf{R}_{22}^{-H/2}\\ &= \rho\frac{\sigma_x\mathbf{R}_{n_x}^{-1/2}\mathbf{s}_x\cdot\sigma_y\left(\mathbf{R}_{n_y}^{-1/2}\mathbf{s}_y\right)^H}{\sqrt{\sigma_x^2\|\mathbf{R}_{n_x}^{-1/2}\mathbf{s}_x\|^2+1}\sqrt{\sigma_y^2\|\mathbf{R}_{n_y}^{-1/2}\mathbf{s}_y\|^2+1}}.\end{aligned} \tag{8}$$

The above *rank-1* coherence matrix again leads to a single non-zero canonical correlation coefficient,

$$k(1) = |\rho|\sqrt{\frac{\eta_x}{1+\eta_x}}\cdot\sqrt{\frac{\eta_y}{1+\eta_y}}, \tag{9}$$

with $\eta_x \triangleq \sigma_x^2\|\mathbf{R}_{n_x}^{-1/2}\mathbf{s}_x\|^2 = \sigma_x^2\mathbf{s}_x^H\mathbf{R}_{n_x}^{-1}\mathbf{s}_x$ and $\eta_y \triangleq \sigma_y^2\|\mathbf{R}_{n_y}^{-1/2}\mathbf{s}_y\|^2 = \sigma_y^2\mathbf{s}_y^H\mathbf{R}_{n_y}^{-1}\mathbf{s}_y$ being the output SNRs at array-x and array-y in *colored noise fields*, respectively. Again, at high SNRs, $\eta_x,\ \eta_y \gg 1$, we have $k(1) \to |\rho|$, meaning that the canonical correlation coefficient reveals the spatial coherence (normalized) among two sets of data vectors from two spatially separated arrays.

4. EXTENSION TO TWO SOURCES TWO ARRAYS

When two signal sources are present in the data sets from two arrays, the correlation matrices take the following forms,

$$\begin{aligned}\mathbf{R}_{xx} &= \sigma_{x,1}^2\mathbf{s}_{x,1}\mathbf{s}_{x,1}^H + \sigma_{x,2}^2\mathbf{s}_{x,2}\mathbf{s}_{x,2}^H + \sigma_{n_x}^2\mathbf{I}\\ \mathbf{R}_{yy} &= \sigma_{y,1}^2\mathbf{s}_{y,1}\mathbf{s}_{y,1}^H + \sigma_{y,2}^2\mathbf{s}_{y,2}\mathbf{s}_{y,2}^H + \sigma_{n_y}^2\mathbf{I}\\ \mathbf{R}_{xy} &= \rho_1\sigma_{x,1}\sigma_{y,1}\mathbf{s}_{x,1}\mathbf{s}_{y,1}^H + \rho_2\sigma_{x,2}\sigma_{y,2}\mathbf{s}_{x,2}\mathbf{s}_{y,2}^H\end{aligned}$$

Matrix $\mathbf{R}_{xx}$ can be represented in its orthogonal eigen-modes in descending order, i.e.,

$$\mathbf{R}_{xx} = \sum_{i=1}^{N_x}\lambda_{x,i}\mathbf{v}_{x,i}\mathbf{v}_{x,i}^H,$$

where the first two principal eigen modes span the signal subspace, $\text{span}(\mathbf{v}_{x,1},\mathbf{v}_{x,2}) = \text{span}(\mathbf{s}_{x,1},\mathbf{s}_{x,2})$. Under the commonly held condition: $\|\mathbf{s}_{x,1}\| = \|\mathbf{s}_{x,2}\| = \|\mathbf{s}_x\|$, two orthogonal vectors, $\mathbf{s}_{x,\Sigma} \triangleq \mathbf{s}_{x,1}+\mathbf{s}_{x,2}$ and $\mathbf{s}_{x,\Delta} \triangleq \mathbf{s}_{x,1}-\mathbf{s}_{x,2}$, can be constructed from the two signal modes. Therefore, the two principal eigen-vectors can be formed as,

$$\mathbf{v}_{x,1} = \frac{\mathbf{s}_{x,\Sigma}}{\|\mathbf{s}_{x,\Sigma}\|},\quad \mathbf{v}_{x,2} = \frac{\mathbf{s}_{x,\Delta}}{\|\mathbf{s}_{x,\Delta}\|}. \tag{10}$$

