TWO-DIMENSIONAL PHASE UNWRAPPING USING SEMIDEFINITE RELAXATION

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ABSTRACT

In many imaging applications, the continuous phase information of the measured signal is wrapped to a single period of 2π, resulting in phase ambiguity. In this paper we consider the two-dimensional phase unwrapping problem and propose a Maximum a Posteriori (MAP) framework for estimating the true phase values based on the wrapped phase data. In particular, assuming a joint Gaussian prior on the original phase image, we show that the MAP formulation leads to a binary quadratic minimization problem. The latter can be efficiently solved by semidefinite relaxation (SDR). We compare the performances of our proposed method with the existing L^1/L^2-norm minimization approaches. The numerical results demonstrate that the SDR approach significantly outperforms the existing phase unwrapping methods.

1. INTRODUCTION

Phase unwrapping is a classical problem which arises from many branches of applied physics and engineering, such as optical interferometry, magnetic resonance imaging, and synthetic aperture radar. In these applications, the true phase values of the reflected signals from an imaged object are "wrapped" in a common range of $[-\pi, \pi]$. This results in not only discontinuities in the measured phase values, but also ambiguities that are an integer multiples of 2π. Thus, a common problem in these applications is to restore the true phase values from the measured phase data which are wrapped and corrupted by noise. Such a process is called "phase unwrapping."

Phase unwrapping is essentially ill-posed due to the ambiguity resulted from the wrapping operator. To solve the phase unwrapping problem, additional properties need to be utilized in the unwrapping process. One commonly used property is the continuity of the true phase values. The continuity requires the phase differences of neighboring pixels to be within the range of half of the period (namely, π). When such condition is satisfied for noise-free data, the "aliasing" problem caused by the wrapping operator can be circumvented and the true phase values can be recovered up to a global constant. This is the so-called Itoh's approach, which was introduced in the early 1980's [1].

The continuity condition of the true phase can be violated in practice either due to abrupt changes in the underlying physical objects being imaged, or because of the impulsive measurement noise. As a result, "aliasing" problem still exists and ambiguity occurs when one tries to reverse the wrapping operation. Hence, other properties must be exploited in the formulation of the phase unwrapping problem. For instance, in a 2-dimensional (2-D) image, the gradient field satisfies the so-called network flow constraint [2]: the sum of gradient vectors along each loop in an image must be zero. Incorporating this network flow constraint, various phase unwrapping methods have been proposed in the literature. Among them, the most well known ones are the so called L^p-methods which aim to estimate the gradient field of the true phase image by minimizing an error function measured in L^p-norm [3]. For instance, due to its simplicity, the L^2-norm phase unwrapping has been well studied and widely used in many applications [4, 5, 6, 7, 8]. In addition to the L^p-norm approach, several other methods have also been proposed including belief propagation [9], network programming [10], and neural networks method [11].

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In this paper, we study the 2-D phase unwrapping problem. Assuming a joint Gaussian prior on the unwrapped phase value, we obtain a statistical model for the phase ambiguity and propose a Maximum a Posteriori (MAP) framework for estimating the true phase values from its wrapped counterpart. At each sampling point, the phase ambiguity is an integer multiple of the period 2π and therefore can be modeled by an integer constraint. As a result, the MAP framework formulation can be turned into a quadratic minimization problem with integer variables and a linear network flow constraint.

Due to the integer constraints, the resulting optimization problem is NP-hard [12, 13]. To facilitate numerical solution, we further simplify and reformulate this nonlinear integer programming problem, effectively reducing it to a binary quadratic minimization problem. We then apply the tools of semidefinite relaxation (SDR) approach introduced in [14, 15]. The SDR approach basically convexifies the binary quadratic minimization problem to a semidefinite programming that can be solved in polynomial time [16]. Moreover, a further randomization step is used to extract a binary solution from the optimal solution of the relaxed semidefinite programming. To demonstrate the effectiveness of our proposed approach, we compare its performance with the existing L^1/L^2-norm minimization approaches. The numerical results demonstrate that the SDR approach significantly outperforms these existing phase unwrapping methods.

