DESIGN OF ROBUST SUPERDIRECTIVE BEAMFORMERS AS A CONVEX OPTIMIZATION PROBLEM

Edwin Mabande, Adrian Schad, and Walter Kellermann

University of Erlangen-Nuremberg, Multimedia Communications and Signal Processing
Cauerstraße 7, 91058 Erlangen, Germany
{emabande,schad,wk}@LNT.de

ABSTRACT

Broadband data-independent beamforming designs aiming at constant beamwidth often lead to superdirective beamformers for low frequencies, if the sensor spacing is small relative to the wavelengths. Superdirective beamformers are extremely sensitive to spatially white noise and to small errors in the array characteristics. These errors are nearly uncorrelated from sensor to sensor and affect the beamformer in a manner similar to spatially white noise. Hence the White Noise Gain (WNG) is a commonly used measure for the robustness of beamformer designs. In this paper, we present a method which incorporates a constraint for the WNG into a least-squares beamformer design and still leads to a convex optimization problem that can be solved directly, e.g. by Sequential Quadratic Programming. The effectiveness of this method is demonstrated by design examples.

Index Terms— Superdirective Beamformer, White Noise Gain, Sequential Quadratic Programming

1. INTRODUCTION

When sensor arrays are employed for sampling wavefields, signal processing of the sensor data allows for spatial filtering which facilitates a better extraction of a desired source signal and suppression of unwanted interference signals [1]. Beamforming represents a class of such multichannel signal processing algorithms. Fig. 1 depicts a linear array with N sensors positioned at $\mathbf{p}_n$, $n = 0, \ldots, N-1$ in space. The signal captured by the n-th sensor is first converted to a discrete-time signal $x(l, \mathbf{p}_n)$ by an A/D converter and then filtered by convolution with an impulse response $w_n(l)$. The output $y(l)$ of the beamformer is obtained by the summation of all the filter outputs.

Superdirective beamforming is often desirable due to its inherent ability to provide high directivity with a small array aperture [2]. However the high sensitivity of these beamforming designs to spatially white noise, mismatch between sensor characteristics, and position errors, limits their application in practice [1]. The errors due to mismatch between sensor characteristics and position errors are nearly uncorrelated from sensor to sensor and affect the beamformer in a manner similar to spatially white noise, and therefore the White Noise Gain (WNG) is a useful measure of robustness of the beamformer [2]. For data-dependent adaptive beamformers, diagonal loading was suggested for increasing robustness with the frequency variant loading factor obtained via iterative design schemes [1]. In the case of the data-independent broadband least-squares frequency-invariant beamforming design proposed in [3], Tikhonov regularization was suggested. Unfortunately, the relationship between the regularization parameters used in the regularization procedure and a desired WNG constraint value is nontrivial. In this paper a beamforming design which directly constrains the WNG to lie above a given lower limit by directly solving a convex optimization problem is presented. The major advantage of this new method over previous attempts to jointly satisfy the constraint of undistorted signal response from the desired look direction and the WNG constraint is that it does not rely on other parameters which are nontrivially related to a given WNG constraint value but directly uses the given WNG constraint value to obtain an optimum design. For simplification of the following treatment, we use the common assumptions that waves propagate in a free field, that the sources are in the farfield relative to the array, and that all sensors are omnidirectional.

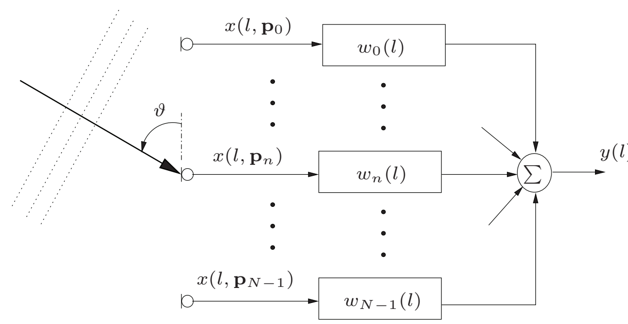

Fig. 1. Beamforming with a linear array

This work was supported by the European Commission within the DICIT project under contract number 034624.

2. SUPERDIRECTIONAL BEAMFORMING

A beamformer is characterized here by the beamformer response which describes the wavefield in the farfield produced for a given complex harmonic signal with frequency ω as parameter. The response of a filter-and-sum beamformer (FSB) with N sensors in a linear array is given by [3]

$$B(\omega, \vartheta) = \sum_{n=0}^{N-1} W_n(\omega) e^{-j\omega\tau_n(\vartheta)}, \quad (1)$$

where $W_n(\omega) = \sum_{l=0}^{L-1} w_n(l) \exp(-jl\omega)$, $\tau_n(\vartheta) = d_n \cos\vartheta / c$, d_n is the distance of the n-th sensor to the center of the array, L is the FIR filter length and ϑ is the angle of arrival relative to the sensor array axis. The magnitude square of the beamformer response is known as the beampattern of the beamformer [1].

