MIMO MULTI-CHANNEL BEAMFORMING IN DOUBLE-SCATTERING CHANNELS

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ABSTRACT

In this paper, we investigate the performance of multipleinput multiple-output (MIMO) multi-channel beamforming (MB) systems in double-scattering channels. In particular, we first derive new expressions for the marginal ordered eigenvalue distributions of the double-scattering channel matrices. Based on these results, we present analytical expressions for the symbol error rate of MIMO MB. It is demonstrated that MB sub-channels can achieve full spatial diversity even in the presence of double scattering, if the number of effective scatterers is greater than or equal to the maximum number of transmit and receive antennas.

Index Terms- Beamforming, eigenvalues, distribution

1. INTRODUCTION

Multiple-input multiple-output (MIMO) multi-channel beamforming (MB) systems, designed to transmit along the eigendirections of the MIMO channel, have been shown to be the optimal linear solution under many performance criteria of interest [1]. The performance of MIMO MB depends directly on the statistical characteristics of the ordered eigenvalues of the random MIMO channel matrix. This has recently led researchers to study the ordered eigenvalue distributions of various classes of MIMO channel models (see eg. [2–5]).

Existing work in this area, however, has focused mainly on channel scenarios for which the scattering is rich enough to render *full-rank* channel matrices. Recently, it has been shown that the channel may in fact exhibit rank-deficient behavior, due to a lack of physical scattering. This has led to the development of the *double-scattering* model [6]. This model, described by a product of complex random matrices, is a generalized representation which embraces the rich-scattering (eg. uncorrelated Rayleigh) and completely rank-deficient (ie. pinhole) channel as special cases. For the double-scattering model, there are currently very few prior results. In the context of MIMO MB, the only related results are presented in [7–9], which considered the special case of single-channel beamforming.

In this paper, we investigate the performance of MIMO MB under double-scattering channels with spatial correlation

at the receiver (or transmitter). Based on the new Cauchy-Binet Formula [5], we extend the marginal distribution of unordered eigenvalue [10] and the largest eigenvalue [7,9] to that of all ordered eigenvalues. In particular, we derive exact and asymptotic expressions for the marginal cumulative distribution functions (CDF) of all the ordered eigenvalues of the double-scattering channel model. Based on these CDFs, the analytical symbol error rate (SER) is also presented. Our results demonstrate that full diversity order is achieved even in the presence of double-scattering as long as the number of scatterers is not less than the maximum number of transmit and receive antennas.

2. CHANNEL MODEL AND PROBLEM FORMULATION

We consider double-scattering channels with spatial correlation at either the transmitter or receiver. Note that, due to the reciprocity of MIMO channel, correlation at either the receiver or transmitter is equivalent. As a result, we only consider receive correlation in the paper. The channel model is

$$\boldsymbol{H} = \frac{1}{\sqrt{N_s}} \boldsymbol{\Sigma}^{1/2} \boldsymbol{H}_1 \boldsymbol{H}_2 \tag{1}$$

where $\boldsymbol{H}_1 \in \mathbb{C}^{N_r \times N_s}$ and $\boldsymbol{H}_2 \in \mathbb{C}^{N_s \times N_t}$ are mutually independent complex Gaussian matrices with independent and identically distributed (i.i.d.), zero-mean, unit-variance elements, and $\boldsymbol{\Sigma} \in \mathbb{C}^{N_r \times N_r}$ is a Hermitian positive definite matrix. N_r and N_t denote the numbers of receive and transmit antennas, N_s denotes the number of effective scatterers, and we also use the following notations:

$$W_{1} = H_{1}^{H} \Sigma H_{1}$$

$$W_{2} = N_{s} H^{H} H = H_{2}^{H} W_{1} H_{2}$$

$$S = \min(N_{r}, N_{s}), \quad T = \max(N_{r}, N_{s})$$

$$M = \min(S, N_{t}), \quad N = \max(S, N_{t})$$

$$\sigma = (\sigma_{1}, \dots, \sigma_{N_{r}}) = \operatorname{eigs}(\Sigma), \quad \sigma_{1} \ge \dots \ge \sigma_{N_{r}}$$

$$\eta = (\eta_{1}, \dots, \eta_{S}) = \operatorname{eigs}(W_{1}), \quad \eta_{1} \ge \dots \ge \eta_{S}$$