2. 2-D PHASE UNWRAPPING PROBLEM

Assume that the principle period for the wrapped phase data is $(-\pi, \pi]$. The 2-D phase unwrapping problem is to restore the true phase $\boldsymbol{\phi}_{ij}$ (on a $M \times N$ grid) from the wrapped data $\boldsymbol{\psi}_{ij}$, which is $\boldsymbol{\phi}_{ij}$ shifted to the interval $(-\pi, \pi]$. In particular, we have

$$\boldsymbol{\psi}_{ij} = \boldsymbol{\phi}_{ij} - 2\pi \boldsymbol{k}_{ij}, \quad \text{for all } 1 \le i \le M,\ 1 \le j \le N, \tag{1}$$

where $\boldsymbol{\psi}_{ij} = \boldsymbol{\phi}_{ij}^{\mathcal{W}}$, and $\boldsymbol{k}_{ij} = \boldsymbol{\phi}_{ij}^{\mathcal{D}} \in \mathbb{Z}$ is an integer. Note that $(\cdot)^{\mathcal{W}}$ denotes the wrapping operator, e.g., $\boldsymbol{\phi}_{ij}^{\mathcal{W}}$ denotes the wrapped

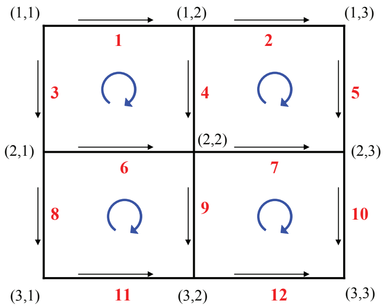

Fig. 1. Illustration of edge ordering.

part of ϕ_{ij}, and $(\cdot)^{\mathcal{D}}$ denotes the integer component (or the integer quantity involved in the wrapping operation). The problem of decoding ϕ from ψ is basically ill-posed due to the wrapping operator. Almost all existing phase unwrapping techniques start from a continuity assumption stated below.

Definition 1 *For an image $\boldsymbol{f}$ on a $M \times N$ grid, we say that $\boldsymbol{f}$ satisfies the* **continuity** *condition if for all $1 \le \ell \le L$, it holds that*

$$|(\nabla \boldsymbol{f})_\ell| < \pi, \tag{2}$$

where $\nabla \boldsymbol{f}$ is the gradient field of $\boldsymbol{f}$, consisting of $L \overset{\text{def}}{=} 2MN - M - N$ quantities that are differences of neighboring pixels.

The differences of the neighboring pixels in $\boldsymbol{f}$ are put into a vector of dimension L in $\nabla \boldsymbol{f}$ with the following index order (see Fig. 1 for the case of $M = N = 3$). Starting from the pixel at $(1, 1)$, the horizontal gradient field on the $N - 1$ horizontal edges in the first row are indexed as the first $N - 1$ components. The vertical gradient field elements are appended subsequently as the following $M - 1$ components of $\nabla \boldsymbol{f}$. After the first $2N - 1$ elements with one or both vertices on the first row, we then start from the pixel at $(2, 1)$ and repeat the same indexing for the second row, until reaching the pixel at $(M, 1)$.

For an image $\boldsymbol{f}$ of on an $M \times N$ grid, it generates a unique gradient vector $\nabla \boldsymbol{f}$ of dimension L. Conversely, for any vector $\boldsymbol{v}$ of dimension L, what condition it should satisfy so that it is a gradient field of some image $\boldsymbol{f}$ on an $M \times N$ grid? Such condition is the so-called network flow constraints [2], i.e., the directed sum of the gradient fields along the edges of any closed loop in the grid is 0. It is actually sufficient to choose only the closed loops consisting of unit squares (see Fig. 1). Since there are $(M - 1)(N - 1)$ unit squares in the whole image, we obtain $(M - 1)(N - 1)$ linear constraints, which can be presented in terms of a network flow matrix $\mathbf{A}$ below.

Definition 2 *For a vector $\boldsymbol{v}$ of dimension L, $\boldsymbol{v}$ satisfies the* **network flow constraints** *if $\mathbf{A}\boldsymbol{v} = \mathbf{0}$.*

When ϕ satisfies the continuity condition, the true phase value ϕ can be restored up to a global constant. Indeed, by computing gradients in (1) and then omitting the indices, we obtain $\nabla\psi = \nabla\phi - 2\pi\nabla k$. Thus

$$(\nabla\boldsymbol{\phi})^{\mathcal{W}} = (\nabla\boldsymbol{\psi})^{\mathcal{W}}. \tag{3}$$

When ϕ satisfies the continuity condition, the above formula implies that $\nabla\boldsymbol{\phi} = (\nabla\boldsymbol{\psi})^{\mathcal{W}}$ since $|\nabla\boldsymbol{\phi}| < \pi$ componentwise, i.e., the gradient field $\nabla\boldsymbol{\phi}$ is simply the wrapped part of $\nabla\psi$. After restoring $\nabla\boldsymbol{\phi}$, we can recover $\boldsymbol{\phi}$ from $\nabla\boldsymbol{\phi}$ up to a global constant.

