COMPRESSED SENSING AND MULTISTATIC SAR

Jonathan D. Coker and Ahmed H. Tewfik

University of Minnesota Department of Electrical and Computer Engineering

ABSTRACT

We demonstrate that the remarkable advantages of compressed sensing remain in force when the information operator is constrained to obey the physical rules of a multistatic SAR measurement. The design guidelines of the SAR information operator for ℓ_2 reconstructions is compared to those provided for generic ℓ_1 reconstructions. We report little or no degradation in compression performance when using an information operator obeying SAR sampling constraints. Simulations for a Shepp-Logan image show an image is faithfully reconstructed when the number of measurements is about a third of the number of image pixels, using a minimum total-variation technique. We observed high sensitivity in performance and algorithm convergence to small perturbations in the measurement vectors.

Index Terms— Multistatic synthetic aperture radar, radar tomography, compressed sensing

1. INTRODUCTION

Using the well-known results of compressed sensing (CS) [1, 2], an image can be reconstructed from projections of the image onto several basis functions. Remarkably, if the image is sparse enough in any known set of basis vectors, then the number of projections m required to reconstruct the image is substantially less than the number of pixels n in the image.

Conditions on the the *m* projection vectors (which when taken together form an *information operator*) for reconstruction methods to be successful were formally developed by Donoho [2] and others. For large enough problems, there are plenty of feasible information operators. In particular, an $m \times n$ matrix of independent, identically-distributed (i.i.d.) random entries is overwhelmingly likely to provide the necessary reconstruction information, so long as *m* is large enough compared to the sparsity of the image's representation. In cases where known random projections are physically easy to measure, this property is a very useful result.

However, not all physical measurements can be thought of as random projections of this type. In [3], we developed a measurement framework and reconstruction technique for the multistatic synthetic aperture radar (SAR) image problem, for both deterministic and stochastic vehicle trajectories, in which the measurement process is also modeled as m projections of an n-pixel image. Under the SAR physical sampling model, each projection vector is necessarily quite sparse and contains significant correlations from index to index. Thus, we set up a problem under CS in which not only the image, but also the information operator, is assumed to be sparse.

In our previous work, we assumed that m > n and solved the resulting maximum likelihood problems by minimization in ℓ_2 . The central focus of the present work is to determine whether, in cases where m < n, the principles of CS and ℓ_1 reconstruction remain useful under the non-random physical constraints placed on SAR multistatic measurements. We shall demonstrate that they do.

The paper is organized as follows. In section 2, we review the basic framework of the SAR measurement process and translate the problem in to the language of the CS information operator. In section 3 we discuss the design of the information operator for ℓ_2 reconstructions and contrast these principles with those provided by Donoho for ℓ_1 reconstructions. Section 4 gives results for a Shepp-Logan image simulation in which the image is faithfully reconstructed when m is about a third of n using a minimum total-variation technique. Section 5 is a discussion of the advantages particular to multistatic SAR for CS and the effects of its practical impediments such as noise. Section 6 contains final remarks.

2. BACKGROUND

The SAR measurement framework used here is a version of the *approximate inverse*, developed for downlooking multistatic SAR in [4]. These results can be briefly summarized as follows.

In one transmission burst, an omnidirectional, bistatic receiver/transmitter pair, over a flat plane of reflectivity function f, generates a series of measurements which are the line integrals of f over an expanding set of ellipses. The geometry of these ellipses depend on the coordinates of the UAVs above the plane. A typical example of these sampling ellipses (Figure 1) shows several measurement ellipses generated during a single transmit burst between a bistatic pair. Each ellipse is the locus of points, in the reflectivity plane, for which photons leaving the transmitter and gathered at the receiver have traveled an equal distance. The physical measurement (that is,



Fig. 1. A representative example of four ellipses of integration, each representing a row of the matrix **A**. The light dotted curves represent contours of some reflectivity function f(x, y).

the signal strength at the receiver at a particular instant corresponding to a given ellipse) is modeled as the line integral of the reflectivity function f(x, y) around the ellipse. That measurement is an element of the measurement vector $\mathbf{r_e}$.

