EXPLOITING MULTIPLE SHIFT INVARIANCES IN HARMONIC RETRIEVAL

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ABSTRACT

A novel algorithm for estimating multi-dimensional damped harmonics is proposed. A matrix polynomial is formed from the weighed sum of multiple shift invariances contained in the data model. Necessary and sufficient conditions are derived that reveal that the damped harmonics can be uniquely determined from the roots of the matrix polynomial. The proposed algorithm reveal a seamless link between two important classes of search free subspace methods. The classical rooting based harmonic estimation methods that exploit the complete invariance structure and the single invariance ESPRIT algorithms. We show that both approaches can be expressed under this general framework by an appropriate choice of the weights.

Index Terms— Direction-of-arrival estimation, damped harmonic retrieval, shift invariance, ESPRIT, root-MUSIC

1. INTRODUCTION

The multi-dimensional (MD) damped harmonic retrieval (HR) problem is underlying a variety of important engineering applications. Classical examples include sensor array processing for radar and sonar applications, geophysics, mobil communications, chemical spectroscopy, diagnostic imaging, and condition monitoring of rotating machinery.

Numerous search-free subspace based methods have been proposed that exploit the underlying Vandermonde structure of the signal model and yield estimates of the MD harmonics as closed-form solutions, e.g. as the solution of a generalized eigenvalue problem [1, 2]. In general this contributions can be classified in two categories: the rooting based methods as e.g. the 2D root-MUSIC algorithm [3] and the ESPRIT based methods [1, 2, 4]. While the former methods take advantage of the full Vandermonde structure contained in the data, the ESPRIT algorithm only exploits a specific shift-invariance, thus leading to reduced asymptotic performance. However, an attractive feature of the ESPRIT-type methods is that they naturally generalize to damped harmonic scenarios [5, 6]. This is not the case for the rooting-based methods.

In this paper the approach in [7, 8] is further generalized to include appropriate weighing of the individual shift invariances. New identifiability results for the rooting based weighed multiple invariance (WMI) approach are derived, that give further insight in the underlying estimation problem and yield a smooth transition between the single invariance (SI) and multiple invariance (MI) approach. The paper is organized as follows. The signal model is formulated in section 2. In section 3 the WMI-root ESPRIT-algorithm is derived from the MI equations. In section 4 uniqueness conditions for the roots of the associated matrix polynomials are provided. Implementation specific aspects of the proposed algorithm are discussed in section 5. Simulation results are given in section 6 before section 7 concludes the paper.

Notation: The transpose is denoted by ^{T}, the complex conjugate Hermitian transpose by ^{H}. The expectation operation is denoted by $\mathrm{E}\{\cdot\}$, $\odot$ denote element-wise multiplication, $\otimes$ denotes the Kronecker product, $\circ$ denotes the Khatri-Rao product, $\mathrm{diag}\{\cdot\}$ converts a vector to a diagonal matrix with the vector place on the main diagonal, $\mathrm{vec}\{\cdot\}$ converts a matrix into a vector stacking the columns of the matrix, $\boldsymbol{I}_k$ denotes the $k \times k$ identity matrix, and $\mathbf{0}_{k,l}$ denote the $k \times l$ matrix containing zeros in all elements. We write $\boldsymbol{A} > (\geq)\boldsymbol{B}$ for two arbitrary square Hermitian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ if $\boldsymbol{A} - \boldsymbol{B}$ is positive (semi) definite, respectively.

2. SIGNAL MODEL

A superposition of P complex exponentials is observed along two independent sampling axis labeled as a-axis and b-axis. Let $a_p \in \mathbb{C}$ with $a_p \leq 1$ denote the pth signal generator. While a uniform sampling grid is assumed along the a-axis, for sake of generality, we impose no restrictions to the sampling and generator structure along the b-axis. In matrix representation the signal model associated with the nth observation reads

$$\boldsymbol{Y}_n \;=\; \sum_{p=1}^{P} \boldsymbol{b}_p \boldsymbol{a}_p^T x_{p,n} + \boldsymbol{N}_n \;\in \mathbb{C}^{L \times K} \qquad (1)$$