However, the continuity condition can not hold in general in many applications. The rapid change in the observed data, or the noise contamination can easily invalidate the continuity assumption. If this happens, both $(\nabla\boldsymbol{\psi})^{\mathcal{W}}$ and $(\nabla\boldsymbol{\psi})^{\mathcal{D}}$ may violate the network flow conditions, and thus are even not legitimate gradient fields. Therefore, the reconstruction of $\boldsymbol{\phi}$ using $(\nabla\boldsymbol{\psi})^{\mathcal{W}}$ as its gradient field is not well-founded.

There are various methods of estimating $\nabla\boldsymbol{\phi}$ by compensating $(\nabla\boldsymbol{\psi})^{\mathcal{W}}$ using the network flow constraints. A representative approach is the so-called L^p-norm phase unwrapping [3], which aims to find a gradient field $\nabla\boldsymbol{\phi}$ that satisfies the network flow constraints and also has the closest L^p-distance to $(\nabla\boldsymbol{\phi})^{\mathcal{W}}$. This leads to the following mathematical formulation:

$$\begin{aligned} \min_{\nabla\boldsymbol{\phi}} \quad & \left|\left|\nabla\boldsymbol{\phi} - (\nabla\boldsymbol{\psi})^{\mathcal{W}}\right|\right|_p \\ \text{s.t.} \quad & \mathbf{A}\nabla\boldsymbol{\phi} = 0, \end{aligned} \tag{4}$$

where $\mathbf{A}$ enforces the network flow conditions on $\nabla\boldsymbol{\phi}$, and

$$\left|\left|\nabla\boldsymbol{\phi} - (\nabla\boldsymbol{\psi})^{\mathcal{W}}\right|\right|_p \overset{\text{def}}{=} \left(\sum_{\ell=1}^{L} |(\nabla\boldsymbol{\phi})_\ell - (\nabla\boldsymbol{\psi})_\ell^{\mathcal{W}}|^p\right)^{1/p}.$$

3. MAXIMUM A POSTERIORI PHASE UNWRAPPING WITH INTEGER CONSTRAINTS

To recover the gradient field $\nabla\boldsymbol{\phi}$ using $(\nabla\boldsymbol{\psi})^{\mathcal{W}}$, a multiple of 2π needs to be compensated on $(\nabla\boldsymbol{\psi})^{\mathcal{W}}$ for those components in $\nabla\boldsymbol{\phi}$ which do not satisfy the continuity condition (i.e., are out of the range of $(-\pi, \pi]$). In fact, it is easy to see that

$$\nabla\boldsymbol{\phi} - (\nabla\boldsymbol{\psi})^{\mathcal{W}} = 2\pi\left(\nabla\boldsymbol{k} + (\nabla\boldsymbol{\psi})^{\mathcal{D}}\right), \tag{5}$$

which implies that $\nabla\boldsymbol{\phi} - (\nabla\boldsymbol{\psi})^{\mathcal{W}}$ are integer multiples of 2π. In the L^p-norm phase unwrapping approach described in (4), due to the lack of integer constraints, the compensated quantity $\nabla\boldsymbol{\phi} - (\nabla\boldsymbol{\psi})^{\mathcal{W}}$ may not be integer multiples of 2π, which leads to inaccurate quantities compensations to $(\nabla\boldsymbol{\psi})^{\mathcal{W}}$. This results in inaccurate estimation of $\nabla\boldsymbol{\phi}$. In the sequel, we propose a new formulation of 2-D phase unwrapping. We assume a Gaussian prior on the gradient field $\nabla\boldsymbol{\phi}$, and adopt integer constraints to reflect that fact that $\nabla\boldsymbol{\phi} - (\nabla\boldsymbol{\psi})^{\mathcal{W}}$ are integer multiples of 2π.