Using the simplest possible discretization of f(x, y), the function f(x, y) is approximated by a checkerboard function $\hat{f}(x, y)$, which is constant within each grid cell, and whose value within that cell is specified by an element of the vector $\hat{\mathbf{f}}$. In such a case, an approximation of the line integral is the sum of $\hat{f}_j A_{ij}$, where A_{ij} is the segment length of the i^{th} sampling ellipse passing through the j^{th} grid cell corresponding to \hat{f}_j . By setting this approximation of the line integral to be equal to the corresponding radar measurement in \mathbf{r}_e , for many measurement ellipses arising from (possibly) many such bistatic pairs, the SAR measurement process is simply described as

$$\mathbf{A}\,\mathbf{f} = \mathbf{r}_e,\tag{1}$$

where **A** comprises the known A_{ij} values and is an *m*measurement by *n*-pixel measurement matrix, $\hat{\mathbf{f}}$ is an *n*-pixel vector of unknown pixels comprising the image, and \mathbf{r}_e is the known measurement vector of length *m*. Each component of \mathbf{r}_e measurement vector is therefore a projection of \mathbf{f} onto the rows of **A**. If the measurement noise in each element of \mathbf{r}_e can be thought of as i.i.d. Gaussian disturbances, then a maximum likelihood estimate of the values $\hat{\mathbf{f}}$ is the standard least-squares solution

$$\hat{\mathbf{f}}_{ML} = (\mathbf{A}^{\mathrm{T}} \mathbf{A})^{-1} \mathbf{A}^{T} \mathbf{r}_{e}$$
(2)

$$= \mathbf{A}^+ \mathbf{r}_e \tag{3}$$

$$= \mathbf{V} \boldsymbol{\Sigma}^+ \mathbf{U}^T \mathbf{r}_e \tag{4}$$

where \mathbf{A}^+ is the pseudoinverse of \mathbf{A} , \mathbf{U} and \mathbf{V} are orthogonal matrices, and Σ^+ is a diagonal matrix whose elements are given by the reciprocals of the singular values on the diagonal of the matrix Σ in the singular value decomposition $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$. When m > n, this solution is the standard ℓ_2 minimal solution for error. If m < n, then the solution form becomes the pseudoinverse, giving the minimum- ℓ_2 solution for $\hat{\mathbf{f}}$.

=

3. INFORMATION OPERATORS

The matrix **A** has the precise form of a CS *information operator*. In standard CS, the matrix **A** is often assumed to be dense and possess random i.i.d. entries. Donoho proves that information operators so constructed are very likely to be suitable for compressed sensing [2].

In the present work, we shall focus on "random sampling" SAR trajectories [3], in which the *locations* of the unmanned aerial vehicles (UAVs) comprising the trajectories are randomly chosen above the reflectivity plane. Under our SAR measurement framework, what we call "random sampling" always describes a physical line integral over an ellipse in the plane with a random location within the image area. Because this is a line integral over an area, the corresponding projection is itself quite sparse. The sparsity is inversely proportional to the number of pixels n. Furthermore, the non-zero elements of each projection vector are correlated, such that the information operator is not i.i.d. even though the physical locations may be random. Thus, the resulting operator A for SAR applications is neither i.i.d. nor dense. We name such operators *SAR/CS information operators*.

In [3] we showed that the best design for the matrix \mathbf{A} , when m > n and ℓ_2 minimization is used for reconstruction, is that \mathbf{A} should obey a modified Welch bound condition; that is, that the columns of \mathbf{A} are orthogonal and furthermore have equal energy. Equivalently, we may say that the columns of good ℓ_2 -reconstruction \mathbf{A} matrices form a *tight frame* for the subspace of \mathbb{R}^m spanned by those columns. We showed that while it is not strictly possible to design such matrices under the physical constraint of the SAR system, except in impractical degenerate cases, it is possible to develop useful trajectory design guidelines which approximate this condition for \mathbf{A} .

