COMPUTING PERFORMANCE GUARANTEES FOR COMPRESSED SENSING

Kiryung Lee and Yoram Bresler

Coordinated Science Laboratory and Department of ECE
University of Illinois at Urbana-Champaign
1308 W. Main St., Urbana, IL 61801

ABSTRACT

There are various conditions on the CS matrix for unique and stable recovery. These include universality, or spark, and UUP. Furthermore, quantitative bounds on the stability depend on related properties of the CS matrix. The construction of good CS matrices – satisfying the various properties – is key to successful practical applications of compressive sensing. Unfortunately, verifying the satisfiability of any of these properties for a given CS matrix involves infeasible combinatorial search. Our methods use ℓ_1 and semidefinite relaxation into a convex problem. Given a set of candidate CS matrices, our approach provides tools for the selection of good CS matrices with verified and quantitatively favorable performance.

***Index Terms*—** Spark, Uniform Uncertainty Principle, Basis Pursuit, Semidefinite Programming, Compressive Sampling

1. INTRODUCTION

Compressed sensing, or compressive sampling (CS) addresses under-determined inverse problems for sparse signals. In the prototype problem, for $x \in \mathbb{C}^n$ with *sparsity* (number of non-zero elements) $\|x\|_0 \leqslant q \ll n$ and $B \in \mathbb{C}^{p\times n}$, we want to reconstruct x from $y = Bx$. For some CS matrices B and $p > 2q$, x (or, for $p > q$, almost all x) can be exactly reconstructed, by finding the minimizer for $\|x\|_0$ subject to $Bx = y$ [1, 2]. Similar results hold for essentially sparse, or compressible signals x, for p proportional to $q \log n$. [3, 4]. Moreover, x can be also reconstructed by solving a convex problem (basis pursuit) [3, 4]. Related results involve sparse sampling of both finite and infinite-dimensional signals that have unknown but sparse spectrum [1, 2, 5].

There are various conditions on the CS matrix B that guarantee unique and stable recovery, or recovery using basis pursuit. These include so-called *universality* [1, 2, 5], later called *spark* [6] which is a full-rank condition on every q, $q \leqslant p$ column minor of B, and the more restrictive *UUP (uniform uncertainty principle)*, which implies a sufficient condition for exact recovery using basis pursuit [7]. Bounds on the stability of the solution with respect to measurement noise, and with respect to un-modeled small non-zero elements in x also depend on related properties of B [1, 8, 9].

The construction of good CS matrices – satisfying the various properties – is key to successful practical applications of compressive sensing. Feng and Bresler [1] used exhaustive search and random selection of sampling patterns generating CS matrices with low condition number in their work on spectrum-blind sampling, and Candes *et al* [10] and Donoho *et al* [3] showed that random CS matrices generated by the uniform or Gaussian distribution satisfy the UUP with a high probability. However, none of these properties can be guaranteed for any instance of a CS matrix generated at random. In other words, we may face failure. This can have serious consequences if the cost of reconstruction failure is very high.

Tao recently posted an open problem on constructing good CS matrices [11]. The objective is either to devise a deterministic construction of CS matrices that obey the UUP or to reduce the computational complexity of determining satisfiability of the UUP for a given CS matrix. We propose an efficient (polynomial-time) scheme to verify the UUP.

In addition to the verification of the UUP, we consider the quantification of the stability of the recovery problem for a given B. Donoho *et al* [8] and Candes *et al* [9] derived bounds on the error in ℓ_1-reconstruction. They used quantities whose computation is NP-hard. Feng and Bresler [1] considered the worst-case condition number over all possible sparse x instances, which is also NP-hard to compute. Our methods use ℓ_1-relaxation for the sparsity $\|x\|_0$, and additional relaxation into a convex problem that is solved in polynomial time by semidefinite programming. Given a set of candidate CS matrices generated at random, or using any other heuristic, our approach provides tools for the selection of good CS matrices with verified and quantitatively favorable performance.

2. PERFORMANCE GUARANTEES

Let $B \in \mathbb{R}^{p\times n}$ be a given matrix. Consider a minor of B, $B_{\bullet,J}$ formed from a subset indexed by $J \subset \{1, \cdots, n\}$ of its columns. We refer to the number of columns of $B_{\bullet,J}$, given by $|J|$, the cardinality of J, as the *size* of minor $B_{\bullet,J}$. Let $\mathrm{supp}(x) \triangleq \{k \in \{1, \cdots, n\} : x_k \neq 0\}$ be the support set of x. We can then rewrite the compressed sensing problem as: find x such that $y = B_{\bullet,\mathrm{supp}(x)} x_{\mathrm{supp}(x)}$.

