# A SIMPLE ITERATIVE ALGORITHM FOR RANGE–BASED LOCALIZATION

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**ABSTRACT**

The range–based localization problem often arises in TOA or RSSI based position estimation schemes. It is well–known that such a localization problem can be formulated as a nonlinear least-squares (NLS) estimation problem. In this paper, we formulate the problem as a constrained optimization problem, which is equivalent to the general NLS problem. By using a greedy optimization strategy, we derive a simple iterative algorithm with closed–form expressions for the NLS localization, which can be implemented in a distributed way. Simulation results show that the localization performance of the proposed localization algorithm is very close to the Cramer–Rao lower bound.

***Index Terms—*** Range–based localization, Nonlinear least–squares estimation, Iterative algorithm

## 1. INTRODUCTION

With the emergence of various location–based services and other potential applications in wireless communication networks, positioning in wireless networks has received a great deal of attention in the past decade [1]. In general, positioning involves two steps. The first step is to estimate position–related signal parameters, e.g., time of arrival (TOA), received signal strength (RSS), time difference of arrival (TDOA), etc, based on which the position is estimated in the second step. Among the above signal parameters, TOA and RSS can be translated into range or distance measurements based on signal propagation models. Localization using range measurements is referred to as range–based localization. In the rest of paper, we assume range–measurements have been obtained from the estimated TOA or RSS signal parameters.

The principle of the range–based localization is rather simple. Given a set of range measurements $r_i$ $(i = 1, 2, \cdots, n)$ between a target node whose location is unknown, denoted by $\boldsymbol{x} \in \mathbb{R}^d$ ($d$=2 or 3), and several reference nodes with known location, denoted by $\boldsymbol{a}_i \in \mathbb{R}^d$ $(i = 1, 2, \cdots, n)$, the objective of the range–based localization is to estimate the position of the target node from the given range measurements. Mathematically, the range–based localization problem can be interpreted as a problem of solving a system of nonlinear equations given by

$$\|\boldsymbol{x} - \boldsymbol{a}_i\| = r_i,\ i = 1, 2, \cdots, n. \tag{1}$$

The problem (1) can be easily solved when range–measurements are noise–free. However, due to the error in range–measurements, the system of nonlinear equations are inconsistent, that is, there is no solution for the problem (1). Hence, the range–based localization problem is often approached by solving the following nonlinear linear–squares (NLS) estimation problem:

$$\min_{\boldsymbol{x}} J(\boldsymbol{x}) = \sum_{i=1}^{n} (||\boldsymbol{x} - \boldsymbol{a}_i|| - r_i)^2 \tag{2}$$

which is known as the best approach if the errors in range measurements are i.i.d Gaussian with zero mean. Solving the problem (2) perfectly is not easy. Common approaches to the problem are local search methods such as gradient–descent method or Gauss–Newton method [1] which fall into the category of iterative algorithms. These methods generally have to perform non–exact line search (e.g., backtracking line search [3]) at each iteration in order to choose a right stepsize that guarantees the nonincreasing of the value of the NLS cost function in (2). It is known that the non–exact line search requires evaluating the cost function a few times at each iteration, which results in the inefficiency of the above iterative algorithms.

A better NLS localization approach, called approximate maximum likelihood (AML), was proposed in [4]. In essence, the AML method is also of iterative form. For 2-D localization case (i.e., $\boldsymbol{x}$=$(x, y)$), starting from some initial $(x, y)$, the authors of [4] first changes the gradient equations (derived by setting the gradient of the NLS cost function $\nabla J(\boldsymbol{x})$ equal to zero) into two linear equations in the unknown $(x, y)$, from which $x$ and $y$ can be respectively expressed as a linear function of $(x^2 + y^2)$. Substituting the derived expressions of $x$ and $y$ into $x^2 + y^2$ yields a quadratic equation in $(x^2 + y^2)$. After solving the quadratic equation for the value of $(x^2 + y^2)$, $x$ and $y$ are then obtained by substituting the value of $(x^2+y^2)$ into the linear expressions. Repeating the above procedure with the new values of $(x, y)$ $q$ times (the authors repeat five times), the AML method derives $q$ values of $(x, y)$. The authors select the $(x, y)$ that give the smallest value of the cost function as the final solution. In the AML method, a *root selection routine* requiring cost function evaluation is necessary when there are two positive roots or no positive roots for the quadratic equation. The AML method's computational complexity is equivalent to that of evaluating the cost function $3q$ times. A drawback of the AML method is that *there is no theoretical result about the final solution*.

