# PACKETIZED VIDEO TRANSMISSION FOR OFDM WIRELESS SYSTEMS WITH DYNAMIC ORDERED SUBCARRIER SELECTION ALGORITHM

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## ABSTRACT

In this paper, we proposed a dynamic ordered subcarrier selection algorithm (DOSSA) for OFDM based video transmission system. The proposed scheme is shown to achieve lower bit error rate (BER) than the previously proposed OSSA by first selecting a fraction of the subcarriers with highest channel gain. The content information is then exploited in order to extend the OSSA to achieve unequal error protection (UEP) for packets of different importance. Simulation results show that system that utilizes the proposed scheme can achieve higher PSNR, especially at low SNR, compared to those that use the equal error protection (EEP) OSSA.

Index Terms- OFDM, OSSA, DOSSA, UEP, cross-layer design

### I. INTRODUCTION

Multicarrier systems, such as Orthogonal Frequency Division Multiplexing (OFDM), has been used extensively in a variety of wireless communications protocols because of its ability to achieve high data rate using low-complexity transceiver. This has proliferated video communications using mobile devices, making it possible to send high quality video at anytime and anywhere.

In video communications, the video is first compressed and then packetized before it is transmitted across a fading channel. Besides utilizing error concealment techniques at the source coder, the design of the transmission algorithm also plays a crucial role in increasing reliability, as well as throughput, of the transmitted data. Traditionally, the task of designing the source coder and transmission scheme can remain separate due to the abstraction provided by the Open Systems Interconnection (OSI) 7-layer model [1]. This model attempts to abstract common features that are common to all approaches in data communications and organize them into layers or modules such that each layer only worries about the layer directly above it and the one directly below it. This alleviates designers of the intricacy of the other layers. This model has worked well in the past when the parameters of the communication link remain static, which is not the case for mobile communications. As a result, suboptimal performance is often encountered when systems based on this model are deployed. This has led to the development of cross-layer design, which if designed appropriately, can lead to increased transmission efficiency and reliability.

Several cross-layer design schemes have been previously proposed for video communications using OFDM based on the Ordered Subcarrier Selection Algorithm (OSSA) proposed in [2]. [3] exploited unequal error protection (UEP) techniques during

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channel coding in order to assign different levels of importance for layered video. However, there is no optimization involved for the assignment of code rate to a corresponding video layer. [4] jointly optimized the diversity of channel gain among different subcarriers, which are selected by the OSSA, and channel coding rates with layered video of different importance. However, it requires more accurate channel state information (CSI) feedback. The OSSA selects the top strongest subcarriers for data transmission. The advantage of the OSSA is that it has low implemenation complexity due to the fact that power allocation and bit-loading are uniform across all the selected subcarriers. Moreover, since the OSSA requires less CSI compared to other adaptive bit-loading algorithms or waterfilling based power allocation algorithms, the impact of delayed CSI feedback is alleviated. However, it does not guarantee an efficient usage of channel capacity and it cannot tradeoff between link reliability and transmission rate, which is important for video communications.

In this paper, we proposed a novel subcarrier selection scheme for video communications based on the OSSA which can tradeoff link reliability and transmission rate by taking into account the content of the transmitted data. This is achieved by releasing the constraint in the OSSA that a fixed number of selected subcarriers for data transmission can only be used. Based on this premise, we proposed a Dynamic OSSA (DOSSA) to dynamically assign the number of selected subcarriers during video transmission according to the importance of each packet. The dynamic assignment is formulated to exploit UEP so that end-to-end video distortion is minimized.

The paper is organized as follows. A detailed description of the proposed DOSSA will be described in Section II, followed by simulation results in Section III. The paper will be concluded in Section IV.