for $n = 1, \ldots, N$. Here, $\boldsymbol{a}_p = [1, a_p, a_p^2, \ldots, a_p^{K-1}]^T$ and $\boldsymbol{b}_p = [b_{p,1}, \ldots, b_{p,L}]^T$ are the generator vectors of the pth signal, and $\boldsymbol{N}_n \in \mathbb{C}^{L \times K}$ is the corresponding noise matrix. After vectorization equation (1) becomes $\boldsymbol{y}_n = \boldsymbol{H}\boldsymbol{x}_n + \boldsymbol{n}_n$

where $\boldsymbol{y}_n = \mathrm{vec}\boldsymbol{Y}_n$, $\boldsymbol{H} = \boldsymbol{A}_K \circ \boldsymbol{B}_L$ is the $KL \times P$ 2D signal generator matrix formed from the $K \times P$ Vandermonde a-axis generator matrix $\boldsymbol{A}_K = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_P]$ and the $L \times P$ unstructured full column rank b-axis matrix $\boldsymbol{B}_L = [\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_P]$, $\boldsymbol{x}_n = [x_{1,n}, x_{2,n}, \ldots, x_{P,n}]^T$ is the $P \times 1$ amplitude vector, and $\boldsymbol{n}_n$ is the $LK \times 1$ vector contains the zero-mean complex white Gaussian noise of variance σ^2. The covariance matrix and its sample estimate read

$$\boldsymbol{R} = \mathrm{E}\left\{\boldsymbol{y}_n\boldsymbol{y}_n^H\right\} = \boldsymbol{H}\boldsymbol{S}\boldsymbol{H}^H + \sigma^2\boldsymbol{I} \tag{2}$$

$$\hat{\boldsymbol{R}} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{y}_n\boldsymbol{y}_n^H \tag{3}$$

respectively, where $\boldsymbol{S} = \mathrm{E}\left\{\boldsymbol{x}_n\boldsymbol{x}_n^H\right\}$ is the full rank $P \times P$ signal amplitude covariance matrix. We write the eigendecompositions of the matrices (2) and (3) as $\boldsymbol{R} = \boldsymbol{U}_S\boldsymbol{\Lambda}_S\boldsymbol{U}_S^H + \boldsymbol{U}_N\boldsymbol{\Lambda}_N\boldsymbol{U}_N^H$ and $\hat{\boldsymbol{R}} = \hat{\boldsymbol{U}}_S\hat{\boldsymbol{\Lambda}}_S\hat{\boldsymbol{U}}_S^H + \hat{\boldsymbol{U}}_N\hat{\boldsymbol{\Lambda}}_N\hat{\boldsymbol{U}}_N^H$ where the diagonal $P \times P$ matrices $\boldsymbol{\Lambda}_S$ and $\hat{\boldsymbol{\Lambda}}_S$ contain the L signal-subspace eigenvalues of $\boldsymbol{R}$ and $\hat{\boldsymbol{R}}$, respectively, and the $(KL - P) \times (KL - P)$ diagonal matrices $\boldsymbol{\Lambda}_N$ and $\hat{\boldsymbol{\Lambda}}_N$ contain the $KL - P$ noise-subspace eigenvalues of $\boldsymbol{R}$ and $\hat{\boldsymbol{R}}$, respectively. In turn, the columns of the $KL \times P$ matrices $\boldsymbol{U}_S$ and $\hat{\boldsymbol{U}}_S$ contain the signal-subspace eigenvectors of $\boldsymbol{R}$ and $\hat{\boldsymbol{R}}$, respectively, whereas the $KL \times (KL - P)$ matrices $\boldsymbol{U}_S$ and $\hat{\boldsymbol{U}}_S$ are composed of the noise-subspace eigenvectors of $\boldsymbol{R}$ and $\hat{\boldsymbol{R}}$, respectively.