In addition to NLS localization approaches, various versions of linear least–squares (LLS) methods have been proposed (see [2] and reference therein). They are suboptimal in general but rather simple. The LLS solutions can be used as a good initial estimate for NLS approaches. The localization performance of various LLS localization methods has been studied in [2], where it is shown that, the LLS method [5] by subtracting the average of quadratic equations (obtained by squaring both sides of (1)) from all equations outperforms the one by subtracting one of quadratic equations.

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In this paper, a new formulation with equality constraints is proposed for range–based localization, which is proved to be equivalent to the general NLS position estimation problem as shown in (2). The new formulation seems to be more complex. However, by using a greedy optimization strategy, an easily realizable iterative localization algorithm is obtained. The algorithm can be implemented in a distributed way[1], which is an advantage over other localization algorithms presented above. Simulation results show that the localization performance of the proposed algorithm is very close to the Cramer–Rao lower bound (CRLB).

## 2. PROBLEM FORMULATION

The new formulation for the range–based localization is given by

$$\begin{aligned} &\min_{\boldsymbol{x},\bar{\boldsymbol{x}}} \sum_{i=1}^{n} ||\boldsymbol{x}-\boldsymbol{x}_i||^2 \\ &\text{subject to } \; ||\boldsymbol{x}_i-\boldsymbol{a}_i||^2 = r_i^2 \;\; i=1,2,\cdots,n \end{aligned} \tag{3}$$

where $\bar{\boldsymbol{x}} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n)$ denotes a vector obtained by stacking all vectors $\boldsymbol{x}_i \in \mathbb{R}^d$. The idea behind this formulation is to find a point $\boldsymbol{x}$ that minimizes the sum of distances to the circles or spheres.

**Proposition**: *The problem (3) is equivalent to the problem (2) in the sense that the solution in $\boldsymbol{x}$ of the problem (3) minimizes $J(\boldsymbol{x})$.*

*Proof*: Define the lagrangian function associated with the problem (3) as

$$L(\boldsymbol{x},\bar{\boldsymbol{x}},\boldsymbol{\lambda}) = \sum_{i=1}^{n} ||\boldsymbol{x}-\boldsymbol{x}_i||^2 + \sum_{i=1}^{n} \lambda_i (||\boldsymbol{x}_i-\boldsymbol{a}_i||^2 - r_i^2) \tag{4}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is the lagrangian multiplier vector. From the Karush–Kuhn–Tucker (KKT) condition, the optimal primal–dual pair, denoted by $(\boldsymbol{x}^*, \bar{\boldsymbol{x}}^*, \boldsymbol{\lambda}^*)$, must satisfy

$$\nabla_x L = 2\sum_{i=1}^{n} (\boldsymbol{x}^* - \boldsymbol{x}_i^*) = 0 \tag{5a}$$

$$\nabla_{x_i} L = 2(\boldsymbol{x}_i^* - \boldsymbol{x}^*) + 2\lambda_i^* (\boldsymbol{x}_i^* - \boldsymbol{a}_i) = 0, \; i=1,2,\cdots,n \tag{5b}$$

$$||\boldsymbol{x}_i^* - \boldsymbol{a}_i||^2 = r_i^2, \; i=1,2,\cdots,n \tag{5c}$$

where $L$ is the abbreviation of $L(\boldsymbol{x},\bar{\boldsymbol{x}},\boldsymbol{\lambda})$.

There may be two cases about $\boldsymbol{x}^*$. One is that $\boldsymbol{x}^*$ is not either of $\boldsymbol{a}_i$, $i=1,2,\cdots,n$, and conversely the other is that $\boldsymbol{x}^*$ is one of $\boldsymbol{a}_i$, $i=1,2,\cdots,n$.