This leads to the detailed spectral decomposition,

$$\begin{aligned}\mathbf{R}_{xx} &\overset{*}{=} \frac{\sigma_x^2}{2}\mathbf{s}_{x,\Sigma}\mathbf{s}_{x,\Sigma}^H + \frac{\sigma_x^2}{2}\mathbf{s}_{x,\Delta}\mathbf{s}_{x,\Delta}^H + \sigma_{n_x}^2\mathbf{I}\\ &= \frac{\sigma_x^2\|\mathbf{s}_{x,\Sigma}\|^2}{2}\mathbf{v}_{x,1}\mathbf{v}_{x,1}^H + \frac{\sigma_x^2\|\mathbf{s}_{x,\Delta}\|^2}{2}\mathbf{v}_{x,2}\mathbf{v}_{x,2}^H + \sigma_{n_x}^2\mathbf{I},\end{aligned} \tag{11}$$

where equality $\overset{*}{=}$ holds when $\sigma_{x,1}^2 = \sigma_{x,2}^2 = \sigma_x^2$. Accordingly the eigenvalues of $\mathbf{R}_{xx}$ can be found as,

$$\lambda_{x,1} = \frac{\sigma_x^2\|\mathbf{s}_{x,\Sigma}\|^2}{2} + \sigma_{n_x}^2;\quad \lambda_{x,2} = \frac{\sigma_x^2\|\mathbf{s}_{x,\Delta}\|^2}{2} + \sigma_{n_x}^2;$$
$$\lambda_{x,i} = \sigma_{n_x}^2\ (i = 3, 4, \ldots, N_x).$$

Hence the spectral decomposition of $\mathbf{R}_{xx}^{-1/2}$ and $\mathbf{R}_{yy}^{-1/2}$ has the following forms,

$$\mathbf{R}_{xx}^{-1/2} = \left[\mathbf{v}_{x,1}\ \mathbf{v}_{x,2}\right]\cdot\begin{bmatrix}\frac{1}{\sqrt{\lambda_{x,1}}}-\frac{1}{\sigma_{n_x}} & 0\\ 0 & \frac{1}{\sqrt{\lambda_{x,2}}}-\frac{1}{\sigma_{n_x}}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{v}_{x,1}^H\\ \mathbf{v}_{x,2}^H\end{bmatrix} + \frac{1}{\sigma_{n_x}}\mathbf{I}, \tag{12}$$

$$\mathbf{R}_{yy}^{-1/2} = \left[\mathbf{v}_{y,1}\ \mathbf{v}_{y,2}\right]\cdot\begin{bmatrix}\frac{1}{\sqrt{\lambda_{y,1}}}-\frac{1}{\sigma_{n_y}} & 0\\ 0 & \frac{1}{\sqrt{\lambda_{y,2}}}-\frac{1}{\sigma_{n_y}}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{v}_{y,1}^H\\ \mathbf{v}_{y,2}^H\end{bmatrix} + \frac{1}{\sigma_{n_y}}\mathbf{I}, \tag{13}$$

$$\begin{aligned}\mathbf{R}_{xy} &= \sigma_x\sigma_y\left(\rho_1\mathbf{s}_{x,1}\mathbf{s}_{y,1}^H + \rho_2\mathbf{s}_{x,2}\mathbf{s}_{y,2}^H\right)\\ &\overset{*}{=} \frac{\sigma_x\sigma_y}{2}\left[\mathbf{s}_{x,\Sigma}\quad \mathbf{s}_{x,\Delta}\right]\cdot\begin{bmatrix}\rho & 0\\ 0 & \rho\end{bmatrix}\cdot\begin{bmatrix}\mathbf{s}_{y,\Sigma}^H\\ \mathbf{s}_{y,\Delta}^H\end{bmatrix}\\ &= \left[\mathbf{v}_{x,1}\quad \mathbf{v}_{x,2}\right]\cdot\begin{bmatrix}\psi_1 & 0\\ 0 & \psi_2\end{bmatrix}\cdot\begin{bmatrix}\mathbf{v}_{y,1}^H\\ \mathbf{v}_{y,2}^H\end{bmatrix},\end{aligned} \tag{14}$$