This work was supported in part by NSF grant No. CCF 06-35234.

2.1. Spark_η

Donoho *et al* [8] introduced Spark_η defined by

$$\mathrm{P1}: \qquad \mathrm{Spark}_\eta \triangleq \max\{q \in \mathbb{Z} : \min_{|J|<q} \sigma_{\min}(B_{\bullet,J}) > \eta\}$$

where $\sigma_{\min}$ denotes the minimum singular value. Using ℓ_0 minimization, exact reconstruction can be achieved for almost all (Lesbegue) x satisfying $\|x\|_0 < \mathrm{Spark}_0$, and for all x satisfying $\|x\|_0 < \mathrm{Spark}_0/2$ [2, 8]. Similar results apply in Spectrum Blind Sampling of continuous-time signals [1]. When the measurement contains noise, the error in the ℓ_0 reconstruction for a given B is bounded in terms of Spark_η.

Theorem 2.1 *[8] Given D and ϵ, set $\eta = 2\epsilon/D$. If $x_i, i = 1, 2$ are two approximate representations both obeying*

$$\|y - Bx_i\|_2 \leqslant \epsilon \quad \textit{and} \quad \|x_i\|_0 < \mathrm{Spark}_\eta(B)/2,$$

then $\|x_1 - x_2\|_2 \leqslant D$.

Unfortunately, the computation of Spark_η is NP-hard. A more relaxed bound [8] in terms of a quantity called coherence is easily computed, but the bound is often too conservative to be useful in practice.

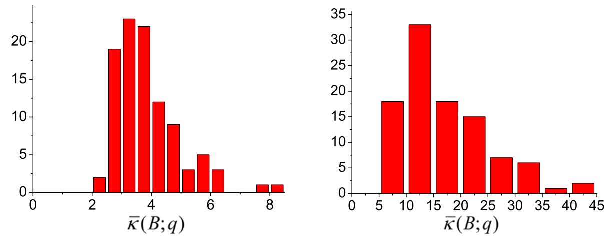
2.2. Worst-Case Condition Number

Let $\kappa(B) = \sigma_{\max}(B)/\sigma_{\min}(B)$ denote the two-norm condition number of matrix B, given by the ratio of its largest to smallest singular values. The condition number $\kappa(B_{\bullet,J})$ of any such minor will depend on the particular subset J of columns selected. For a given upper bound q, $q \leqslant p$, on the size of $B_{\bullet,J}$, we define $\overline{\kappa}(B;q) \triangleq \max_{|J|\leqslant q} \kappa(B_{\bullet,J})$, the *worst-case condition number* of all minors of size at most q.

Even when $\mathrm{supp}(x)$ is correctly recovered (e.g., using basis pursuit or OMP) in the presence of errors in y, these errors can be amplified in the solution $x_{\mathrm{supp}(x)}$ by the condition number of $B_{\bullet,\mathrm{supp}(x)}$, which can be as large as $\overline{\kappa}(B;q)$. In fact, in spectrum blind sampling, the recovery of the spectral support $x_{\mathrm{supp}(x)}$ is asymptotically (with increasing data) insensitive to noise, so that the error in the recovery of the signal of interest is in practice governed by $\overline{\kappa}(B;q)$ [1].

To study the distribution of $\overline{\kappa}(B;q)$ over random B, we computed its histograms (Fig. 1) for iid Gaussian distributed $B_{i,j} \sim N(0,1)$. As these examples demonstrate, as q increases, $\overline{\kappa}(B;q)$ can vary over a wider range between randomly generated B, with a moderate probability of bad B, but non-negligible fraction of very good B with low $\overline{\kappa}(B;q)$. For $q = 4$, $\overline{\kappa}(B;q)$ reaches close to 50, but is only about 7 for some good B. Therefore, an efficient selection strategy is needed for a systematic evaluation of $\overline{\kappa}(B;q)$ over a sufficient sample of candidate B to avoid the catastrophic B. It will instead allow to select B close to the best, and provide an SNR improvement of more than 16dB.