Let us first look at the first case: $\boldsymbol{x}^*$ is not either of $\boldsymbol{a}_i$, $i=1,2,\cdots,n$. From (5b), it is readily known that all $\lambda_i^*$ must not be $-1$ since otherwise $\boldsymbol{x}^*$ is either of $\boldsymbol{a}_i$, $i=1,2,\cdots,n$. This means that $1+\lambda_i \neq 0$ for all $i$. Thus, we have from (5b) $\boldsymbol{x}_i^* = \boldsymbol{a}_i + \frac{\boldsymbol{x}^*-\boldsymbol{a}_i}{1+\lambda_i^*}$, and substituting $\boldsymbol{x}_i^*$ into (5c) yields $|1+\lambda_i^*| = \frac{||\boldsymbol{x}^*-\boldsymbol{a}_i||}{r_i}$, which means $1+\lambda_i^*$ may be either $\frac{||\boldsymbol{x}^*-\boldsymbol{a}_i||}{r_i}$ or $-\frac{||\boldsymbol{x}^*-\boldsymbol{a}_i||}{r_i}$. Correspondingly, $\boldsymbol{x}_i^* = \boldsymbol{a}_i + r_i \frac{\boldsymbol{x}^*-\boldsymbol{a}_i}{||\boldsymbol{x}^*-\boldsymbol{a}_i||}$ or $\boldsymbol{x}_i^* = \boldsymbol{a}_i - r_i \frac{\boldsymbol{x}^*-\boldsymbol{a}_i}{||\boldsymbol{x}^*-\boldsymbol{a}_i||}$. In fact, observing that

$$\begin{aligned} &\left\| \boldsymbol{x} - \left( \boldsymbol{a}_i + r_i \frac{\boldsymbol{x}-\boldsymbol{a}_i}{||\boldsymbol{x}-\boldsymbol{a}_i||} \right) \right\|^2 = (||\boldsymbol{x}-\boldsymbol{a}_i|| - r_i)^2 \\ &< (||\boldsymbol{x}-\boldsymbol{a}_i|| + r_i)^2 = \left\| \boldsymbol{x} - \left( \boldsymbol{a}_i - r_i \frac{\boldsymbol{x}-\boldsymbol{a}_i}{||\boldsymbol{x}-\boldsymbol{a}_i||} \right) \right\|^2 \end{aligned} \tag{6}$$

holds for all $\boldsymbol{x} \neq \boldsymbol{a}_i$, we can determine $1+\lambda_i^* = \frac{||\boldsymbol{x}^*-\boldsymbol{a}_i||}{r_i}$ and $\boldsymbol{x}_i^* = \boldsymbol{a}_i + r_i \frac{\boldsymbol{x}^*-\boldsymbol{a}_i}{||\boldsymbol{x}^*-\boldsymbol{a}_i||}$ when $\lambda_i^* \neq -1$.

Below we are to show that $\boldsymbol{x}^*$ minimizes $J(\boldsymbol{x})$, that is, $\boldsymbol{x}^*$ is a global minimizer of the problem (2). To arrive at a contradiction, we assume $\boldsymbol{x}^*$ is not a global minimizer of (2). Due to the fact that, for any $\boldsymbol{x} \neq \boldsymbol{a}_i$, $i=1,2,\cdots,n$, it must hold that

$$\sum_{i=1}^{n} (||\boldsymbol{x}-\boldsymbol{a}_i|| - r_i)^2 = \sum_{i=1}^{n} \left\| \boldsymbol{x} - \left( \boldsymbol{a}_i + r_i \frac{\boldsymbol{x}-\boldsymbol{a}_i}{||\boldsymbol{x}-\boldsymbol{a}_i||} \right) \right\|^2, \tag{7}$$

the global minimizer of the problem (2) must be one of $\boldsymbol{a}_i$, $i=1,2,\cdots,n$. Without loss of generality, we assume it is $\boldsymbol{a}_1$. Then we have

$$\sum_{i=1}^{n} (||\boldsymbol{a}_1-\boldsymbol{a}_i|| - r_i)^2 < \sum_{i=1}^{n} \left\| \boldsymbol{x}^* - \boldsymbol{a}_i - r_i \frac{\boldsymbol{x}^*-\boldsymbol{a}_i}{||\boldsymbol{x}^*-\boldsymbol{a}_i||} \right\|^2. \tag{8}$$