$$\lambda = (\lambda_{1}, \dots, \lambda_{M}) = \operatorname{eigs}(W_{2}), \quad \lambda_{1} \ge \dots \ge \lambda_{M}$$

where eigs(A) denote the non-zero ordered eigenvalues of A. For a MIMO MB system, the received vector is given by

$$s = HPd + n, \qquad (2)$$

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where $\boldsymbol{H} \in \mathbb{C}^{N_r \times N_t}$ is defined in (1), $\boldsymbol{P} \in \mathbb{C}^{N_t \times L}$ is the pre-coding matrix which maps the $L \leq M$ modulated data symbols $\boldsymbol{d} = (d_1, \ldots, d_L)^T$ onto the N_t transmit antennas, and $\boldsymbol{n} = (n_1, \ldots, n_{N_r})$ is an additive white Gaussian noise (AWGN) vector, with elements having zero mean and unit variance. We assume that each modulated symbol has fixed power ρ_k ($k = 1, \ldots, L$), and that perfect channel state information is available at both the transmitter and receiver. By weighting the received vector \boldsymbol{s} with a (spatial) equalizing matrix $\boldsymbol{Q} \in \mathbb{C}^{N_r \times L}$, we get the statistic decision variables $\hat{\boldsymbol{d}} = \boldsymbol{Q}^H \boldsymbol{s}$, where $\hat{\boldsymbol{d}} = (\hat{d}_1, \ldots, \hat{d}_L)$. It is shown in [1] that by choosing the eigenvectors corresponding to the L largest eigenvalues of $\boldsymbol{H}^H \boldsymbol{H}$ as the columns of the pre-coding and equalizing matrices, the MIMO channel can be decomposed into L parallel scalar (eigen-mode) sub-channels. Each of these sub-channels is described by

$$\hat{d}_k = \sqrt{\lambda_k} \, d_k / N_s + n_k, \quad k = 1, \dots, L \tag{3}$$

The instantaneous signal-to-noise-ratio (SNR) of each subchannel is

$$\gamma_k = \rho_k \lambda_k / N_s, \quad k = 1, \dots, L. \tag{4}$$

Obviously, the MIMO MB system performance depends directly on the marginal distribution of the ordered eigenvalues λ_k . These will be derived in the following section.

3. EIGENVALUE DISTRIBUTIONS OF DOUBLE-SCATTERING CHANNELS

We start by deriving the joint probability density function (PDF) of λ . This will be used subsequently to derive the marginal CDFs of the ordered eigenvalues $\lambda_k, k = 1, ..., M$.

Lemma 1 (Joint PDF). $(x_1 \ge \ldots \ge x_M \ge 0)$

$$f_{\boldsymbol{\lambda}}(\boldsymbol{x}) = \mathcal{K}_1 \mathcal{K}_2 \left| \boldsymbol{K}_{N_r}(\boldsymbol{x}, \boldsymbol{\sigma}) \right| \left| \boldsymbol{V}_M(\boldsymbol{x}) \right| \prod_{i=1}^M x_i^{N-S}$$
(5)

where

$$\begin{aligned} \mathcal{K}_{1} &= \frac{(-1)^{(N_{r}-S)(N_{r}+S-1)/2}}{\left[\prod_{i=1}^{S}(N_{s}-i)!\right] \left[\prod_{i=1}^{N_{r}}\sigma_{i}^{N_{s}}\right] |\boldsymbol{U}_{N_{r}}(\boldsymbol{\sigma})} \\ \mathcal{K}_{2} &= \frac{(-1)^{(S-M)(S+M-1)/2}}{\prod_{i=1}^{M}(N_{t}-i)!} \\ \{\boldsymbol{U}_{N_{r}}(\boldsymbol{\sigma})\}_{i,j} &= (-1/\sigma_{i})^{j-1}, \quad i,j = 1, \dots, N_{r}. \\ \{\boldsymbol{V}_{M}(\boldsymbol{x})\}_{i,j} &= x_{i}^{j-1}, \qquad i,j = 1, \dots, M. \end{aligned}$$