It can be shown (confirmed numerically in Fig. 1) that $\overline{\kappa}(B;q)$ is an increasing function of q (intuitively, with more



(a) (b)

Fig. 1. Histogram of $\overline{\kappa}(B;q)$ for Gaussian random matrices. $n = 8$. $p = 6$. (a) $q = 2$. (min 2.4, max 8.2) (b) $q = 4$. (min 6.8, max 43.9)

columns, there is increased opportunity for greater linear dependence), so that $\overline{\kappa}(B;q) = \max_{|J|=q} \kappa(B_{\bullet,J})$. Therefore, for given B, we wish to compute the largest q such that $\overline{\kappa}(B;q) < \xi$, where $\xi > 1$ is a fixed stability bound, determined by the application. This will determine, for given sensing matrix B, the maximum sparsity $\|x\|_0$ of vectors that can be recovered with a guaranteed condition number less than ξ. This largest q, denoted q^*, is obtained as the solution to the following optimization problem

$$\mathrm{P2}: \qquad q^* = \begin{array}{ll} \max_{q\in\mathbb{Z}_+} & q \\ \text{subject to} & \max_{|J|\leqslant q} \kappa(B_{\bullet,J}) < \xi. \end{array}$$

2.3. Uniform Uncertainty Property

The restricted isometry constant δ_q defined by Candes and Tao [7] can be expressed in the following form :

$$\delta_q = \min_\delta\{\delta : 1-\delta \leqslant (\sigma(B_{\bullet,J}))^2 \leqslant 1+\delta, \forall J, |J| \leqslant q\}, \quad (1)$$

and provides a sufficient condition for exact ℓ_1-reconstruction, as well as an error bound for noisy measurements.

Theorem 2.2 *[7] If $\delta_{2q} + \delta_{3q} < 1$, then ℓ_1-minimization exactly recovers every x with $\|x\|_0 \leqslant q$ from $y = Bx$.*

Theorem 2.3 *[9] If $\delta_{3q} + 3\delta_{4q} < 2$, for any x that obeys $\|y - Bx\|_2 \leqslant \epsilon$ and $\|x\|_0 \leqslant q$, the error in the reconstruction $\hat{x}$ achieved through ℓ_1-minimization is bounded by $\|\hat{x} - x\|_2 \leqslant C_q\epsilon$, where C_q depends only on δ_{4q}.*

In both cases, computation of δ_q, which is NP-hard, is necessary in order to verify the satisfiability for a given CS matrix. In the next section, we provide a lower bound for r^* defined by the following problem.

$$\mathrm{P3}: \qquad r^* = \begin{array}{ll} \max_{q\in\mathbb{Z}_+} & q \\ \text{subject to} & \sum_{k=1}^{\ell} c_k \delta_{d_k q} < 1, \end{array} \quad (2)$$

where $c_k > 0, d_k \in \mathbb{Z}$ are constants for $k = 1, \cdots, \ell$. Using this lower bound, we can verify the sufficient conditions for unique and stable ℓ_1-recovery.

3. EFFICIENT COMPUTATION OF THE BOUNDS

Unfortunately, because they require combinatorial search over all subset J of $\{1,\ldots n\}$, the computation of any of the performance bounds (equivalently, the solution of any of the problems P1 – P3) is NP-hard, and thus infeasible except for small examples. We derive efficient (polynomial-time) algorithms to compute tight bounds for these performance bounds, which thus provide computable performance guarantees.

3.1. Spark_η

We derive an efficient algorithm to compute a lower bound for Spark_η using a sequence of problem reformulations. Setting $A = B^T B$, define the set

$$T_L^\alpha \triangleq \{q \in \mathbb{Z} : \min_{|J|\leqslant q} \lambda_{\min}(A_{J,J}) > \alpha\}.$$

For any fixed α, let $q_L^*(\alpha) \triangleq \max(T_L^\alpha)$ denote the maximum element in the set T_L^α. Then it can be easily verified that $\mathrm{Spark}_\eta = q_L^*(\eta^2) + 1$.

Proposition 3.1 *An alternative way to compute $q_L^*(\alpha)$ is given by the following problem.*

$$\mathrm{P1}': \quad \begin{array}{rll} q_L^*(\alpha) = & \min_{x\in\mathbb{R}^n} & \|x\|_0 - 1 \\ & \text{subject to} & x^T A x \leqslant \alpha \\ & & \|x\| = 1 \end{array} \qquad (3)$$

Proof (Outline) It can be shown that $q_L^*(\alpha) = \min\{q-1 : \min_{|J|\leqslant q} \lambda_{\min}(A_{J,J}) \leqslant \alpha\}$. The result then follows by the Courant-Fisher variational characterization of $\lambda_{\min}$. ∎

Problem $\mathrm{P1}'$ is still NP hard. Consider its ℓ_1-relaxation,

$$\mathrm{P1}'': \quad \begin{array}{ll} \min_{x\in\mathbb{R}^n} & \|x\|_1^2 - 1 \\ \text{subject to} & x^T A x \leqslant \alpha \\ & \|x\| = 1 \end{array} \qquad (4)$$

Since by the Cauchy-Schwartz inequality $\|x\|_1 \leqslant \sqrt{\|x\|_0}$ when $\|x\| = 1$, the minimum achieved in $\mathrm{P1}''$ lower bounds that of $\mathrm{P1}'$.