It is not difficult to verify that the point given by $(\boldsymbol{a}_1, \bar{\boldsymbol{y}})$ where

$$\begin{aligned} &\bar{\boldsymbol{y}} = (\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_n), \\ &\boldsymbol{y}_i = \boldsymbol{a}_i + r_i \frac{\boldsymbol{a}_1-\boldsymbol{a}_i}{||\boldsymbol{x}-\boldsymbol{a}_i||}, \; i=2,3,\cdots,n, \\ &\boldsymbol{y}_1 \text{ is any } \boldsymbol{x}_1 \text{ that satifies } ||\boldsymbol{x}_1-\boldsymbol{a}_1|| = r_1 \end{aligned}$$

satisfies all equality constraints of the problem (3). Moreover, one can see that $(\boldsymbol{a}_1, \bar{\boldsymbol{y}})$ yields a smaller value of the objective function in (3) than the point $(\boldsymbol{x}^*, \bar{\boldsymbol{x}}^*)$ in terms of (8), which contradicts to the fact that $(\boldsymbol{x}^*, \bar{\boldsymbol{x}}^*)$ is the global minimizer of the problem (3). Therefore, $\boldsymbol{x}^*$ minimizes $J(\boldsymbol{x})$.

Then let us look at the second case: $\boldsymbol{x}^*$ is one of $\boldsymbol{a}_i$, $i=1,2,\cdots,n$. We assume without loss of generality $\boldsymbol{x}^* = \boldsymbol{a}_1$. Then we must have

$$\begin{aligned} &\boldsymbol{x}_i^* = \boldsymbol{a}_i + r_i \frac{\boldsymbol{a}_1-\boldsymbol{a}_i}{||\boldsymbol{a}_1-\boldsymbol{a}_i||}, \; i=2,3,\cdots,n, \\ &\boldsymbol{x}_1^* \text{ is any } \boldsymbol{x}_1 \text{ that satifies } ||\boldsymbol{x}_1-\boldsymbol{a}_1|| = r_1, \\ &\sum_{i=1}^{n} (||\boldsymbol{a}_1-\boldsymbol{a}_i|| - r_i)^2 \leq \sum_{i=1}^{n} (||\boldsymbol{a}_j-\boldsymbol{a}_i|| - r_i)^2, \; j=2,3,\cdots,n, \end{aligned}$$

and for any $\boldsymbol{x} \neq \boldsymbol{a}_i$, $i=1,2,\cdots,n$

$$\sum_{i=1}^{n} (||\boldsymbol{a}_1-\boldsymbol{a}_i|| - r_i)^2 \leq \sum_{i=1}^{n} \left\| \boldsymbol{x} - \boldsymbol{a}_i - r_i \frac{\boldsymbol{x}-\boldsymbol{a}_i}{||\boldsymbol{x}-\boldsymbol{a}_i||} \right\|^2.$$

This means that $\boldsymbol{x}^*$ minimizes $J(\boldsymbol{x})$.

From the above analysis, we conclude that the problem (3) is equivalent to the problem (2). ■

Although the second case has been discussed in the proof above, this case nearly always doesn't occur in practical scenarios. Thus an (nearly always) valid assumption can be made that the global minimizer of the problem (2) is a stationary point of $J(\boldsymbol{x})$ (i.e., not either of $\boldsymbol{a}_i$, $i=1,2,\cdots,n$), which is also implicitly used in gradient–related localization algorithms, e.g., the AML [4]. Under the assumption, we propose the following algorithm based on a greedy optimization strategy.