$$\{\boldsymbol{K}_{N_{r}}(\boldsymbol{x},\boldsymbol{\sigma})\}_{i,j} = \begin{cases} 2(\sigma_{i}x_{j})^{(N_{s}-N_{t})/2}K_{N_{s}-N_{t}}(2\sqrt{x_{j}}/\sigma_{i}), \\ i = 1, \dots, N_{r}; j = 1, \dots, M. \\ (-1)^{S-j}\sigma_{i}^{T+j-N_{r}-N_{t}}(T+j-N_{r}-N_{t}-1)!, \\ i = 1, \dots, N_{r}; j = M+1, \dots, S. \\ (-1/\sigma_{i})^{N_{r}-j}, \\ i = 1, \dots, N_{r}; j = S+1, \dots, N_{r}. \end{cases}$$

where $K_a(x)$ is the modified Bessel function of the second kind [11].

Proof. See Appendix A for a sketch of the proof. \Box

Theorem 1 (Marginal CDF). $(z \ge 0, 1 \le k \le M)$

$$F_{\lambda_k}(z) = \mathcal{K}_1 \mathcal{K}_2 \sum_{l=0}^{k-1} \sum_{\beta_1 < \dots < \beta_{k-l-1} \atop \beta_{k-l} < \dots < \beta_M} |\mathbf{\Upsilon}_{N_r}(z, \boldsymbol{\beta})| \quad (6)$$

where the second summation is over the combination of $(\beta_1 < \cdots < \beta_{k-l-1})$ and $(\beta_{k-l} < \cdots < \beta_M)$, with $\beta = (\beta_1, \ldots, \beta_M)$ being a permutation of $(1, \ldots, M)$, and $\Upsilon_{N_r}(z, \beta)$ is given in (7) at the top of next page, with $i, j = 1, \ldots, N_r$.

Proof. See Appendix B for a sketch of the proof. \Box

In addition to the exact expressions for the marginal ordered eigenvalue CDFs, it is also of importance to consider asymptotic expansions of the CDFs around the origin.

Theorem 2 (Asymptotic Expression). $(z \ge 0, 1 \le k \le M)$

$$F_{\lambda_k}(z) = c_k z^{(N+1-k)(M+1-k)} + o(z^{(N+1-k)(M+1-k)})$$
(8)

where c_k is a positive constant.

Proof. Omitted due to space constraint. The proof will be presented in an extended journal version of this paper. \Box

4. PERFORMANCE ANALYSIS OF MIMO MB SYSTEMS IN DOUBLE-SCATTERING

The average SER of many general modulation formats (BPSK, BFSK, *M*-PAM, etc.) can be expressed as [2] SER = $\mathbb{E}_{\gamma} \left[aQ(\sqrt{2b\gamma}) \right]$, where γ is the average receive SNR, $Q(\cdot)$ is the Gaussian *Q*-function, and *a* and *b* are modulation-specific constants (eg. for BPSK, a = 1, b = 1). Thus, the SER of the *k*-th sub-channel of MIMO MB can be expressed as (k = 1, ..., L)

$$\operatorname{SER}_{k} = \frac{a\sqrt{b}}{2\sqrt{\pi}} \int_{0}^{\infty} x^{-1/2} e^{-bx} F_{\lambda_{k}}(N_{s} x/\rho_{k}) \, dx \quad . \tag{9}$$

Substituting (6) into (9), we obtain the analytical SER of each sub-channel. Evaluating a closed-form solution for the above expression seems difficult, however, Eq. (9) can be evaluated numerically, which is more efficient than by using Monte Carlo simulation methods. Additionally, since independent symbols are sent on each sub-channel, the global SER (i.e., the average SER of all sub-channels) can be obtained as [2]: $SER_{global} = \sum_{k=1}^{L} SER_k/L$.

To obtain further insights, we now investigate the subchannel SER at high SNR. For (9), it is obvious that as $\rho_k \rightarrow \infty$, the SER is dominated by the behavior of $F_{\lambda_k}(z)$ around the origin (i.e., z = 0). As such, substituting (8) into (9),