The WNG is given by [4]

$$A(\omega) = \frac{|\mathbf{w}_\mathrm{f}^T(\omega)\mathbf{d}(\omega)|^2}{\mathbf{w}_\mathrm{f}^H(\omega)\mathbf{w}_\mathrm{f}(\omega)}, \quad (2)$$

where $\mathbf{d}^T(\omega) = [\exp(-j\omega\tau_0(\vartheta_\mathrm{o})), \dots, \exp(-j\omega\tau_{N-1}(\vartheta_\mathrm{o}))]$ and $\mathbf{w}_\mathrm{f}(\omega) = [W_0(\omega), \dots, W_{N-1}(\omega)]^T$ denote the so-called steering vector and the frequency responses of the filters $w_n(l)$, respectively. The angle ϑ_o denotes the desired look direction. The WNG quantifies a beamformer's ability to suppress spatially white noise as it expresses the gain of the beamformer for the desired signal from the desired look direction relative to the amplification of spatially white noise. Therefore $A(\omega) < 1$ effectively corresponds to an amplification of spatially white noise at frequency ω.

The logarithmic array gain [4], termed the directivity index, of the delay-and-sum beamformer (DSB) reaches its maximum of $10 \log N$ if the sensor spacing is $\lambda/2$. Beamforming designs that lead to a directivity index greater than $10 \log N$ are called superdirective. The WNG for superdirectional beamformers is typically very small, e.g. $A(\omega) < 10^{-3}$, at low frequencies [2]. Consequently these beamformers are highly sensitive to small errors in the array characteristics.

The design used for beamforming here is the least-squares beamformer (LSB). The idea behind the design is to optimally approximate a desired response, $\hat{B}(\omega, \vartheta)$, by $B(\omega, \vartheta)$ in the LS sense. Typically, a numerical solution is obtained by discretizing the frequency range into P frequencies ω_p, $p = 0, \dots, P-1$ and the angular range into M angles ϑ_m, $m = 0, \dots, M-1$, and solving the resulting set of linear equations numerically. The beamformer design problem then reads:

$$\hat{B}(\omega_p, \vartheta_m) \stackrel{!}{=} \sum_{n=0}^{N-1} W_n(\omega_p) e^{-j\omega_p\tau_n(\vartheta_m)} \quad (3)$$

Reformulating (3) in matrix notation leads to

$$\hat{\mathbf{b}}(\omega_p) \stackrel{!}{=} \mathbf{G}(\omega_p)\mathbf{w}_\mathrm{f}(\omega_p),$$

where $\hat{\mathbf{b}}(\omega_p) = [\hat{B}(\omega_p, \vartheta_0), \dots, \hat{B}(\omega_p, \vartheta_{M-1})]^T$ and $[\mathbf{G}(\omega_p)]_{mn} = e^{-j\omega_p\tau_n(\vartheta_m)}$. Since the number of discretized angles is typically greater than the number of sensors, $M > N$, the problem is therefore overdetermined. The least-squares solution to this problem, which gives the smallest quadratic error by definition, is given by:

$$\min_{\mathbf{w}_\mathrm{f}(\omega_p)} \|\mathbf{G}(\omega_p)\mathbf{w}_\mathrm{f}(\omega_p) - \hat{\mathbf{b}}(\omega_p)\|_2^2 \quad (4)$$

In order to ensure that the desired signal from a given angle ϑ_o remains undistorted, the linear constraint

$$\mathbf{w}_\mathrm{f}^T(\omega_p)\mathbf{d}(\omega_p) = 1, \quad (5)$$

must be satisfied and defines a linearly constrained problem in conjunction with (4). This problem is to be solved for each frequency ω_p. A least-squares frequency-invariant beamformer design (LSFIB) is obtained by choosing the same desired response for all frequencies i.e. $\hat{\mathbf{b}}(\omega_p) = \hat{\mathbf{b}}$. This design inherently leads to superdirective beamformers for low frequencies if the wavelengths are larger than twice the sensor spacing [1] and is therefore very sensitive to small random errors encountered in real-world applications.