# **II. THE PROPOSED ALGORITHM**

In the derivation below, we shall assume that all video sequences are in CIF (352x288 pixels) format and are encoded in the H.264 standard [5] with a frame rate of 30 frames per second. There is a total of I video frames in a group of pictures (GOP), where I = 8. In each GOP, the first frame is an I-frame, which is followed by  $N_P$ number of P-frames, where  $N_P = 7$ . These frames are packetized such that each slice only contains a single row of macroblocks (MBs). There are J slices per video frame, where J = 18 (a typical MB contains 16x16 pixels). We denote the  $j^{th}$  slice of the  $i^{th}$  frame as the  $(i, j)^{th}$  slice. For packetized video transmission, each packet only contains a single slice.

### **II-A. Bit-loading and Power Allocation**

We will first analyze the average BER when the top K strongest subcarriers are used together during data transmission. Assuming the channel coefficients are generated independently with a Rayleigh distribution, the channel gain  $\lambda_n = |H_n|^2$ , for n = $1, 2, \ldots, N$ , will consequently be exponentially distributed [6]. In addition, the fading process is normalized such that  $E[\lambda_n] = 1$ . According to the OSSA [2], the subcarriers are first ordered based on its channel gain in ascending order. The probability distribution of the channel gain,  $f_n(\lambda_n)$ , for  $\lambda_n$ ,  $n = 1, 2, \ldots, N$ , is derived in [2], [7]. Assuming M-QAM modulation with Gray coding is employed for all subcarriers, the BER for the  $n^{th}$  subcarrier can thus be approximated as [2], [6]

$$P_{b_n} \cong \frac{\sqrt{M} - 1}{\sqrt{M} \log_2 \sqrt{M}} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0} \frac{3 \log_2(M) R_c \lambda_n \eta}{2(M-1)}}\right), \quad (1)$$

where  $\eta \triangleq N/(N+N_{CP})$  is defined as the transmission efficiency. N denotes the FFT length, and  $N_{CP}$  denotes length of the cyclic prefix.  $R_c$  is the code rate with the use of channel coding. erfc(x) is the complementary error function, and  $E_b/N_0$  is the transmit SNR.

Under the total transmission power constraint and the uniform power allocation over the selected subcarriers, the transmission power per bit in our proposed DOSSA, denoted as  $E_{b,K}$ , is a function of K, which is written as

$$E_{b,K} = \frac{N}{K} \frac{\log_2(M_{ref})}{\log_2(M)} E_b, \tag{2}$$

where  $M_{ref}$  is the reference modulation for the reference system, i.e. a system that does not employ any subcarrier selection. Replacing  $E_b$  with  $E_{b,K}$  in (1), we can rewrite (1) as

$$P_{b_n,K} = \frac{\sqrt{M} - 1}{\sqrt{M}\log_2 \sqrt{M}} erfc\left(\sqrt{\frac{NE_b}{KN_0} \frac{3\log_2(M_{ref})R_c\lambda_n\eta}{2(M-1)}}\right).$$

Using  $f_n(\lambda_n)$ , the average BER can be written as [2], [7]

$$\bar{P}_{b_n,K} = \int_0^\infty P_{b_n,K} f_n(\lambda_n) d\lambda_n.$$
(3)

Finally, the average BER across the K strongest subcarriers can be obtained by averaging across these K subcarriers, and thus we have

$$\bar{P}_{b,K} = \frac{1}{K} \sum_{n=N-K+1}^{N} \bar{P}_{b_n,K}.$$

The tradeoff between K and the analytical result of  $\overline{P}_{b,K}$  for N = 64 is illustrated in Figure 1.