$$\{ \boldsymbol{\Upsilon}_{N_{r}}(z,\beta) \}_{i,j} = \\ \begin{cases} (N-S+j-1)! \sum_{p=0}^{N-S+j-1} \frac{2 \, \sigma_{i}^{(N_{s}-N_{t}+N-S+j-p)/2}}{p!} z^{\frac{N_{s}-N_{t}+N-S+j+p}{2}} K_{N_{s}-N_{t}+N-S+j-p} \left(2\sqrt{z/\sigma_{i}} \right), j = \beta_{1}, \dots, \beta_{k-l-1}. \\ (N-S+j-1)! \left[\sigma_{i}^{N_{s}-N_{t}+N-S+j} (N_{s}-N_{t}+N-S+j-1)! - \sum_{p=0}^{N-S+j-1} \frac{2 \, \sigma_{i}^{(N_{s}-N_{t}+N-S+j-p)/2}}{p!} z^{\frac{N_{s}-N_{t}+N-S+j+p}{2}} K_{N_{s}-N_{t}+N-S+j-p} \left(2\sqrt{z/\sigma_{i}} \right) \right], j = \beta_{k-l}, \dots, \beta_{M}. \\ (-1)^{S-j} \sigma_{i}^{T+j-N_{r}-N_{t}} (T+j-N_{r}-N_{t}-1)!, j = M+1, \dots, S. \\ (-1/\sigma_{i})^{N_{r}-j}, j = S+1, \dots, N_{r}. \end{cases}$$

we have the following result on the diversity order of the k-th sub-channel (k = 1, ..., L)

$$d_k = (N+1-k)(M+1-k).$$
(10)

It was shown that [2], for the $N_r \times N_t$ MIMO MB systems under rich (ie. full-rank) scattering, the diversity order of the *k*-th sub-channel is $d_k = (N_r + 1 - k)(N_t + 1 - k)$. Here, we find that if the number of effective scatterers is greater than or equal to the maximum number of transmit and receive antennas (i.e., $N_s \ge \max(N_r, N_t)$), then each sub-channel achieves the same (maximum possible) diversity order, even in the presence of double scattering.

Next, numerical simulations are carried out to verify our derivations. The following channel is used to give a simple example: $N_r = 2, N_s = 5, N_t = 3$. Since the result in this paper can be applied to arbitrary correlation, here we use the *exponential correlation model* [12] to give a simple illustration, i.e., $\{\Sigma\}_{i,j} = \delta^{|i-j|}$ for $\delta \in [0,1), i, j = 1, \ldots, N_r$. The SER of each sub-channel of a MIMO MB system is plotted in Fig. 1, where coherent BPSK and equal power allocation are assumed. The analytical results are calculated from (9), while the simulated curves are generated based on 10^6 channel realizations. As Fig. 1 illustrates, the two results fit very well. Furthermore, diversity orders of 6 and 2 are observed, as predicted by (10).

5. CONCLUSION

We investigated the performance of MIMO MB systems under double-scattering channels. Our results were based on new expressions which we presented for the marginal CDFs of the ordered eigenvalues of the double-scattering random channel matrix. We presented analytical SER expressions, and demonstrated the interesting result that if $N_s \geq \max(N_r, N_t)$, then full diversity order is obtained even in the presence of double-scattering.

A. PROOF SKETCH OF LEMMA 1

Lemma 2 ([13], [14]). Let $W = H_w^H \Theta H_w$, (where $H_w \in \mathbb{C}^{r \times t}$ is a complex Gaussian matrix with statistically independent, zero-mean, unit-variance elements, $\Theta \in \mathbb{C}^{r \times r}$ is



Fig. 1. Sub-channel SER of the MIMO MB system ($\delta = 0.5$)

a Hermitian, positive definite, deterministic matrix), $m = \min(r, t)$, and $n = \max(r, t)$. Also, let $\boldsymbol{w} = (w_1, \ldots, w_m)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_r)$ denote the non-zero ordered eigenvalues of \boldsymbol{W} and $\boldsymbol{\Theta}$, respectively. Then, the joint PDF of \boldsymbol{w} can be expressed as $(x_1 \ge \cdots \ge x_m \ge 0)$

$$f_{\boldsymbol{w}}(\boldsymbol{x}) = \mathcal{K} |\boldsymbol{E}_{r}(\boldsymbol{x}, \boldsymbol{\theta})| |\boldsymbol{V}_{m}(\boldsymbol{x})| \prod_{i=1}^{m} x_{i}^{n-r}$$
(11)