The following argument supports the expectation that the ℓ_2 tight-frame condition may also help to guarantee CS sparse reconstruction. When m < n, the same trajectory guidelines tend to make the columns of **A** approximate a tight frame for all of \mathbb{R}^m . Donoho does not explicitly demonstrate that information operators Ψ , which themselves constitute a tight

frame, are guaranteed to meet his conditions; however, he notes that the multiple of Ψ , $\Phi\Psi^T$, does meet these conditions if (and we suggest *only* if) Φ does. Letting Φ equal Ψ , we have $\Psi\Psi^T = \alpha \mathbf{I}_m$ for tight frames, which is clearly a good (if trivial) information operator for a problem in \mathbb{R}^m . Now following the argument in reverse direction, we can expect that the original Ψ to meet his conditions because it is a tight frame.

Donoho's conditions (CS1-CS3) on the information operator ensure, among other things, that good ℓ_1 reconstruction **A** matrices possess a degree of linear independence between small groups of columns, and that linear combinations of small groups of columns must look something like noise. Further, direct comparison of these conditions to SAR/CS operators is beyond the scope of the present work.

4. RESULTS

The signature feature of CS reconstruction is that the number of projections *m* required to faithfully reconstruct a sparse image can be substantially *less* than the number of pixels *n*. The now-standard method of *basis pursuit* [5], for example, can reconstruct a sparse image by finding that **x** with minimum ℓ_1 norm which is consistent with the measurement information $\mathbf{Ax} = \mathbf{r}$.

To demonstrate whether the advantages of CS survive SAR's physical sampling constraints in a good trajectory, we follow [6] in which Shepp-Logan phantom images are reconstructed by minimizing the gradient or total variation (TV) in an image under the information-operator constraint Ax = r. In continuous-function language, this is a minimization of the objective function

$$\int \int ||\nabla f||_1 dx dy \tag{5}$$

under the constraint that f must be *exactly* consistent with the projection information. The Shepp-Logan and other artificial piecewise-constant phantoms are particularly well suited (perhaps, too well suited) to this technique because their gradients are even more sparse than the image itself. We used the very useful software package " ℓ_1 -magic" of Candés and Romberg [7] to generate a minimum-TV reconstruction from a random-location SAR A matrix and its projections r. Because there is no measurement noise in this simulation, we used the *minimum TV with equality constraints* version of ℓ_1 reconstruction.

Figure 2 shows the results for a 75×75 image, using 25 randomly-located bistatic transmit events, generating a total of 1765 measurements; here, **A** is a 1765×5625 matrix. For comparison purposes, the minimum- ℓ_2 reconstruction from the standard pseudoinverse is shown. The minimum-TV reconstruction is visually indistinguishable from the original image, indicating that CS concepts can remain in force de-



Fig. 2. Noiseless minimum- ℓ_2 norm reconstruction (right) and minimum-TV reconstruction (left) for a $75 \times 75 = 5625$ pixel image and m = 1765 random-location SAR measurements.

spite the sparsity and correlations of the information operator **A**, enforced by the physical constraints of the SAR system.

The previous result gave good reconstruction when the number of measurements was roughly a third of the number of pixels in the image. We have generally found this relationship to be consistent at various image resolutions. Figure 3 shows the standard deviation of the difference between the reconstructed and perfect image, versus the number of *noiseless* measurements m, for an n = 2500 pixel image. Reconstruction performance is shown for the random SAR/CS operator, and for a standard random Gaussian operator of the same size. As expected, there is little or no degradation in the efficacy of the random SAR/CS operator, when compared to that of the standard operator.

The data demonstrates that performance begins to degrade markedly when the number of measurements is less than a third of the number of pixel images. We hasten to point out, however, that such a breakpoint depends strongly on the test image and its sparsity (and the sparsity of its gradient) and is not expected to remain constant for all images of practical interest.

5. NOISE AND SAR/CS

The driving potential of CS in multistatic SAR is for marked reductions in the required number of measurements. These gains can translate into reduced transmit/receive power consumption; less demanding communication bandwidths between the unmanned aerial vehicles (UAVs) and the central processing location; and a higher level of security in covert applications. However, measurement disturbances and noise are an important impediment in practical SAR systems which highlight particular problems for ℓ_1 -constrained reconstructions, and which must be overcome.