By reformulating $\mathrm{P1}''$ as the following equivalent semidefinite programming (SDP) problem over the cone S_n^+ of positive semi-definite $n \times n$ real matrices, we replace the nonconvex constraint of $\|x\| = 1$ by a linear constraint.

$$\mathrm{P1}''': \quad \begin{array}{ll} \min\limits_{X\in S_n^+} & \mathbf{1}^T |X| \mathbf{1} - 1 \\ \text{subject to} & \mathbf{tr}(AX) \leqslant \alpha \\ & \mathbf{tr}(X) = 1 \\ & \mathbf{rank}(X) = 1 \end{array} \qquad (5)$$

The last constraint is necessary because we derived $\mathrm{P1}'''$ through the relation $X = xx^T$. Unlike the other linear constrains, the rank constraint is difficult. We relax further by dropping the rank constraint. Because the feasible set is again increased, the achieved minimum lower bounds the actual minimum of $\mathrm{P1}'$, as stated below.

Theorem 3.2 *Let $s_L^*(\alpha)$ denote the minimum achieved in the relaxed version of Problem $\mathrm{P1}'''$ without the rank constraint. Then $q_L^*(\alpha) \geqslant \lceil s_L^*(\alpha)\rceil$.*

As a corollary, $\mathrm{Spark}_\eta \geqslant \lceil s_L^*(\eta^2)\rceil + 1$.

3.2. Worst-Case Condition Number

With $A = B^T B$ and $\gamma = \xi^2$, problem P2 can be rewritten as

$$\mathrm{P2}': \quad \begin{array}{ll} \max_{q\in\mathbb{Z}_+} & q \\ \text{subject to} & \max_{|J|\leqslant q} \kappa(A_{J,J}) < \gamma. \end{array} \qquad (6)$$

Define the set

$$T_U^\beta \triangleq \{q \in \mathbb{Z} : \max_{|J|\leqslant q} \lambda_{\max}(A_{J,J}) < \beta\}, \qquad (7)$$

and $q_U^*(\beta) \triangleq \max(T_U^\beta)$. Our first goal is to relate the solution of $\mathrm{P2}'$ to q_L^* defined in Section3.1 and to q_U^*.

Lemma 3.3 *Let $A \in S_n^+$ and $\gamma \geqslant 1$. Then $\kappa(A) < \gamma$ if and only if $\exists \alpha > 0, \beta > 0$ such that $\lambda_{\min}(A) > \alpha, \lambda_{\max}(A) < \beta$, and $\beta \leqslant \gamma\alpha$.*

Proposition 3.4 *Let $q^*(\alpha,\beta) \triangleq \min\{q_L^*(\alpha), q_U^*(\beta)\}$. Then q^* in P2 satisfies $q^* \geqslant q^*(\alpha,\beta), \forall \alpha, \beta \leqslant \gamma\alpha$.*

Proof (Outline) It is easy to see that $q^*(\alpha,\beta) \in T_L^\alpha \cap T_U^\beta$. Let $T_0 \triangleq \{q : \max_{|J|\leqslant q} \kappa(A_{J,J}) < \gamma\}$. Then $q^* = \max(T_0)$. $q^*(\alpha,\beta) \in T_L^\alpha \cap T_U^\beta$ and $\beta \leqslant \gamma\alpha$ imply $q^*(\alpha,\beta) \in T_0$ by Lemma 3.3. ∎

In order to obtain the tightest lower bound, we maximize $q^*(\alpha,\beta)$ over $\forall \alpha, \beta \leqslant \gamma\alpha$. Without changing the maximum, we can restrict the maximization to the set $F \triangleq \{(\alpha,\beta) : \lambda_{\min}(A) \leqslant \alpha \leqslant \lambda_{\max}(A)/\gamma, \beta = \gamma\alpha\}$. Let $q^{**} \triangleq \max_{(\alpha,\beta)\in F} q^*(\alpha,\beta)$. Then it follows from Proposition 3.4 that $q^{**} \leqslant q^*$.