where the above identity $\overset{*}{=}$ holds when $\rho_1 = \rho_2 = \rho$, and,

$$\psi_1 = \frac{\sigma_x\sigma_y}{2}\rho\,\|\mathbf{s}_{x,\Sigma}\|\cdot\|\mathbf{s}_{y,\Sigma}\|,\quad \psi_2 = \frac{\sigma_x\sigma_y}{2}\rho\,\|\mathbf{s}_{x,\Delta}\|\cdot\|\mathbf{s}_{y,\Delta}\|. \tag{15}$$

Consequently, the coherence matrix $\mathbf{C}$ can be decomposed as,

$$\begin{aligned}\mathbf{C} &= \mathbf{R}_{xx}^{-1/2}\mathbf{R}_{xy}\mathbf{R}_{yy}^{-H/2}\\ &= \left[\mathbf{v}_{x,1}\ \mathbf{v}_{x,2}\right]\cdot\begin{bmatrix}\frac{\psi_1}{\sqrt{\lambda_{x,1}\lambda_{y,1}}} & 0\\ 0 & \frac{\psi_2}{\sqrt{\lambda_{x,2}\lambda_{y,2}}}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{v}_{y1}^H\\ \mathbf{v}_{y2}^H\end{bmatrix}.\end{aligned} \tag{16}$$

Therefore, the non-zero canonical correlation coefficients, $k(1)$ and $k(2)$, can be obtained as,

$$\begin{aligned}k(1) &= |\rho|\sqrt{\frac{\eta_{x,1}}{1+\eta_{x,1}}}\cdot\sqrt{\frac{\eta_{y,1}}{1+\eta_{y,1}}},\\ k(2) &= |\rho|\sqrt{\frac{\eta_{x,2}}{1+\eta_{x,2}}}\cdot\sqrt{\frac{\eta_{y,2}}{1+\eta_{y,2}}},\end{aligned} \tag{17}$$

with $\eta_{x,1} \triangleq \sigma_x^2\|\mathbf{s}_{x,\Sigma}\|^2/2\sigma_{n_x}^2$, $\eta_{y,1} \triangleq \sigma_y^2\|\mathbf{s}_{y,\Sigma}\|^2/2\sigma_{n_y}^2$, $\eta_{x,2} \triangleq \sigma_x^2\|\mathbf{s}_{x,\Delta}\|^2/2\sigma_{n_x}^2$, and $\eta_{y,2} \triangleq \sigma_y^2\|\mathbf{s}_{y,\Delta}\|^2/2\sigma_{n_y}^2$ being the equivalent output SNRs of two orthogonal decomposition of signal modes of array-x and array-y, respectively. Our findings are verified once again that at reasonably high SNRs, we have $k(2) \rightarrow k(1) \rightarrow |\rho|$, meaning that the canonical correlation coefficients reveal the spatial coherence among two sets of data vectors from two spatially separated arrays.

5. SIMULATIONS

Simulation results are presented to demonstrate our analytical results on CCA in comparison with the empirical CCA (with varying snapshot numbers and SNRs). We choose two uniform linear arrays (ULAs) with $N_x = N_y = 8$. The impinging angles of the sources are arbitrarily chosen and fixed. As shown in the following figures, we plot the values of the canonical correlation coefficients, both the analytical results in solid lines in eqs.(9) and (17), and the simulated results in scattered points where the empirical CCA are done in combination with subspace pre-processing. We have shown $k(1)$ and $k(2)$ as functions of the nominal spatial correlation ρ, under various SNRs per sample for both single source in colored noise fields (Fig.1) and two sources in white noise fields (Fig.2).