3. THE WMI-ROOT ESPRIT ALGORITHM

In the derivation of the invariance properties inherent in signal model we observe that the generator matrix can be expressed as $\boldsymbol{H} = \left[\boldsymbol{B}_L^T, (\boldsymbol{B}_L\boldsymbol{\Delta}_a)^T, \ldots, (\boldsymbol{B}_L\boldsymbol{\Delta}_a^{K-1})^T\right]^T$, where $\boldsymbol{\Delta}_a = \mathrm{diag}\{a_1, a_2, \ldots, a_P\}$ is the diagonal shifting matrix. We further define the upper- und lower-kL-row-masked signal generator matrix as

$$\overline{\boldsymbol{H}}_k = \overline{\boldsymbol{A}}_k \circ \boldsymbol{B} = \left(\overline{\boldsymbol{J}}_{K,k} \otimes \boldsymbol{I}_L\right)\boldsymbol{H}_K \tag{4}$$

$$\underline{\boldsymbol{H}}_k = \underline{\boldsymbol{A}}_k \circ \boldsymbol{B} = \left(\underline{\boldsymbol{J}}_{K,k} \otimes \boldsymbol{I}_L\right)\boldsymbol{H}_K, \tag{5}$$

respectively, where $\overline{\boldsymbol{A}}_k = \overline{\boldsymbol{J}}_{K,k}\boldsymbol{A}_k$ and $\underline{\boldsymbol{A}}_k = \underline{\boldsymbol{J}}_{K,k}\boldsymbol{A}_k$ with the $K \times K$ masking and shifting matrices

$$\overline{\boldsymbol{J}}_{K,k} = \begin{bmatrix} \boldsymbol{0}_{K-k,k} & \boldsymbol{0}_{k,k} \\ \boldsymbol{I}_{K-k} & \boldsymbol{0}_{K-k,k} \end{bmatrix} \; ; \; \underline{\boldsymbol{J}}_{K,k} = \begin{bmatrix} \boldsymbol{0}_{k,k} & \boldsymbol{0}_{K-k,k} \\ \boldsymbol{0}_{K-k,k} & \boldsymbol{I}_{K-k} \end{bmatrix}. \tag{6}$$

Thus the matrices $\overline{\boldsymbol{H}}_k$ and $\underline{\boldsymbol{H}}_k$ are formed from $\boldsymbol{H}$ by setting the entries in the kL first and last rows to zero, respectively. Further, in the formation of $\underline{\boldsymbol{H}}_k$ the non-zero rows obtained from masking $\boldsymbol{H}$ are shifted up by k rows to match the corresponding non-zero rows in $\overline{\boldsymbol{H}}_k$. Equipped with these definitions we are now ready to express the multiple shift invariances as

$$\underline{\boldsymbol{H}}_k = \overline{\boldsymbol{H}}_k\boldsymbol{\Delta}_a^k \tag{7}$$

for $k = 1, \ldots, K-1$. The classical ESPRIT algorithm makes use of a SI, e.g. the algorithm is deduced from (7) for $k = 1$. The algorithm proposed in this contribution however is designed to exploit multiple shifts an therefore extract more information contained in the rich structure of the signal model. Towards this aim we multiply the matrices in (7) from the left with the row masked matrices $\underline{\boldsymbol{H}}_k^H$ in (5) to obtain shift invariance equations of identical dimensions $P \times P$ independent of the choice of k. Later, the analysis of section 4 will reveal why this specific choice of left multiplication is particularly suitable. Subtracting the right side of the invariance equation we obtain

$$\underline{\boldsymbol{H}}_k^H\underline{\boldsymbol{H}}_k - \underline{\boldsymbol{H}}_k^H\overline{\boldsymbol{H}}_k\boldsymbol{\Delta}_a^k = \boldsymbol{0}_{P,P} \tag{8}$$

for $k = 1, \ldots, K-1$. The idea of the proposed algorithm is to form a linear combination of the shift invariance equations (8), hence

$$\sum_{k=1}^{K-1} c_k\left(\underline{\boldsymbol{H}}_k^H\underline{\boldsymbol{H}}_k - \underline{\boldsymbol{H}}_k^H\overline{\boldsymbol{H}}_k\boldsymbol{\Delta}_a^k\right) = \boldsymbol{0}_{P,P} \tag{9}$$