#### **II-B.** Objective Function

Since the probability of packet loss depends on the number of selected subcarriers, the expected value of the video distortion for the packet containing the  $(i, j)^{th}$  slice can be formulated as a function of  $K_{i,j}$ , where  $K_{i,j}$  denotes the top  $K_{i,j}^{th}$  strongest subcarriers which are used during the transmission of the  $(i, j)^{th}$  slice. We assume the encoder has full knowledge of the error concealment scheme utilized at the receiver, so the transmitter can estimate the distortion resulting from the packet loss during video transmission. We shall adopt the expected decoder side distortion model proposed in [8], [9], and thus, the expected decoder side distortion for the  $(i, j)^{th}$  slice can be written as

$$E[D_{K_{i,j}}] = P_{K_{i,j}}^{\ell} D_{i,j}^{C} \gamma + (1 - P_{K_{i,j}}^{\ell}) D_{i,j}^{Q}$$



Fig. 1. Analytical results: BER vs. number of selected subcarriers, K (K = 17 to 64), for N = 64, M = 16,  $M_{ref} = 4$ ,  $R_c = 1$  and  $\eta = 0.8$ .

where  $D_{i,j}^C$  is the concealment distortion if the packet containing the  $(i, j)^{th}$  slice is lost;  $D_{i,j}^Q$  is the quantization distortion if the packet containing the  $(i, j)^{th}$  slice is perfectly received; and  $\gamma$  is the error propagation factor that takes into account the distortion caused by the loss of packets. The value of  $\gamma$  is set to the number of frames before the arrival of the next I-frame.  $P_{K_{i,j}}^\ell$  is the probability of packet loss, which can be expressed as

$$P_{K_{i,j}}^{\ell} \approx 1 - (1 - \bar{P}_{b,K_{i,j}})^{L_{i,j}},$$

where  $L_{i,j}$  denotes the packet size in terms of the number of bits for the packets containing the  $(i, j)^{th}$  slice.

In order to minimize the end-to-end video distortion, we can in fact solve for the optimum value of K for different packet while maintaining the average transmission rate by minimizing the overall expected value of the video distortion. This can be expressed as

$$\min_{\mathbf{K},M} \sum_{i,j=1}^{I,J} E\left[D_{K_{i,j}}\right]$$
s.t.  $\left(\sum_{i,j=1}^{I,J} r_{i,j}/K_{i,j}\right)^{-1} = K_{avg},$   
 $1 \le K_{i,j} \le N, K_{i,j} \in \mathbb{Z},$  (4)

where  $\mathbf{K} = [K_{1,1}, K_{1,2}, \cdots, K_{1,J}, \cdots, K_{I,1}, \cdots, K_{I,J}]^T$  is a column vector containing  $K_{i,j}$  for all the corresponding slices in the GOP, and  $r_{i,j} = L_{i,j}/(\sum_{m,n=1}^{I,J} L_{m,n})$  is the size ratio of the packet containing the  $(i, j)^{th}$  slice in the GOP.  $K_{avg} = \frac{N \log_2(M_{ref})}{\log_2(M)}$  denotes the average number of subcarriers selected in the DOSSA and is a function of M in order to make the average transmission rate in the proposed algorithm to be the same as that of the reference system.

## **II-C.** Optimization

Since it is difficult to jointly optimize **K** and *M*, we like to simplify (4) by eliminating the variable *M*. This can be achieved by letting  $K_{avg}$  to be equal to the optimum value of *K* in [2], which is derived as  $K_{opt} = \frac{N \log_2(M_{ref})}{\log_2(4M_{ref})}$  with  $M = 4M_{ref}$ . Assuming that the reference modulation used is 4-QAM, i.e.  $M_{ref} = 4$ , then  $K_{opt}$  in [2] would be equal to N/2 and M = 16 in order to maintain the same transmission rate as the reference system. With

M = 16, (4) can be rewritten as

$$\min_{\mathbf{K}} \sum_{i,j=1}^{I,J} E\left[D_{K_{i,j}}\right]$$
s.t.  $\left(\sum_{i,j=1}^{I,J} r_{i,j}/K_{i,j}\right)^{-1} = \frac{N}{2},$   
 $1 \le K_{i,j} \le N, K_{i,j} \in \mathbb{Z}.$  (5)

Since all the elements in **K** are integers, (5) becomes a nonlinear, discrete constrained minimization problem. Since the feasible region for (5) is quite large, i.e.  $N^{IJ}$ , it is infeasible to search the entire region to obtain the global optimal solution. Hence, we propose to use the same concept as that in Lagrangian relaxation [10], [11] to first obtain an initial solution, which will then be applied to the *discrete first-order method* [12] in order to iteratively find a suboptimal solution.