## 3. GREEDY LOCALIZATION ALGORITHM

The proposition tells us the problem (2) can be addressed by solving the problem (3), for which a greedy algorithm is available. Starting from a point $\boldsymbol{x} \neq \boldsymbol{a}_i$ ($i=1,2,\cdots,n$), we first solve (3) for $\boldsymbol{x}_i$s

[1] Distributed localization methods are beneficial to localization in power–limited wireless sensor networks [6]. As compared to the centralized localization methods which require the transmission of all range–measurements from reference nodes (i.e., sensor nodes) to a fusion center, the distributed methods are energy–efficient since they avoid the transmission of all ranges.

while fixing $\boldsymbol{x}$. Equivalently, we minimize $\|\boldsymbol{x}_i - \boldsymbol{x}\|^2$ subject to $\|\boldsymbol{x}_i - \boldsymbol{a}_i\|^2 = r_i^2$ for each $i$. Using the Lagrangian method, we derive

$$\boldsymbol{x}_i = \boldsymbol{a}_i + r_i \frac{\boldsymbol{x} - \boldsymbol{a}_i}{\|\boldsymbol{x} - \boldsymbol{a}_i\|}, \; i = 1, 2, \cdots, n. \tag{9}$$

Next, we solve (3) for $\boldsymbol{x}$ while fixing $\boldsymbol{x}_i$s and obtain $\boldsymbol{x} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i$. It is readily known that repeating the above procedure will result in nonincreasing of the objective function in (3). On the other hand, because the objective function has a lower bound (and is continuous), the algorithm by repeating the above procedure must converge. In particular, the algorithm can be simply written as in an iterative form

$$\boldsymbol{x}^{k+1} = \frac{1}{n}\sum_{i=1}^{n}\left(\boldsymbol{a}_i + r_i \frac{\boldsymbol{x}^k - \boldsymbol{a}_i}{||\boldsymbol{x}^k - \boldsymbol{a}_i||}\right). \tag{10}$$

Assume the algorithm converges to $\boldsymbol{x}$. Based on (10), we have

$$\boldsymbol{x} = \frac{1}{n}\sum_{i=1}^{n}\left(\boldsymbol{a}_i + r_i \frac{\boldsymbol{x} - \boldsymbol{a}_i}{||\boldsymbol{x} - \boldsymbol{a}_i||}\right). \tag{11}$$

By simple manipulation on (11), it is obtained

$$\nabla J(\boldsymbol{x}) = 2\sum_{i=1}^{n}(||\boldsymbol{x} - \boldsymbol{a}_i|| - r_i)\frac{\boldsymbol{x} - \boldsymbol{a}_i}{||\boldsymbol{x} - \boldsymbol{a}_i||} = 0. \tag{12}$$

This means the convergence point $\boldsymbol{x}$ is a stationary point of $J(\boldsymbol{x})$.

One can see that the iterative algorithm is closed–form and can be easily implemented since it does not require choosing a stepsize at each iteration. Moreover, it can be implemented in a distributed way as follows. Each reference node $i$ but the last node $n$ computes $\boldsymbol{x}_i^k = \boldsymbol{a}_i + r_i \frac{\boldsymbol{x}^k - \boldsymbol{a}_i}{||\boldsymbol{x}^k - \boldsymbol{a}_i||}$ based on the current estimate $\boldsymbol{x}^k$ and passes the current estimate $\boldsymbol{x}^k$ and the sum of all calculated $\boldsymbol{x}_i^k$, i.e., $\sum_{j=1}^{i}\boldsymbol{x}_j^k$, to the next node $(i+1)$. The last node computes $\boldsymbol{x}_n^k$ and $\boldsymbol{x}^{k+1} = \frac{1}{n}\left(\sum_{j=1}^{n-1}\boldsymbol{x}_j^k + \boldsymbol{x}_n^k\right)$, and then passes $\boldsymbol{x}^{k+1}$ to the first node. Repeat the above process until the convergence criterion $\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^k\| \le \epsilon/(2n)$ is met. Here $\epsilon$ is a small scalar. Note that

$$\begin{aligned}\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^k\| &= \left\|\frac{1}{n}\sum_{i=1}^{n}\left(\boldsymbol{a}_i + r_i\frac{\boldsymbol{x}^k - \boldsymbol{a}_i}{||\boldsymbol{x}^k - \boldsymbol{a}_i||}\right) - \boldsymbol{x}^k\right\| \\ &= \frac{1}{2n}\|\nabla J(\boldsymbol{x}^k)\|.\end{aligned} \tag{13}$$

Hence, the convergence criterion is equivalent to $\|\nabla J(\boldsymbol{x}^k)\| \le \epsilon$.