for linear weighting coefficients $c_k \in \mathbb{R} \geq 0, c_1 \neq 0$. Replacing the diagonal matrix $\boldsymbol{\Delta}_a$ in (9) by the scale identity matrix $a\boldsymbol{I}$ we obtain a matrix polynomial (MP) in a. It is immediate from (9) that the pth column of the resulting MP

$$\begin{aligned}\boldsymbol{M}(a) &= \sum_{k=1}^{K-1} c_k\left(\underline{\boldsymbol{H}}_k^H\underline{\boldsymbol{H}}_k - \underline{\boldsymbol{H}}_k^H\overline{\boldsymbol{H}}_k a^k\right) \\ &= \sum_{k=1}^{K-1} c_k\left(\underline{\boldsymbol{H}}_k^H\underline{\boldsymbol{H}}_k - \underline{\boldsymbol{H}}_k^H\underline{\boldsymbol{H}}_k\boldsymbol{\Delta}_a^{-k}a^k\right)\end{aligned} \tag{10}$$

is identical zero for $a = a_p$, for $p = 1, \ldots, P$ and non-zero otherwise. In other words, the MP $\boldsymbol{M}(a)$ becomes rank deficient if a is identical to one of the true generators. Thus a necessary condition for a to be a true generator is that a is a root of the MP $\boldsymbol{M}(a)$. Conditions under which a root a of the MP $\boldsymbol{M}(a)$ is also sufficient for a to be a true generator are given in the following section. In practice the signal generator matrix $\boldsymbol{H}$ shall be estimated from the signal eigenvectors $\boldsymbol{U}_S$. From the covariance model (2) and its eigendecomposition, it can readily be shown that the signal generator matrix $\boldsymbol{H}$ and the signal eigenvectors $\boldsymbol{U}_S$ span the same P dimensional signal subspace [1, 2]. Thus there exists a full-rank $P \times P$ mixing matrix $\boldsymbol{K}$ such that $\boldsymbol{H} = \boldsymbol{U}_S\boldsymbol{K}$. Multiplying the left and the right side of $\boldsymbol{M}(a)$ with $\boldsymbol{K}^H$ and $\boldsymbol{K}$, respectively, we obtain the MP

$$\begin{aligned}\boldsymbol{G}(a) &= \sum_{k=0}^{K-1} \boldsymbol{G}_k a^k = \boldsymbol{K}^H\boldsymbol{M}(a)\boldsymbol{K} \\ &= \sum_{k=1}^{K-1} c_k\left(\boldsymbol{K}^H\underline{\boldsymbol{H}}_k^H\underline{\boldsymbol{H}}_k\boldsymbol{K} - \boldsymbol{K}^H\underline{\boldsymbol{H}}_k^H\overline{\boldsymbol{H}}_k a^k\boldsymbol{K}\right) \\ &= \sum_{k=1}^{K-1} c_k\left(\underline{\boldsymbol{U}}_{S,k}^H\underline{\boldsymbol{U}}_{S,k} - \underline{\boldsymbol{U}}_{S,k}^H\overline{\boldsymbol{U}}_{S,k}a^k\right)\end{aligned} \tag{11}$$

Here $\overline{\boldsymbol{U}}_{S,k} = \left(\overline{\boldsymbol{J}}_{K,k} \otimes \boldsymbol{I}_L\right) \boldsymbol{U}_S$ and $\underline{\boldsymbol{U}}_{S,k} = \left(\underline{\boldsymbol{J}}_{K,k} \otimes \boldsymbol{I}_L\right) \boldsymbol{U}_S$ are the upper- and lower-kL-row-masked signal eigenvector matrices of dimensions $KL \times P$ that result from $\boldsymbol{U}_s$ from masking (setting the entries to zero) the upper and lower $(K-k)L$ rows of the signal eigenvector matrix $\boldsymbol{U}_S$, respectively. Since $\boldsymbol{K}$ is non-singular the rank properties of $\boldsymbol{M}(a)$ translate directly to $\boldsymbol{G}(a)$, i.e. $\boldsymbol{G}(a)$ becomes rank deficient if a is identical to one of the true generators. Thus the true generators $a_1, \ldots, a_P$ can be determined as the roots of the MP (11).