3.1. Maximum a Posteriori Formulation

We assume a Gaussian prior on $\nabla\boldsymbol{\phi}$, i.e., the $L = 2N(N - 1)$ elements of $\nabla\boldsymbol{\phi}$ have a joint Gaussian distribution with zero mean and covariance $\boldsymbol{\Omega}$. The pdf of $\nabla\boldsymbol{\phi}$ is thus:

$$f_{\nabla\boldsymbol{\phi}}(\nabla\boldsymbol{\phi}) = \frac{1}{\sqrt{(2\pi)^L \det(\boldsymbol{\Omega})}} \exp\left(-\frac{(\nabla\boldsymbol{\phi})^T\boldsymbol{\Omega}^{-1}\nabla\boldsymbol{\phi}}{2}\right). \tag{6}$$

Our goal is to estimate $\nabla\boldsymbol{\phi}$ from $\boldsymbol{\psi}$, or from $\nabla\boldsymbol{\psi}$. Because $(\nabla\boldsymbol{\phi})^{\mathcal{W}} = (\nabla\boldsymbol{\psi})^{\mathcal{W}}$ (see (5)), our remaining task is to estimate $(\nabla\boldsymbol{\phi})^{\mathcal{D}}$. We can actually obtain the prior distribution of $(\nabla\boldsymbol{\phi})^{\mathcal{D}}$ using the distribution of $\nabla\phi$. Thus, treating $(\nabla\boldsymbol{\phi})^{\mathcal{D}}$ as the parameter to be estimated, we are able to derive the maximum a posteriori (MAP) estimator of $(\nabla\boldsymbol{\phi})^{\mathcal{D}}$ based on $\nabla\boldsymbol{\psi}$.

It can also be shown that given $(\nabla\boldsymbol{\psi})^{\mathcal{W}}$, $(\nabla\boldsymbol{\psi})^{\mathcal{D}}$ is deterministic, by using the fact that $\boldsymbol{\psi} \in (-\pi, \pi]$. Thus, we can formulate our estimator of $(\nabla\boldsymbol{\phi})^{\mathcal{D}}$ based on $(\nabla\boldsymbol{\psi})^{\mathcal{W}}$ only. Since $\nabla\boldsymbol{\phi} = (\nabla\boldsymbol{\phi})^{\mathcal{W}} + 2\pi(\nabla\boldsymbol{\phi})^{\mathcal{D}}$ and $(\nabla\boldsymbol{\phi})^{\mathcal{W}} = (\nabla\boldsymbol{\psi})^{\mathcal{W}}$, we obtain that:

$$\begin{aligned} & f_{(\nabla\boldsymbol{\psi})^{\mathcal{W}},(\nabla\boldsymbol{\phi})^{\mathcal{D}}}(\boldsymbol{x}, \boldsymbol{d}) = f_{\nabla\boldsymbol{\phi}}(\boldsymbol{x} + 2\pi\boldsymbol{d}) \\ & = \frac{1}{\sqrt{(2\pi)^L \det(\boldsymbol{\Omega})}} \exp\left(-\frac{(\boldsymbol{x} + 2\pi\boldsymbol{d})^T\boldsymbol{\Omega}^{-1}(\boldsymbol{x} + 2\pi\boldsymbol{d})}{2}\right). \end{aligned} \tag{7}$$

We thus obtain that the MAP estimator of $(\nabla\boldsymbol{\phi})^{\mathcal{D}}$ can be obtained by solving

$$
\begin{aligned}
\min_{(\nabla\boldsymbol{\phi})^{\mathcal{D}}} \quad & \left|\left|(\nabla\boldsymbol{\psi})^{\mathcal{W}} + 2\pi(\nabla\boldsymbol{\phi})^{\mathcal{D}}\right|\right|_{\boldsymbol{\Omega}^{-1}} \\
\text{s.t.} \quad & \mathbf{A}\left((\nabla\boldsymbol{\psi})^{\mathcal{W}} + 2\pi(\nabla\boldsymbol{\phi})^{\mathcal{D}}\right) = 0; \\
& (\nabla\boldsymbol{\phi})^{\mathcal{D}} \in \mathbb{Z}^L
\end{aligned} \tag{8}
$$

where $(\nabla\boldsymbol{\psi})^{\mathcal{W}}$ is a data matrix computed from the measured phase values $\boldsymbol{\psi}$, and for a vector $\boldsymbol{v}$, $||\boldsymbol{v}||_{\boldsymbol{\Omega}^{-1}} \stackrel{\text{def}}{=} \boldsymbol{v}^T\boldsymbol{\Omega}^{-1}\boldsymbol{v}$. We can write the first constraint as:

$$
\mathbf{A}(\nabla\boldsymbol{\phi})^{\mathcal{D}} = \mathbf{A}(\nabla\boldsymbol{\psi})^{\mathcal{D}},
$$

since $\mathbf{A}\left((\nabla\boldsymbol{\psi})^{\mathcal{W}} + 2\pi(\nabla\boldsymbol{\psi})^{\mathcal{D}}\right) = \mathbf{A}(\nabla\boldsymbol{\psi}) = 0$.

3.2. Binary Integer Constraints

Solving (8) involves an integer constraint on $(\nabla\boldsymbol{\phi})^{\mathcal{D}}$. This is the major difficulty in the numerical implementation. In fact, the integer constraints $(\nabla\boldsymbol{\phi})^{\mathcal{D}} \in \mathbb{Z}^L$ makes the resulting problem NP-hard. Thus, a relaxation method is needed when one tries to devise a computationally efficient approach to solve (8). Theoretically $(\nabla\boldsymbol{\phi})^{\mathcal{D}}$ can take any integer values since it is the integer part of the Gaussian distributed random variable $\nabla\boldsymbol{\phi}$. To reduce complexity, we approximate (8) by restricting the values that each component $(\nabla\boldsymbol{\phi})^{\mathcal{D}}$ can take.

We consider the case that the phase shift is mild, i.e., $(\nabla\boldsymbol{\phi})_\ell \in (-2\pi, 2\pi)$ for all $1 \leq \ell \leq L$ in this work. More general cases such as $(\nabla\boldsymbol{\phi})_\ell \in (-R\pi, R\pi)$ for all $1 \leq \ell \leq L$, for a given $R > 2$, will be investigated in an expanded version of this work. Technically, we ignore the low probability events of either when $(\nabla\boldsymbol{\phi})_\ell \in (-\infty, -2\pi]$ or $(\nabla\boldsymbol{\phi})_\ell \in (2\pi, +\infty)$. With such assumptions, we obtain that

$$
\begin{cases}
(\nabla\boldsymbol{\phi})_\ell^{\mathcal{D}} \in \{-1, 0\} & \text{if} \quad 0 \leq (\nabla\boldsymbol{\phi})_\ell^{\mathcal{W}} \leq \pi; \\
(\nabla\boldsymbol{\phi})_\ell^{\mathcal{D}} \in \{+1, 0\} & \text{if} \quad -\pi < (\nabla\boldsymbol{\phi})_\ell^{\mathcal{W}} < 0
\end{cases}
$$

since $(\nabla\boldsymbol{\phi})_\ell = (\nabla\boldsymbol{\phi})_\ell^{\mathcal{W}} + 2\pi(\nabla\boldsymbol{\phi})_\ell^{\mathcal{D}}$.

Let us further introduce a matrix $\mathbf{S} \stackrel{\text{def}}{=} \operatorname{diag}([\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_M])$ where

$$
\begin{cases}
\boldsymbol{s}_\ell = -1 & \text{if} \quad 0 \leq (\nabla\boldsymbol{\phi})^{\mathcal{W}} \leq \pi; \\
\boldsymbol{s}_\ell = 1 & \text{if} \quad -\pi < (\nabla\boldsymbol{\phi})^{\mathcal{W}} < 0,
\end{cases}
$$

We also introduce new variables $\boldsymbol{z} = \mathbf{S}\left[(\nabla\boldsymbol{\phi})^{\mathcal{D}}\right]$ and $\mathbf{A}_s = \mathbf{AS}$. Then we obtain that $\boldsymbol{z}_\ell \in \{1, 0\}$ for all ℓ, and (8) is transformed into a binary quadratic minimization problem in the new variable $\boldsymbol{z}$ below:

$$
\begin{aligned}
\min_{\boldsymbol{z}} \quad & \left|\left|(\nabla\boldsymbol{\psi})^{\mathcal{W}} + 2\pi\mathbf{S}\boldsymbol{z}\right|\right|_{\boldsymbol{\Omega}^{-1}} \\
\text{s.t.} \quad & \mathbf{A}_s\boldsymbol{z} = \mathbf{A}(\nabla\boldsymbol{\psi})^{\mathcal{D}}, \\
& \boldsymbol{z}_\ell \in \{0,\, 1\}; \quad 1 \leq \ell \leq L.
\end{aligned} \tag{9}
$$