## 4. PERFORMANCE EVALUATION

In this section, simulations are conducted to show the localization performance of the proposed algorithm. We will compare it with other localization algorithms[2] including AML [4], LLS [5]. Note that, we use the LLS solution as the initial point of our algorithm. Also, the CRLB [2] is provided as reference. In simulations, nodes are located within the region $[0, 100]$unit$\times[0, 100]$unit, and the range measurements are corrupted by i.i.d Gaussian noise with zero mean and variance of $\sigma^2$.

### 4.1. Convergence performance

Here, we compare the convergence performance of the proposed algorithm with that of the gradient–descent (GD) algorithm. In simulations, $n$, $\epsilon$ and $\sigma$ are taken to be 5, 0.1 and 10, respectively. For the GD algorithm, the convergence criterion is $\|\nabla J(\boldsymbol{x})\| \le 0.1$. Hence, the convergence criteria for two algorithms are the same. Many trials are conducted, in each of which, nodes are randomly placed. It is found that, the proposed algorithm may require more iterations to converge than the GD algorithms. However, due to the backtracking line search method used at each iteration of the GD algorithm, the computational complexity of the GD algorithm is nearly always higher than the proposed algorithm (similar results can also be found for the Gauss–Newton method). An example comparison is shown in Fig. 1. In the example, the GD algorithm with[3] $\alpha = 0.25$ and $\beta = 0.5$ ($\alpha = 0.1$ and $\beta = 0.8$) equivalently evaluates the cost function 23 (98) times while the proposed algorithm equivalently evaluates the cost function 13 times[4]. It is also observed that, in general, 20 iterations are sufficient for one to achieve a good position estimate.

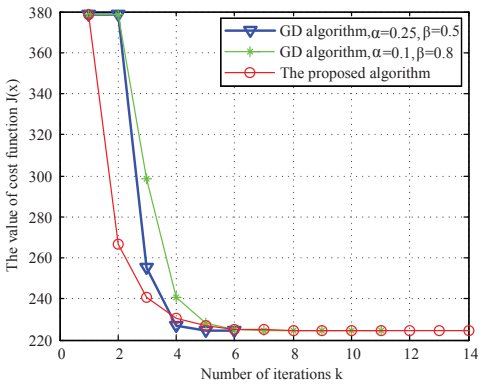


**Fig. 1**. An example comparison of convergence.

### 4.2. Localization performance

Below we compare the localization performance of the proposed algorithm with other algorithms. We first consider the general case where reference nodes are not located on a straight line. This case can be further divided into two subcases where it is known that the LLS algorithm may behave quite differently. The one is that in which the target node is located *inside* the convex hull of reference nodes and the other is that in which the target node is *outside* the convex hull of reference nodes. An example is shown in Fig. 2, where target nodes are denoted by triangles and reference nodes are denoted by dots; the "inside" case is labeled with blue color and the "outside" case is labeled with red color.

In various noise levels ($\sigma = 2, 4, 6, 8, 10$), 5000 Monte Carlo simulations are conducted for each case as shown in Fig. 2. The performance is evaluated in terms of the root mean squared error (RMSE) given by RMSE $= \sqrt{\frac{1}{5000}\sum_{i=1}^{5000}\|\boldsymbol{x}_{est}^i - \boldsymbol{x}\|^2}$ where $\boldsymbol{x}_{est}^i$ denotes the estimate of the $i$-th simulation. The simulation results for the "inside" case and the "outside" case are respectively presented in Fig. 3 and Fig. 4. The plots show that, in both cases, the localization performance of the proposed algorithm is very close to the CRLB and is a bit higher than that of the AML method.

[2]Due to page limitation, we will not give a comparison with distributed localization algorithms.

[3]$\alpha$ and $\beta$ are two control parameters in the backtracking line search method [3]. The choice of $\alpha$ and $\beta$ influences the convergence rate of the GD algorithm. Note, in general, it is not easy to make a better choice.

[4]Note that, the computational complexity of each iteration of the proposed algorithm and of the main update step of the GD algorithm (i.e.,$\boldsymbol{x}^{k+1}=\boldsymbol{x}^k-\mu_k\nabla J(\boldsymbol{x}^k)$) are both equivalent to that of evaluating the cost function one time.