where

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$$\begin{split} \mathcal{K} &= \frac{(-1)^{(r-m)(r+m-1)/2}}{[\prod_{i=1}^{m}(t-i)!] [\prod_{i=1}^{r}r_{i}^{t}] |\boldsymbol{U}_{r}(\boldsymbol{r})|} \\ \{\boldsymbol{U}_{r}(\boldsymbol{\theta})\}_{i,j} &= (-1/\theta_{i})^{j-1}, \quad i, j = 1, \dots, r. \\ \{\boldsymbol{V}_{m}(\boldsymbol{x})\}_{i,j} &= x_{i}^{j-1}, \quad i, j = 1, \dots, m. \\ \boldsymbol{E}_{r}(\boldsymbol{x},\boldsymbol{\theta})\}_{i,j} &= \begin{cases} e^{-x_{j}/\theta_{i}}, \ i = 1, \dots, r; j = 1, \dots, m. \\ (\frac{-1}{\theta_{i}})^{r-j}, \ i = 1, \dots, r; j = m+1, \dots, r. \end{cases} \end{split}$$

Lemma 2 is a simple unification of the results of [13] and [14]. Specifically, if $r \le t$, it reduces to (17) of [13]; if r > t, it reduces to (25) of [14] (after some algebraic manipulations). Given Lemma 2, we can derive the joint distribution of λ by applying the following three steps:

(i) Obtain the joint PDF of the non-zero ordered eigenvalues η of W_1 by directly applying Lemma 2;

(ii) Evaluate the joint PDF of λ , conditioned on η . To this end, note that if W_1 is rank deficient, (i.e., $N_r < N_t$), one can prove that λ are the eigenvalues of $\tilde{H}_2^H D_{W_1} \tilde{H}_2$, where $D_{W_1} = \text{diag}(\eta), \tilde{H}_2 \in \mathbb{C}^{N_t \times S}$ is a complex Gaussian matrix with i.i.d., zero-mean, unit-variance elements. Thus, the joint PDF of λ , conditioned on η , can again be obtained via application of Lemma 2.

(iii) Evaluate the unconditional joint PDF of λ by integrating the conditional PDF from Step (ii) with respect to the distribution of η from Step (i). This integration can be carried out with the help of the generalized Cauchy-Binet type integral identities (see eg. [13]).

B. PROOF SKETCH OF THEOREM 1

According to the definition of the marginal CDF, we have

$$F_{\lambda_k}(z) = \Pr(z \ge \lambda_k) = \sum_{l=0}^{k-1} \Pr(\lambda_1 \ge \dots \ge \lambda_{k-l-1} \ge z$$
$$\ge \lambda_k \ge \dots \ge \lambda_m \ge 0) = \sum_{l=0}^{k-1} \int_{\mathcal{D}_l} f_{\lambda}(x) \, dx \tag{12}$$

where $D_l = \{\infty > x_1 \ge \cdots \ge x_{k-l-1} \ge z \ge x_{k-l} \ge \cdots \ge x_m \ge 0\}$. Next, we use the new generalized Cauchy-Binet formula from [5, Lemma 1] to integrate as follows

$$\begin{split} \int_{\mathcal{D}_{l}} f_{\lambda}(x) \, dx &= \sum_{\substack{\beta_{1} < \cdots < \beta_{k-l-1} \\ \beta_{k-l} < \cdots < \beta_{M}}} \\ \int_{z}^{\infty} 2(\sigma_{i}x)^{(N_{s}-N_{t})/2} K_{N_{s}-N_{t}}(2\sqrt{x/\sigma_{i}}) x^{N-S+j-1} \, dx, \\ j &= \beta_{1}, \dots, \beta_{k-l-1}, \\ \int_{0}^{z} 2(\sigma_{i}x)^{(N_{s}-N_{t})/2} K_{N_{s}-N_{t}}(2\sqrt{x/\sigma_{i}}) x^{N-S+j-1} \, dx, \\ j &= \beta_{k-l}, \dots, \beta_{M}, \\ (-1)^{S-j} \sigma_{i}^{T+j-N_{r}-N_{t}} (T+j-N_{r}-N_{t}-1)!, \\ j &= M+1, \dots, S, \\ (-1/\sigma_{i})^{N_{r}-j}, \qquad j = S+1, \dots, N_{r}. \end{split}$$

Combining [11, Eq. (8.432.6)] and [11, Eq. (8.350.1)], we evaluate closed-form solutions for the remaining integrals in the above equation, which completes the proof.

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