The standard ℓ_2 solutions of equation 4 give a very simple, proportional relationship between the measurement noise power and the average pixel noise power [8]. Of course, CS



Fig. 3. Pixel error versus the number of measurements for an n = 2500 image for SAR/CS and for a standard CS information operator. The image has unity scale.

works best without measurement noise, which can markedly erode its gains, as in [9]. In our case, we found the reconstruction and convergence performance of the minimum-TV algorithm to be very sensitive to small additive disturbances in the projections; so sensitive, in fact, that a direct comparison of noise performance between the ℓ_1 and ℓ_2 methods is not meaningful. Problems persist even when the disturbances are 4 or 5 orders of magnitude smaller than the maximum measurement magnitude. Undoubtedly, this behavior is due to the use of equality constraints in the minimum-TV algorithm.

Alternative methods for sparse reconstruction in the presence of noise is a current topic of research, e.g. [10]; some methods are already available. An alternative reconstruction method, *minimum TV with quadratic constraints*, could be used to find the minimum-TV f such that the sparse solution is within some distance being consistent with the measurements, as measured by the ℓ_2 -norm. Such research will undoubtedly be applicable for numerical solutions to the noisy SAR problem using CS techniques.

6. CONCLUSIONS

We have demonstrated that the use of a UAV trajectory (a random-location trajectory), shown to be nearly optimal for ℓ_2 reconstruction when the number of measurements exceeds the number of pixels, also give excellent ℓ_1 reconstruction under noiseless conditions when $m \approx \frac{1}{3}n$ for the Shepp-Logan phantom. There appears to be little or no degradation in compressibility performance when using the SAR/CS operator, compared to a standard CS information operator.

The current paucity of analytic performance tools for CS-related reconstruction methods (such as those tools from standard linear algebra used for the ℓ_2 methods) will make it more challenging to directly develop trajectory guidelines

for the SAR problem using CS reconstructions, if, indeed, such guidelines *are* different than those derived for ℓ_2 . Our results suggest that the design guidelines for SAR trajectories for ℓ_2 solutions produce information operators which, if not optimal, at least produce serviceable ℓ_1 reconstructions.

Future work in this area could include an in-depth comparison of ℓ_1 and ℓ_2 information-operator design methods for the SAR problem; a study of the image sparsity versus the number of SAR measurements *m* needed to achieve a performance goal under practical SAR measurement and image conditions; and a study of the efficacy of CS reconstruction algorithms designed to work in the presence of noise and SAR/CS operators.

7. REFERENCES

- E. J. Candés and T. Tao, "Near-optimal signal recovery from random projections: Universal encoding strategies," Tech. Rep., Applied and Computational Mathematics, Calif. Inst. Technol., 2004.
- [2] D. L. Donoho, "Compressed sensing," *IEEE Trans. on Information Theory*, vol. 52, no. 4, 2006.
- [3] Jonathan D. Coker and Ahmed H. Tewfik, "Random sampling strategies in multistatic SAR," *ICASSP*, 2008.
- [4] Jonathan D. Coker and Ahmed H. Tewfik, "Multistatic SAR image reconstruction based on an ellipticalgeometry Radon transform," *International Waveform Diversity and Design Conference*, 2007.
- [5] Shaobing Scott Chen, *Basis Pursuit*, Ph.D. thesis, Stanford University, November 1995.
- [6] E.J. Candes, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information," *Information Theory, IEEE Transactions on*, vol. 52, no. 2, pp. 489– 509, Feb. 2006.
- [7] " ℓ_1 -MAGIC," www.l1-magic.org.
- [8] Jonathan D. Coker and Ahmed H. Tewfik, "On the design of vehicle trajectories for high quality multistatic SAR images," *RadarCon*, 2008.
- [9] Robert Nowak Rui Castro, Jarvis Haupt and Gil Raz, "Finding needles in noisy haystacks," *ICASSP*, 2008.
- [10] E. Candes and J. Romberg, "Robust signal recovery from incomplete observations," *Image Processing*, 2006 *IEEE International Conference on*, pp. 1281–1284, Oct. 2006.