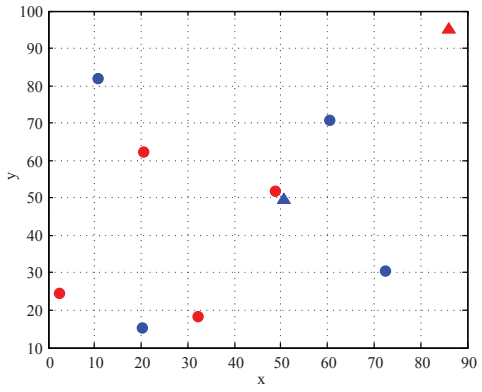


**Fig. 2**. The node placement of two general localization cases.

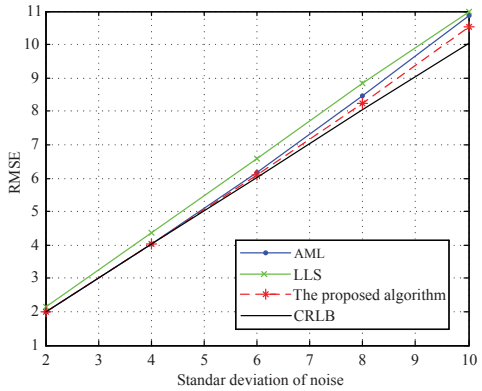


**Fig. 3**. The localization performance of the "inside" case.

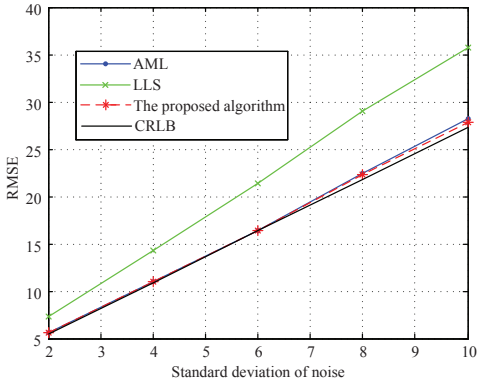


**Fig. 4**. The localization performance of the "outside" case.

We then consider the special case where reference nodes are collinear, for which, all LLS methods as well as the initialization method used in the AML are infeasible. Ambiguity occurs in this case because the cost function has two global minimizer. However, the ambiguity can be eliminated if there is some prior information on the location of the target node or there are additional reference nodes that is not on the straight line. In simulations, four reference nodes are fixed at (20,30), (40,30), (50,30) and (90,30), and the target node is at (65,85). We assume it has been known that the target node lies above the line formed by the reference nodes. Hence, in each simulation our algorithm as well as the AML can be initialized from a good initial estimate given by $\boldsymbol{x}^0 = \boldsymbol{a}_i + (0, r_i)$ where $i = \arg\min_j r_j$. The simulation result is plotted in Fig. 5, which shows that our algorithm can achieve the CRLB while the AML method behaves quite badly. This is due to that our algorithm must converge to the nearest stationary point to the initial point while the AML method may jump to the other stationary point that also minimizes $J(\boldsymbol{x})$.

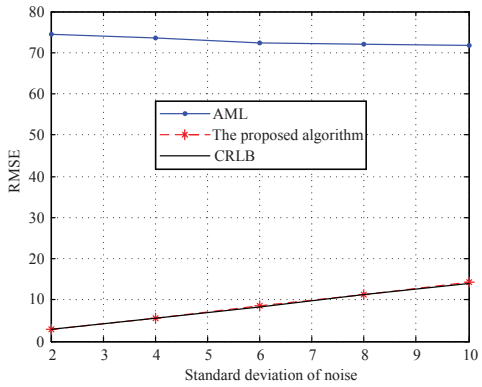


**Fig. 5**. The localization performance of the "collinear" case.

## 5. CONCLUSION

We have proposed a new formulation for the range–based localization and a simple iterative localization algorithm with near–optimal performance. As compared to the AML method, our algorithm is more flexible in the sense that our algorithm can be directly used in the three (or even higher) dimensional case. Moreover, the new formulation maybe very useful since with this formulation the NLS localization problem (2) can be completely solved by using *Gröbner bases* [7] to the KKT equation (5).

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