In this contribution we focus only on the estimation of the generators $a_1, \ldots, a_P$. However, it is important to note that if the generators along the a-axis are known (or estimated) then the mixing matrix $\boldsymbol{K}$ and the corresponding generators along the b-axis, hence the entries of the matrix $\boldsymbol{B}$, can also be estimated from relation (11). We refer to [7, 8] for details.

4. UNIQUENESS CONDITIONS

Theorem 1: Provided that $\underline{\boldsymbol{H}}_1$ is full column rank and

$$c_1 \geq c_2 \geq \ldots \geq c_{K-1} \geq 0; \quad c_1 > 0 \tag{12}$$

then the $P \times P$ matrix polynomials $\boldsymbol{M}(a)$ and $\boldsymbol{G}(a)$ evaluated inside and on the unit-circle (e.g. for $|a| \leq 1$) become singular if and only if a corresponds to one of the true generators $a_1, \ldots, a_P$.

Proof of Theorem 1: The MP in (10) factorizes as

$$\begin{aligned} \boldsymbol{M}(a) &= \left(\sum_{k=1}^{K-1} \sum_{m=0}^{k-1} c_k \underline{\boldsymbol{H}}_k^H \underline{\boldsymbol{H}}_k \boldsymbol{\Delta}_a^{-m} a^m \right) \left(\boldsymbol{I} - \boldsymbol{\Delta}_a^{-1} a \right) \\ &= \boldsymbol{W}(a) \left(\boldsymbol{I} - \boldsymbol{\Delta}_a^{-1} a \right) \end{aligned} \tag{13}$$

The right MP in the factorization (13), i.e. $\left(\boldsymbol{I} - \boldsymbol{\Delta}_a^{-1} a\right)$, becomes singular if and only if a is equal to one of the true generators $a_1, \ldots, a_P$. Next we show that the left MP in (13) is positive definite and therefore non-singular for $|a| \leq 1$. Towards this aim we rewrite $\boldsymbol{W}(a)$ as

$$\begin{aligned} \boldsymbol{W}(a) &= \sum_{k=1}^{K-1} \sum_{m=0}^{k-1} c_k \underline{\boldsymbol{H}}_k^H \underline{\boldsymbol{H}}_k \boldsymbol{\Delta}_a^{-m} a^m \\ &= \underline{\boldsymbol{H}}_1^H \left[\boldsymbol{L}_{K-1}(a) \otimes \boldsymbol{I}_L \right] \underline{\boldsymbol{H}}_1 \end{aligned} \tag{14}$$

where $\boldsymbol{C}_{K-1} = \sum_{k=1}^{K-1} c_k \underline{\boldsymbol{J}}_{K-1,k} \sum_{m=0}^{k-1} \boldsymbol{D}_{K-1,m}$, the $(K-1) \times (K-1)$ matrix $\boldsymbol{D}_{K-1,m}$ lower diagonal with all entries equal zero and the entries on the mth lower subdiagonal equal to one, and

$$\boldsymbol{L}_{K-1}(a) = \boldsymbol{C}_{K-1} \otimes \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ a & 1 & \ddots & & \vdots \\ a^2 & a & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a^{K-2} & \cdots & a^2 & a & 1 \end{bmatrix} \tag{15}$$

The matrix $\boldsymbol{W}(a)$ is positive definite if its Hermitian part is positive definite [9]. Hence

$$\begin{aligned} \boldsymbol{x}^H \left(\boldsymbol{W}(a) + \boldsymbol{W}^H(a) \right) \boldsymbol{x} &= \\ \boldsymbol{x}^H \underline{\boldsymbol{H}}_1^H \left[\left(\boldsymbol{L}_{K-1}(a) + \boldsymbol{L}_{K-1}^H(a) \right) \otimes \boldsymbol{I}_L \right] \underline{\boldsymbol{H}}_1 \boldsymbol{x} &= \\ \tilde{\boldsymbol{x}}^H \left[\left(\boldsymbol{L}_{K-1}(a) + \boldsymbol{L}_{K-1}^H(a) \right) \otimes \boldsymbol{I}_L \right] \tilde{\boldsymbol{x}} &> 0 \end{aligned} \tag{16}$$