3. ROBUSTNESS CONSTRAINT

Since the WNG is a measure of the robustness of a beamformer, a robust superdirectional beamforming design may be obtained by constraining the WNG [2]. The idea behind the method presented here is to incorporate a WNG constraint into the LSB design by adding the following quadratic constraint

$$\frac{|\mathbf{w}_\mathrm{f}^T(\omega_p)\mathbf{d}(\omega_p)|^2}{\mathbf{w}_\mathrm{f}^H(\omega_p)\mathbf{w}_\mathrm{f}(\omega_p)} \geq \gamma > 0, \quad (6)$$

where γ is the lower bound for the WNG. It is a user-defined parameter which enables direct control of the robustness of the beamforming design. Thus, a robust LSB (RLSB) beamforming design may be ensured by combining (4), (5) and (6) resulting in

$$\min_{\mathbf{w}_\mathrm{f}(\omega_p)} \|\mathbf{G}(\omega_p)\mathbf{w}_\mathrm{f}(\omega_p) - \hat{\mathbf{b}}(\omega_p)\|_2^2,$$

subject to

$$\frac{|\mathbf{w}_\mathrm{f}^T(\omega_p)\mathbf{d}(\omega_p)|^2}{\mathbf{w}_\mathrm{f}^H(\omega_p)\mathbf{w}_\mathrm{f}(\omega_p)} \geq \gamma, \quad \mathbf{w}_\mathrm{f}^T(\omega_p)\mathbf{d}(\omega_p) = 1, \quad (7)$$

which is a convex problem for the following reasons. The unconstrained least-squares problem (4) is a convex function [5]. The set $\{\mathbf{w}_\mathrm{f}(\omega_p) | \mathbf{d}^T(\omega_p)\mathbf{w}_\mathrm{f}(\omega_p) = 1\}$ is convex because its elements lie in a hyperplane defined by $\mathbf{d}^T(\omega_p)\mathbf{w}_\mathrm{f}(\omega_p) = 1$. Under the assumption that the linear constraint is met, the quadratic constraint becomes $1/\mathbf{w}_\mathrm{f}^H(\omega_p)\mathbf{w}_\mathrm{f}(\omega_p) \geq \gamma$. The set $\{\mathbf{w}_\mathrm{f}(\omega_p) | \mathbf{w}_\mathrm{f}^H(\omega_p)\mathbf{w}_\mathrm{f}(\omega_p) \leq 1/\gamma\}$ is convex because it describes a Euclidean ball with radius $1/\sqrt{\gamma}$, whose center lies at the origin. Since convexity is preserved under intersection, the constrained problem is therefore convex because we minimize a convex function over two convex sets [5].

There is no known analytic solution to (7). The solution of the problem requires an iterative procedure to establish a direction of search at each major iteration. The general idea behind constrained optimization is to transform the problem into an easier subproblem that then can be solved and used as the basis of an iterative process. Sequential Quadratic Programming (SQP) methods may be used for this task [6]. The SQP methods typically make use of a quasi-Newton method in order to generate a Quadratic Programming subproblem whose solution is used to form a search direction for a line search procedure [7]. The SQP method implemented in the MATLAB Optimization Toolbox [7] is used here.

The initialization vector of the SQP method may be obtained by using the output of a random number generator. Although the SQP implementation used here guarantees convergence for a random starting vector, this may lead to slow convergence if the starting point does not satisfy both constraints in (7). The convergence speed is increased significantly by selecting an initialization vector that satisfies them. For example, selecting the initialization vector $\tilde{\mathbf{w}}_{\mathrm{f}}(\omega_p) = \frac{1}{N}\mathbf{d}^*(\omega_p)$, then

$$\tilde{\mathbf{w}}_{\mathrm{f}}^T(\omega_p)\mathbf{d}(\omega_p) = \frac{1}{N}\mathbf{d}^H(\omega_p)\mathbf{d}(\omega_p) = 1, \tag{8}$$

which satisfies the linear constraint in (7). Evaluating the WNG we obtain

$$A(\omega_p) = \frac{|\tilde{\mathbf{w}}_{\mathrm{f}}^T(\omega_p)\mathbf{d}(\omega_p)|^2}{\tilde{\mathbf{w}}_{\mathrm{f}}^H(\omega_p)\tilde{\mathbf{w}}_{\mathrm{f}}(\omega_p)} = \frac{N^2}{\mathbf{d}^{*H}(\omega_p)\mathbf{d}^*(\omega_p)} = N, \tag{9}$$

which corresponds to the directional gain of the uniform linear array (ULA) DSB and forms the borderline between non-superdirective and superdirective beamforming. It is clear that the quadratic constraint in (7) is also satisfied if $\gamma \leq N$. Since $A(\omega_p) = N$ is the maximum attainable WNG for a beamformer with an array consisting of N sensors, which is only obtained with uniform sensor weights [4], the upper limit for the user-defined WNG lower limit is N, i.e. $\gamma_{\max} = N$. The RLSB design may vary from the DSB to a highly sensitive superdirective beamformer as desired by varying γ accordingly. This flexibility becomes of major importance in practice.