3.3. Solution Using the Semidefinite Relaxation

We next solve (9) using the semidefinite relaxation approach decribed in [14, 15]. The standard problem discussed in those two papers are

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & ||\boldsymbol{y} - \mathbf{H}\boldsymbol{x}||^2 \\
\text{s.t.} \quad & \boldsymbol{x}_\ell \in \{-1, 1\}; \quad 1 \leq \ell \leq L.
\end{aligned} \tag{10}
$$

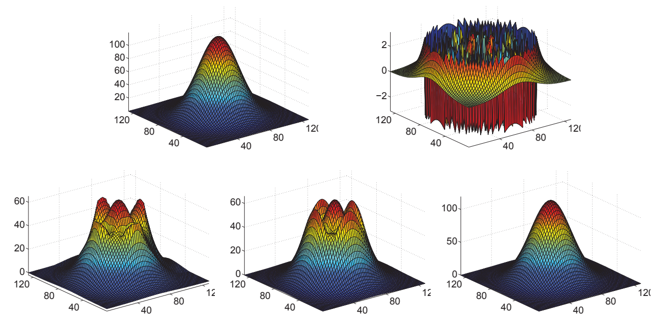

Fig. 2. Unwrapping performance comparison (plots are true phase; wrapped phase, unwrapping by L^2-norm approach, unwrapping by L^1-norm approach, and unwrapping by SDR approach respectively).

To transform the problem in (9) into the standard form in (10), we first penalize the equality constraint $\mathbf{A}_s\boldsymbol{z} = -\mathbf{A}(\nabla\boldsymbol{\psi})^{\mathcal{D}}$ by adding into the objective function a quadratic term $\left|\left|\mathbf{A}_s\boldsymbol{z} + (\nabla\boldsymbol{\psi})^{\mathcal{D}}\right|\right|^2$ multiplied by a multiplier λ and obtain:

$$
\begin{aligned}
\min_{\boldsymbol{z}} \quad & \left|\left|(\nabla\boldsymbol{\psi})^{\mathcal{W}} + 2\pi\mathbf{S}\boldsymbol{z}\right|\right|_{\boldsymbol{\Omega}^{-1}}^2 + \lambda\left|\left|\mathbf{A}_s\boldsymbol{z} - \mathbf{A}(\nabla\boldsymbol{\psi})^{\mathcal{D}}\,)\right|\right|^2 \\
\text{s.t.} \quad & \boldsymbol{z}_\ell \in \{0,\, 1\}; \quad 1 \leq \ell \leq L.
\end{aligned}
$$

We further introduce

$$
\mathbf{H} \stackrel{\text{def}}{=} \frac{1}{2}\mathbf{P}^{1/2}, \quad \boldsymbol{y} \stackrel{\text{def}}{=} -\mathbf{P}^{-1/2}\left(\mathbf{1}^T\mathbf{P} + \mathbf{b}^T\right), \tag{11}
$$

where

$$
\begin{aligned}
\mathbf{P} & \stackrel{\text{def}}{=} 4\pi^2\boldsymbol{\Omega}^{-1} + \lambda\mathbf{A}^T\mathbf{A}, \\
\mathbf{b} & \stackrel{\text{def}}{=} 2\pi\left[(\nabla\boldsymbol{\psi})^{\mathcal{W}}\right]^T\boldsymbol{\Omega}^{-1}\mathbf{S} - \lambda\left[(\nabla\boldsymbol{\psi})^{\mathcal{D}}\right]^T\mathbf{A}^T\mathbf{A}.
\end{aligned}
$$

With the transformations in (11), we have reformulated the binary quadratic minimization problem (9) in the standard form (10). In next section, we use the numerical solver provided in [15] to solve (10) with $\mathbf{H}$ and y given in (11).