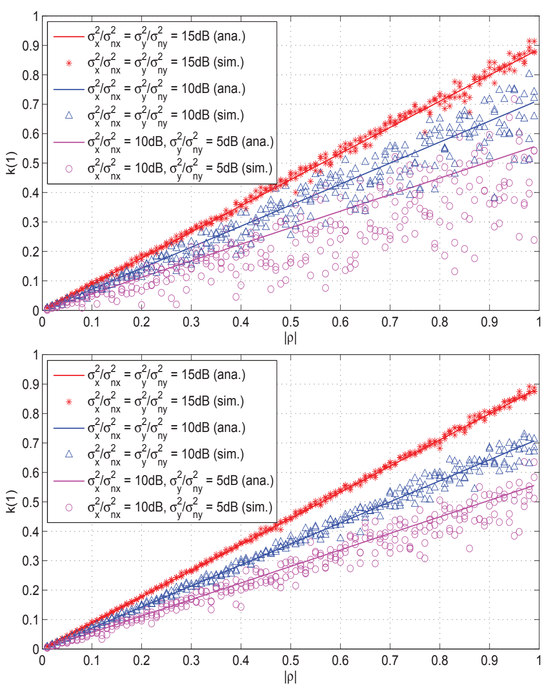

Fig. 1. CCA for a single source in colored Gaussian noise field. Results are for $k(1)$ vs. $|\rho|$ under varying SNRs per sample, where (a) 16 and (b) 80 snapshots are used in the empirical CCA.

6. CONCLUSIONS

In conclusion, CCA provide reliable information about spatial correlations existing among pairs of data sets only when SNRs at both arrays are reasonably high, and the sample support is significantly larger than the data dimensions. Specifically, canonical correlations are a function of the true spatial correlations as well as the SNRs of data sets from both arrays. The quality of empirical CCA depends on data quality (SNRs) on both arrays and sample support.

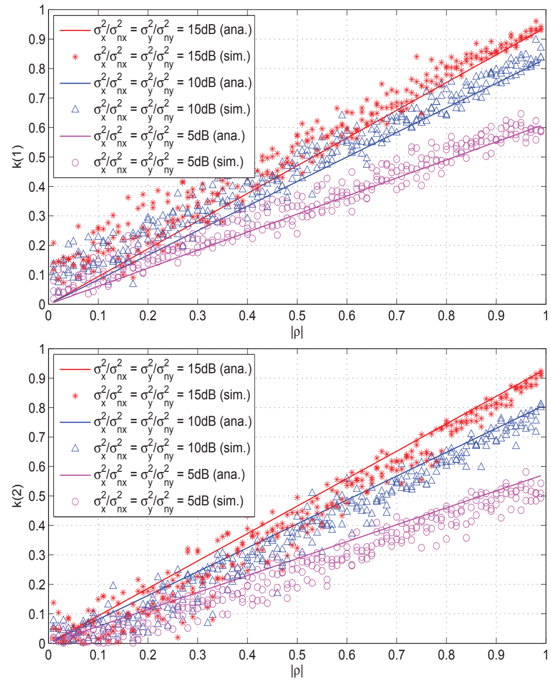

Fig. 2. CCA for two sources in white Gaussian noise field. Results are for $k(1)$ and $k(2)$ vs. $|\rho|$ under varying SNRs per sample, where 160 snapshots are used in the empirical CCA. The impinging angles for the two sources are 15° and 65° respectively, and $\|\mathbf{s}_{x,\Sigma}\|^2/2 = 8.60$, $\|\mathbf{s}_{x,\triangle}\|^2/2 = 7.40$, $\|\mathbf{s}_{y,\Sigma}\|^2/2 = 8.64$, and $\|\mathbf{s}_{y,\triangle}\|^2/2 = 7.36$.

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