for any $\boldsymbol{x} \neq \boldsymbol{0} \in \mathbb{C}^{P \times 1}$ and $\tilde{\boldsymbol{x}} = \underline{\boldsymbol{H}}_1 \boldsymbol{x}$. In equation (16) we made us of the fact that $\underline{\boldsymbol{H}}_1$ has full-column rank. From (16) we observe that the Hermitian part of $\boldsymbol{W}(a)$ is positive definite if the Hermitian matrix $\left(\boldsymbol{L}_{K-1}(a) + \boldsymbol{L}_{K-1}^H(a) \right) \otimes \boldsymbol{I}_L$ is positive definite, which itself is positive definite if the Hermitian matrix $\bar{\boldsymbol{L}}_{K-1}(a) = \boldsymbol{L}_{K-1}(a) + \boldsymbol{L}_{K-1}^H(a)$ is positive definite [9]. In fact it can be proven that $\bar{\boldsymbol{L}}_{K-1}(a) > 0$ for $|a| \leq 1$ provided that the weighting coefficients $c_1, \ldots, c_{K-1}$ satisfy condition (12). (The proof of this statement is based on Gaussian elimination and follows from induction argument. Here we needs to be skipped the proof due to space limitations. It will be included in an accompanying journal paper.) It immediately follows that the Hermitian part of $\boldsymbol{W}(a)$ and therefore MP $\boldsymbol{W}(a)$ itself are positive definite (and therefore non-singular) inside and on the unit-circle and for $c_1 \geq c_2, \geq, c_{K-1} \geq 0$. Hence from the factorization in (13) and with (11) we finally prove *theorem 1*. ■

Remarks: 1.) The condition that $\underline{\boldsymbol{H}}_{K-1}$ has full column rank implies that $P < (K-1)L$. 2.) Condition (12) ensures that the coefficients $\boldsymbol{G}_k$ of the MP in (11) are arranged in decreasing order in the sense that $\boldsymbol{G}_0 \geq \boldsymbol{G}_1 \geq \ldots \geq \boldsymbol{G}_{K-1}$. This can be easily verified from the definitions $\boldsymbol{G}_0 = \sum_{k=1}^{K-1} c_k \underline{\boldsymbol{U}}_{S,k}^H \underline{\boldsymbol{U}}_{S,k}$ and $\boldsymbol{G}_k = c_k \underline{\boldsymbol{U}}_{S,k}^H \underline{\boldsymbol{U}}_{S,k}$ for $k = 1, \ldots, K$. 3.) Appropriate choices for the weights c_k in (12) allow us transit seamless between the different shift invariance approaches. More specific, in the limiting cases, for the choice $c_1 = 1$ and $c_2 = c_3 = \ldots = c_{K-1} = 0$ we obtain the classical ESPRIT algorithm that exploits only a SI approach [1]. On the other hand, for the choice $c_1 = c_2 = \ldots = c_{K-1} = 1$ we obtain the unweighted MI-root ESPRIT algorithm described in [7].

5. IMPLEMENTATION

The WMI-root ESPRIT algorithm for estimating the generators $a_1, \ldots, a_P$ is performed in the following steps.

Step 1: Compute the sample covariance matrix $\hat{\boldsymbol{R}}$ of (3) and determine the associated signal eigenvectors $\hat{\boldsymbol{U}}_S$.

Step 2: Compute of the sample version of the MP in (11) as

$$\hat{\boldsymbol{G}}(a) = \sum_{k=1}^{K-1} c_k \left(\underline{\hat{\boldsymbol{U}}}_{S,k}^H \underline{\hat{\boldsymbol{U}}}_{S,k} - \underline{\hat{\boldsymbol{U}}}_{S,k}^H \overline{\hat{\boldsymbol{U}}}_{S,k} a^k \right) \tag{17}$$

for $\overline{\hat{\boldsymbol{U}}}_{S,k} = \left(\overline{\boldsymbol{J}}_{K,k} \otimes \boldsymbol{I}_L\right) \hat{\boldsymbol{U}}_S$ and $\underline{\hat{\boldsymbol{U}}}_{S,k} = \left(\underline{\boldsymbol{J}}_{K,k} \otimes \boldsymbol{I}_L\right) \hat{\boldsymbol{U}}_S$.