To find the initial solution, we first assume that  $K_{i,j} \in \mathbb{R}, \forall i, j$ . From (5), we can obtain the Lagrangian

$$L(\mathbf{K},\lambda) = \sum_{i,j=1}^{I,J} E[D_{K_{i,j}}] + \lambda \left(\sum_{i,j=1}^{I,J} r_{i,j}\widetilde{K}_{i,j} - \frac{2}{N}\right), \quad (6)$$

where  $\lambda$  is the Lagrange multiplier, and  $\widetilde{K}_{i,j}$  is the multiplicative inverse of  $K_{i,j}$ . Taking derivatives with respect to all  $K_{i,j}$ 's and setting the result to 0, i.e.  $\nabla_{\mathbf{K}} L(\mathbf{K}, \lambda) = 0$ , we obtain

$$-\frac{P_{K_{i,j}}^{\prime\prime}}{\widetilde{K}_{i,j}^{\prime}} \cong \lambda \frac{r_{i,j}}{(D_{i,j}^C \gamma - D_{i,j}^Q)}, \,\forall K_{i,j},$$
(7)

where  $P_{K_{i,j}}^{\prime\ell}$  and  $\widetilde{K}_{i,j}^{\prime}$  denote the first-order difference of  $P_{K_{i,j}}^{\ell}$ and  $\widetilde{K}_{i,j}$ , respectively.

From the above, we observed the following: 1) in order to satisfy the condition in (7),  $\lambda$  is bounded in the interval  $(0, \lambda_{max}]$ , where

$$\lambda_{max} = \min_{i,j} \left\{ \max_{1 \le K_{i,j} \le N} - \frac{P_{K_{i,j}}'}{\tilde{K}_{i,j}'} \frac{(D_{i,j}^C \gamma - D_{i,j}^Q)}{r_{i,j}} \right\},$$
(8)

and 2) for a given  $\lambda$ , which is smaller than  $\lambda_{max}$ , there are at least one and at most two suitable  $K_{i,j}$ 's that satisfy (7). This can be shown in Figure 2 which shows  $-\frac{P_{k_{i,j}}^{K}}{\tilde{K}'_{i,j}}$  vs.  $K_{i,j}$ , for (i,j)=(1,1) and (i,j)=(8,18). From the figure, given  $\lambda^A$  in Figure 2a or  $\lambda^B$  in Figure 2b, we can find two possible solutions,  $K_{i,j}^L$  and  $K_{i,j}^R$ . On the other hand, given  $\lambda^C$  in Figure 2b, only a single possible solution,  $K_{i,j}^L$ , can be found. In that case,  $K_{i,j}^R$  will be equal to N, as indicated in Figure 2b, where N=64.

Although the solution space would be constrained to  $\mathbf{K}^{L}$  and  $\mathbf{K}^{R}$  given a specified  $\lambda$ , there are still two possible solutions for each  $K_{i,j}$ , i.e.  $K_{i,j}^{L}$  or  $K_{i,j}^{R}$ . Therefore, the first task is to check whether or not we can use only  $\mathbf{K}^{L}$  to derive a good initial point for  $\mathbf{K}$ . Defining  $\sigma \triangleq \left(\sum_{i,j=1}^{I,J} r_{i,j}/K_{i,j}^{L}\right)^{-1}$ , where  $\sigma$  is monotonically increasing with  $\lambda$ . If  $\sigma > N/2$  with  $\lambda = \lambda_{max}$ , this implies that the bisection method can be used to quickly find a  $\lambda$  such that  $\sigma$  is as close to N/2 as possible, thereby satisfying the constraint in (5). Thus, we can set  $\lambda^{0}$  equal to  $\lambda$  and  $\mathbf{K}^{0}$  to be  $\mathbf{K}^{L}$  as initial solution for  $\mathbf{K}$  (this usually happens at high SNR).