4. EVALUATION

The proposed robust LSFIB (RLSFIB) design, i.e. RLSB with $\hat{\mathbf{b}}(\omega_p) = \hat{\mathbf{b}}$, was evaluated for microphone array beamforming, by investigating various array geometries and WNG values. FIR filters $w_n(l)$ with length $L = 512$ were used to approximate the frequency response vectors $[W_n(\omega_0), \ldots, W_n(\omega_P)]$ in the Least Squares sense. The main lobe of the desired frequency response was always defined with a 3-dB beamwidth of twenty degrees. Each of the design examples is represented by a figure containing multiple subfigures depicting the beamformer's beampattern and WNG on a logarithmic scale ($A_{\mathrm{dB}} = 10\log_{10} A$ and $\gamma_{\mathrm{dB}} = 10\log_{10}\gamma$).

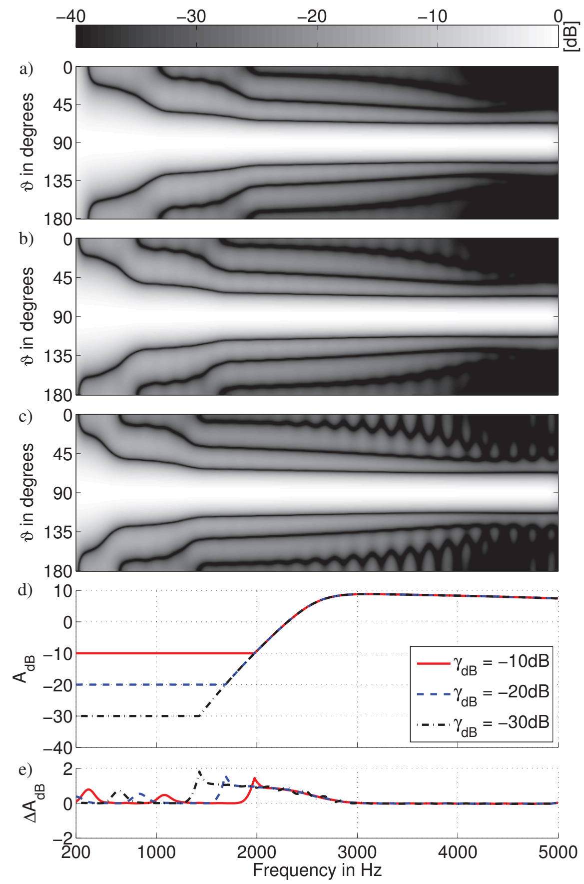

Fig. 2. 8-element ULA; Beampatterns for a) $\gamma_{\mathrm{dB}} = -10$dB, b) $\gamma_{\mathrm{dB}} = -20$dB and c) $\gamma_{\mathrm{dB}} = -30$dB; d) Design-domain-based WNGs; e) WNG deviations due to FIR filter approximations

The RLSFIB design was applied to an eight-element ULA with microphone spacing $d = 0.04$m, where lower and upper cut-off frequencies of 0.2kHz and 5kHz, respectively, were chosen with speech signal capture in mind. The mainlobe of the beamformer was formed at broadside. Figure 2 depicts the beampatterns and corresponding WNGs for three different values of γ_{dB}. The beampatterns show good spatial selectivity across the desired frequency range although the main beams tend to broaden for lower frequencies. The main beams also become broader as the value of γ_{dB} is increased. Therefore, constraining the WNG leads to the expected trade-off between directivity and sensitivity to errors. The constraint of no magnitude distortion in look direction is satisfied in all cases and the phase is perfectly linear. Figure 2d shows that the WNG resulting from the design domain is satisfied perfectly throughout the frequency range for three different values of γ_{dB}. The FIR approximation causes only small positive deviations relative to the ideal behaviour in the design domain as depicted in Figure 2e. The WNG constraint is still satisfied. This confirms the ability of the RLSFIB design to constrain the WNG efficiently.