Step 3: Determine the P smallest roots $\hat{a}_1, \ldots, \hat{a}_P$ of $\hat{\boldsymbol{G}}(a)$

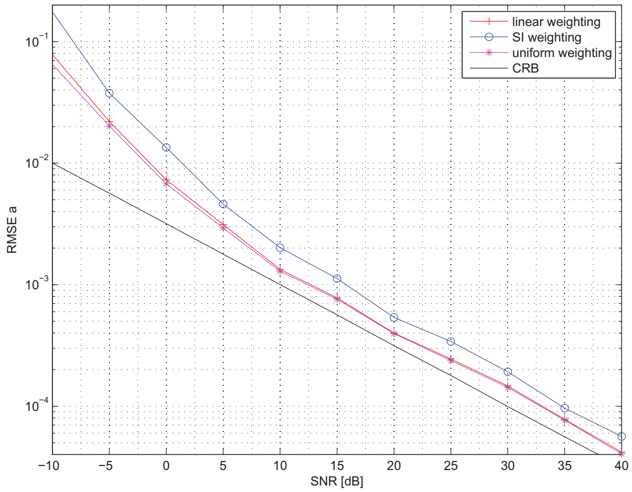

Fig 1: RMSA of a versus SNR

e.g. from the P smallest generalized eigenvalues of the block-companion matrix pair associated with $\hat{\boldsymbol{G}}(a)$ [9, 8].

Remark: As mentioned previously above, the coefficients $\boldsymbol{G}_k$ of of the MP in (11) are arranged in decreasing order $\boldsymbol{G}_0 \geq \boldsymbol{G}_1 \geq \ldots \geq \boldsymbol{G}_{K-1}$. Thus from a numerical point of view it is more convenient to compute the roots of the reverse polynomial, i.e. $\boldsymbol{G}_{\mathrm{rev}}(a) = a^{K-1}\boldsymbol{G}(1/a) = \sum_{k=0}^{K-1} \boldsymbol{G}_{K-1-k}a^k$ and then estimate the true generator from the inverse of the P largest roots of the reverse MP.

6. SIMULATION RESULTS

We compare the estimation performance of the classical SI-root ESPRIT [1] with the weighted WMI-root ESPRIT algorithm for equal weights $c_1, \ldots, c_{K-1}$ and for linearly decreasing weights (between 1 and 0.1). We assume two 2D damped harmonics with parameters given as $a_1 = 0.801 + 0.391j$, $a_2 = 0.838 + 0.356j$, $b_1 = 0.681 + 0.532j$ and $b_2 = 0.692 + 0.623j$. The sample support is $K = 8$, $L = 8$ and $N = 100$. The root-mean-square errors (RMSEs) of the estimated real part of the generators $\hat{a}_1$ and $\hat{a}_2$ are averaged and plotted versus the SNR in Figs. 1. All results are averaged over 100 independent simulation runs and compared to the corresponding Cramer-Rao bound (CRB). From the simulation we observe that the proposed algorithm performs equally well for uniform and for linear weighting, with slight preference to uniform weighing. Further the WMI root-ESPRIT outperforms the SI root-ESPRIT both for the uniform and the linearly decreasing weighting case.

7. CONCLUSIONS

In this paper a novel algorithm for estimating (MD) damped harmonics has been proposed that exploits multiple invariance contained in the data model. Unlike previous methods that rely on joint eigendecomposition of multiple shift invariance matrices, a different approach is taken here. The invariance matrices for different shifts are combined by a weighted summation after appropriate matrix multiplication. The damped harmonics are then obtained from the roots of a matrix polynomial. Necessary and sufficient conditions are derived that ensure uniqueness of the parameter estimation and reveal a seamless link between rooting-based harmonic estimation methods that exploit multiple invariance and (joint) eigendecomposition based methods that only exploit a single-shift invariance.

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