If  $\mathbf{K}^L$  alone does not serve as a good initial point, i.e.  $\sigma < N/2$ , then this implies that we have to take both  $\mathbf{K}^L$  and  $\mathbf{K}^R$  into consideration (this usually happens at low SNR, in which case less subcarriers will be assigned in order to offer more protection to the important packets) when deriving a solution for  $\mathbf{K}$  since



**Fig. 2.**  $-\frac{P_{K_{i,j}}^{\prime\prime}}{K_{i,j}}$  vs.  $K_{i,j}$ : (a) (i, j) = (1, 1), (b) (i, j) = (8, 18). The slice in (a) is from I-frame. The slice in (b) is from P-frame. N = 64 and SNR = 15 dB.

the elements of **K** will be made up of elements from both  $\mathbf{K}^L$ and  $\mathbf{K}^R$ . First, we assign  $\lambda^0$  to be  $\lambda_{max}$ . This is followed by the selection of  $K_{i,j}^L$  and  $K_{i,j}^R$  to be elements of **K**, which is based on the importance of the packets. This is defined as the expected value of the distortion normalized by the slice size. Thus, the importance of the  $(i, j)^{th}$  slice is determined as  $\frac{E[D_{i,j}]}{r_{i,j}}$ . The importance of each slice is ordered so that  $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{IJ-1} \leq \alpha_{IJ}$ , with  $\alpha_n = \frac{E[D_{\alpha(n)}]}{r_{\alpha(n)}}$ , for  $n = 1, 2, \dots, IJ$ , where o(n) is a oneto-one mapping function between the ordered index n and the index pair (i, j) for the packet containing the  $(i, j)^{th}$  slice. Similarly, we define  $o^{-1}(i, j)$  as the inverse mapping of index pair (i, j) to the ordered index n. With the importance defined, a threshold,

$$\beta_T = \operatorname{argmin}_{1 \le \beta \le IJ} \left| \left( \sum_{n=1}^{\beta - 1} \frac{r_{o(n)}}{K_{o(n)}^R} + \sum_{n=\beta}^{IJ} \frac{r_{o(n)}}{K_{o(n)}^L} \right)^{-1} - \frac{N}{2} \right|, \quad (9)$$

is used to determine which elements of  $\mathbf{K}^L$  and  $\mathbf{K}^R$  should be selected to be used in  $\mathbf{K}$ . Specifically,  $K_{i,j}^R$  is assigned to  $K_{i,j}^0$  if the ordering index n is smaller than  $\beta_T$ , otherwise  $K_{i,j}^L$  is assigned to  $K_{i,j}^0$ , i.e.

$$\mathbf{K}_{i,j}^{0} = \begin{cases} K_{i,j}^{R}, & o^{-1}(i,j) < \beta_{T}, \\ K_{i,j}^{L}, & otherwise. \end{cases}$$

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Note that  $\beta_T$  is obtained as in (9) in order to make  $\left(\sum_{i,j=1}^{I,J} r_{i,j}/K_{i,j}\right)^{-1}$  to be as close to N/2 as possible.