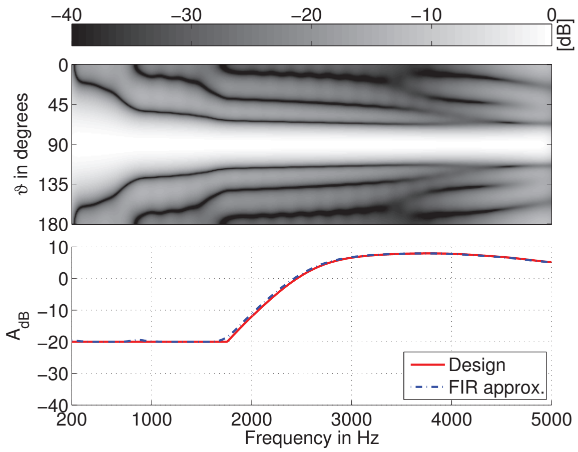

Fig. 3. 8-element NULA, $\gamma_{\text{dB}} = -20\text{dB}$

Due to geometrical constraints which may be encountered in practice uniform spacing in the array is not always possible nor necessary. To this end the RLSFIB design was also investigated for the case of an eight-element nonuniform linear array (NULA) with $\gamma = 0.01$. The actual positioning vector used is [-0.15 -0.09 -0.07 -0.02 0.03 0.06 0.11 0.13]m. The total aperture size and the considered bandwidth is the same as in the 8-element ULA case. The results are depicted in Figure 3. The beampattern shows good spatial selectivity which is comparable to that depicted in Figure 2. The WNG constraint is also satisfied as expected. The convergence behaviour of the SQP method is the same as in the ULA case.

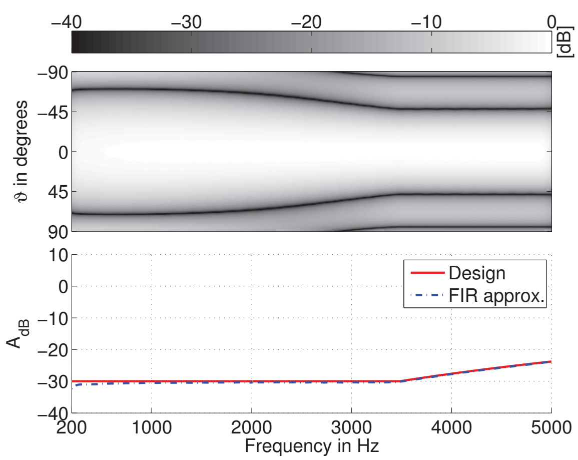

Fig. 4. 3-element ULA, $\gamma_{\text{dB}} = -30\text{dB}$

Superdirectional beamformers are especially desirable due to their ability to obtain a good spatial selectivity with a small array consisting of few sensors satisfying constraints on space and cost. Figure 4 depicts the results for a three-element ULA with $d = 0.008$m and the same bandwidth and FIR filter lengths ($L = 512$) as in the previous examples. The mainlobe of the beamformer was formed at endfire with $\gamma = 0.001$. Although the beamwidth of the mainlobe is large due to the small aperture size, good spatial selectivity is still maintained throughout the frequency range. However, the constraint of no distortion is violated at the lower frequencies. The WNG constraint is satisfied perfectly in the design domain but the FIR approximation cannot fully preserve the WNG due to the limited number of degrees of freedom for the FIR filters. By selecting filters with larger number of filter coefficients e.g. $L \geq 2048$, both constraints can be satisfied.

5. CONCLUSIONS

A method which allows full control of the robustness of a least-squares beamformer design has been presented. The beamformer design has been formulated as a constrained least-squares problem incorporating a linear constraint and a quadratic constraint which in effect constrains the WNG of the resulting design. The constrained least-squares problem was shown to be convex and therefore well established methods for convex optimization, such as the SQP methods, may be used to solve the constrained problem. The results shown confirm that the RLSFIB design is capable of controlling the robustness of the resulting beamformer according to the user's requirements which underlines the flexibility of this design procedure.

6. REFERENCES

[1] M.S. Brandstein and D.B. Ward, Eds., "Microphone arrays: Signal processing techniques and applications," Springer, Berlin, 2001.

[2] H. Cox, R.M. Zeskind, and T. Kooij, "Practical supergain," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-34, pp. 393–398, June 1986.

[3] L.C. Parra, "Steerable frequency-invariant beamforming for arbitrary arrays," *Journal of the Acoustical Society of America*, pp. 3839–3847, June 2006.

[4] W. Herbordt, "Sound capture for human/machine interfaces," Springer, Berlin, 2005.

[5] S. Boyd and L. Vandenberghe, "Convex optimization," Cambridge University Press, 2004.

[6] R. Fletcher, "Practical methods for optimization," John Wiley and Sons Ltd., 1987.

[7] "Optimization toolbox user's guide," The Mathworks, Inc, 2008.