4. NUMERICAL EXPERIMENTS

This section presents numerical results to validate our proposed phase unwrapping algorithm[1]. For simplicity, we choose $\boldsymbol{\Omega}$ to be an identity matrix. In the first experiment as shown in Fig. 2, we choose a Gaussian pdf shaped phase image with $N = 128$. The number of edges with discontinuity (i.e., along which the unwrapped phase data is not continuous, and thus a non-zero amount of compensation is needed to recover $\nabla\boldsymbol{\phi}$ from $(\nabla\boldsymbol{\psi})^{\mathcal{W}}$) is 1004, around 3.09% from the 32512 edges in the whole image. It is clear that our proposed SDR phase unwrapping approach with integer constraints can accurately estimaste the true phase $\boldsymbol{\phi}$ from $\boldsymbol{\psi}$, while both the L^1-norm and L^2-norm approaches fail to recover the true phase data at places where discontinuity exists.

We have implemented the L^1-norm, L^2-norm, and the SDR phase unwrapping approaches in Matlab 7.0 using a Quad Core Intel 2.4 GHz desktop PC. They take 6.19, 9.18, and 14.03 seconds respectively to unwrap the phase image in Fig. 2.

[1]Due to space limit, we only provide examples based on computer simulated data. Additional results on unwrapping synthetic aperture radar imaging data will be provided in an expanded version of this paper.

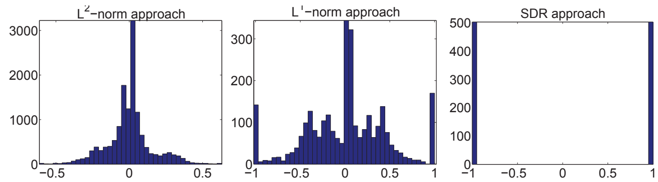

Fig. 3. Histogram of gradient compensations.

The analysis of why the SDR approach performs better is given in Fig. 3. Note that the ambiguity amount always takes integer multiples of 2π. As can be seen from Fig. 3, the SDR approach correctly compensates the integer amount of ambiguity, while both L^2-norm and L^1-norm approaches incorrectly compensated fractional phase values due to the lack of integer constraints. To further study the robustness of our approach, we plot in Fig. 4 the MSE performance of these three approaches on phase images with different qualities. The underlying data is of Gaussian shape, but its height is adjusted to obtain different percentages of edges with discontinuity. As can be seen, our SDR approach outperforms both the L^2-norm and L^1-norm approaches with a large margin when the percentage of edges with discontinuity is within a reasonable region (below 5%).

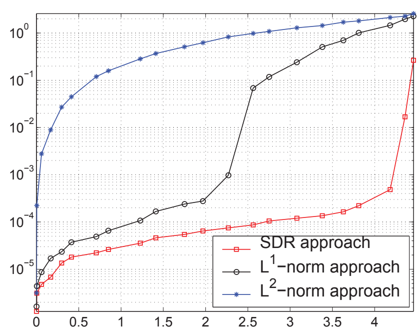

Fig. 4. MSE (per pixel) vs. percentage of edges with discontinuity.

Another example with more complicated shape (to simulate part of a terrain) is given in Fig. 5. We choose $N = 86$ and the phase image ϕ of the shape of the superposition of two symmetric Gaussian pdf skewed by a Gamma distribution. The results again clearly demonstrate that our SDR approach outperforms the L^1-norm and L^2-norm phase unwrapping methods. The computation time for the L^2-norm, L^1-norm, and the SDR approaches is 5.69, 8.37, and 2.87 seconds respectively.

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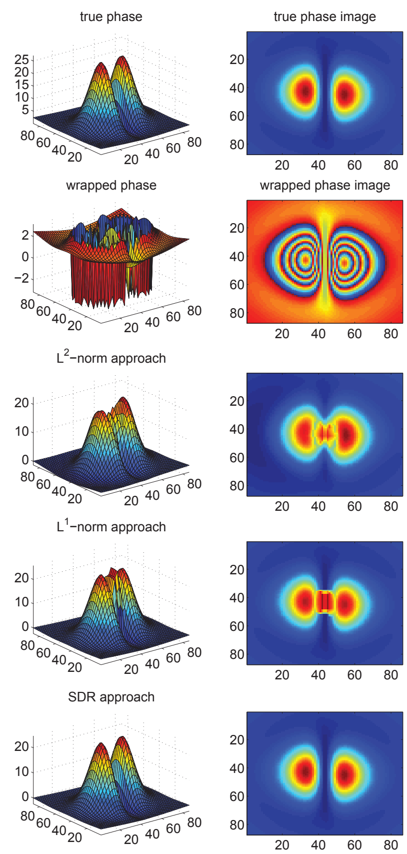

Fig. 5. Unwrapping performance comparison.

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