Once  $\mathbf{K}^0$  is found, we can manipulate (6) to find a suboptimal solution for  $\mathbf{K} \in \mathbb{Z}^{IJ}$  by applying the theory of discrete Lagrangian [12], which transforms the Lagrangian in (6) to a discrete Lagrangian function. This is done by choosing a non-negative transformation function,  $H(\cdot)$ , and applying it to the constraint function  $h(\cdot)$ . Thus, (5) can be reformulated as

$$L_{d}(\mathbf{K}, \lambda) = f(\mathbf{K}) + \lambda H(h(\mathbf{K}))$$
$$= \sum_{i,j=1}^{I,J} E[D_{K_{i,j}}] + \lambda \left| \left( \sum_{i,j=1}^{I,J} r_{i,j} \widetilde{K}_{i,j} - \frac{2}{N} \right) \right|, \quad (10)$$

where  $H(\cdot)$  is the absolute value function. (10) can now be solved using the iterative *discrete first-order method* proposed in [12], resulting in two recursive equations for **K** and  $\lambda$  which are written as

$$\mathbf{K}^{p+1} = \mathbf{K}^{p} + \Delta_{\mathbf{K}} L_d(\mathbf{K}^{p}, \lambda),$$
  
$$\lambda^{p+1} = \lambda^{p} + cH(h(\mathbf{K}^{p})).$$



Fig. 3. Performance comparison between OSSA and DOSSA, with frame index from 1 to 24 at transmit SNR = 15 dB.

 $\Delta_{\mathbf{K}} L_d(\mathbf{K}^p, \lambda)$  is called the *direction of maximum potential drop* [12], which can be derived by searching around the neighborhood of  $\mathbf{K}^p$ . p is the iteration index, c denotes the step size with c > 0. With the non-negative transformation function  $H(\cdot)$ ,  $\lambda$  is guaranteed to increase as p increases since  $\lambda$  would increase by  $c|h(\mathbf{K})|$  at each iteration. The iteration process would converge when  $h(\mathbf{K}) < \epsilon$  is satisfied, where  $\epsilon$  denotes the specified tolerance for convergence [12].

#### **III. SIMULATION RESULTS**

In all of the simulations, we set N = 64 and  $N_{CP} = 16$ . 10 OFDM symbols per transmission frame were used. A bandwidth of 20 MHz was employed.  $R_C$  is equal to 1 because no channel coding was used. In this case,  $\eta = 0.8$ . The channel coefficients are directly generated in the frequency domain. We assume that sufficient cyclic prefix has been added to the system (i.e. channel order  $\leq N_{CP}$ ) so that no ISI and ICI are incurred. The modulation level is 16-QAM for both OSSA and DOSSA based systems. The tolerance parameter,  $\epsilon$ , is set to  $10^{-3}$ .

[3], [4] were not used in our simulations for performance comparison with our proposed scheme because both approaches are based on layered video architecture, which is different from our packet-based video format. In addition, it is infeasible to extend the channel-coding-based UEP scheme to our system, since channel coding cannot support such fine-grained levels of protection for different packets as the proposed DOSSA based system can.

The video sequence used in the simulation is the first 24 frames of a video sequence "Stefan". Figure 3 shows that the video quality of DOSSA is much better than that of OSSA in the first few frames of GOP. The video quality degrades due to the error propagation incurred by packet loss. Although the performance of DOSSA degrades more rapidly than that of OSSA, the average PSNR for DOSSA is still higher than that of OSSA system, as shown in Figure 4 at SNR = 15 dB. The average PSNR of DOSSA is 3.78 dB better than that of OSSA. Figure 4 also shows that the DOSSA outperforms OSSA, especially at low SNR. Clearly, the proposed DOSSA is able to outperform the OSSA in terms of PSNR because the DOSSA exploits knowledge about the video content; allowing it to better assign channel resources for transmission of the video.

#### **IV. CONCLUSION**

In this paper, we proposed the DOSSA that can easily achieve fine-grained UEP levels to serve packets of different importance.



**Fig. 4**. Performance comparison between OSSA and DOSSA, with the PSNR being measured across the first 24 video frames.

Simulation results shows that DOSSA can outperform the OSSA in terms of PSNR, especially at low SNR. Since the optimization problem for the DOSSA is nonlinear and has a large feasible region, we have also proposed an efficient technique for finding a suboptimal solution using the Lagrangian relaxation and discrete first-order methods.

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