Because the computation of $q_U^*(\beta)$ is still NP-hard, we follow analogous steps to those used in the relaxation of Problem $\mathrm{P1}'$ for the computation of $q_L^*(\beta)$, and instead solve

$$\begin{array}{rll} s_U^*(\beta) = & \min\limits_{X\in S_n^+} & \mathbf{1}^T |X| \mathbf{1} - 1 \\ & \text{subject to} & \mathbf{tr}(AX) \geqslant \beta \\ & & \mathbf{tr}(X) = 1. \end{array} \qquad (8)$$

Then we obtain the lower bound $\lceil s_U^*(\beta)\rceil \leqslant q_U^*(\beta)$. Combining the previous results we obtain the final, efficiently computable lower bound.

Theorem 3.5 *Let $s^{**} \triangleq \max_{(\alpha,\beta)\in F} \min\{s_L^*(\alpha), s_U^*(\beta)\}$. Then $\lceil s^{**}\rceil \leqslant q^*$.*

3.3. Uniform Uncertainty Property

Define $V_k^\alpha \triangleq \{q \in \mathbb{Z} : c_k \delta_{d_k q} < \alpha\}$ and let $r_k^*(\alpha) \triangleq \max(V_k^\alpha)$ for $k = 1, \cdots, \ell$. Since δ_q is nondecreasing in q, $x \in V_k^\alpha$ if and only if $x \leqslant \max(V_k^\alpha)$ for each k. Hence $r_k^*(\alpha) = \min((V_k^\alpha)^C) - 1$ for each k.

Let

$$u^*(c) = \begin{array}{ll} \min_{q \in \mathbb{Z}_+} & q - 1 \\ \text{subject to} & \delta_q \geqslant c. \end{array}$$

Then $r_k^*(\alpha) = \lceil u^*(\alpha/c_k)/d_k \rceil$ for $k = 1, \cdots, \ell$.

Proposition 3.6 *$\delta_q \geqslant c$ if and only if $\min_{|J| \leqslant q} \lambda_{\min}(A_{J,J}) \leqslant 1 - c$ or $\max_{|J| \leqslant q} \lambda_{\max}(A_{J,J}) \geqslant 1 + c$.*

It follows from Proposition 3.6 that $u^*(c) = \min\{q_L^*(1 - c), q_U^*(1 + c)\}$. Let $t_k^*(\alpha) \triangleq \min\{s_L^*(1 - \alpha/c_k), s_U^*(1 + \alpha/c_k)\}/d_k$. Then $\lceil t_k^*(\alpha) \rceil \leqslant r_k^*(\alpha)$ for $k = 1, \cdots, \ell$.

Lemma 3.7 *For $\{x_k\}_{k=1}^\ell \subset \mathbb{R}$, $\sum_{k=1}^\ell x_k < 1$ if and only if $\exists \{\alpha_k\}_{k=1}^\ell \subset \mathbb{R}$ such that $\sum_{k=1}^\ell \alpha_k \leqslant 1$ and $x_k < \alpha_k$ for $k = 1, \cdots, \ell$.*

Proposition 3.8 *Let $\theta \triangleq [\theta_1, \cdots, \theta_\ell]^T \in \mathbb{R}^\ell$ and define $t^*(\theta) \triangleq \min_k t_k^*(\theta_k)$. Then r^* in (2) satisfies $r^* \geqslant \lceil t^*(\theta) \rceil$ for all θ such that $\mathbf{1}^T\theta \leqslant 1$.*

Proof (Outline) Let $V_0 \triangleq \{q \in \mathbb{Z} : \exists \theta \in \mathbb{R}^\ell, \mathbf{1}^T\theta \leqslant 1, c_k \delta_{d_k q} < \theta_k, \forall k = 1, \cdots, \ell\}$. Then $r^* = \max(V_0)$ from Lemma 3.7. Since $\mathbf{1}^T\theta \leqslant 1$ implies $\cap_{k=1}^\ell V_k^{\theta_k} \subseteq V_0$, $t^*(\theta) \leqslant \min_k r_k^*(\theta_k) = \max(\cap_{k=1}^\ell V_k^{\theta_k}) \leqslant \max(V_0)$. ∎

For the tightest lower bound, we maximize $t^*(\theta)$ over $\forall \mathbf{1}^T\theta \leqslant 1$. Without changing the maximum, we can restrict the maximization to the set $G \triangleq \{\theta \in \mathbb{R}^\ell : \mathbf{1}^T\theta = 1 \text{ and } \delta_1 \leqslant \theta_k \leqslant 1 - (\ell - 1)\delta_1, \forall k = 1, \cdots, \ell\}$. Let $t^{**} \triangleq \max_{\theta \in G} t^*(\theta)$. Then it follows from Proposition 3.8 that $\lceil t^{**} \rceil \leqslant r^*$.

Theorem 3.9 *Let $t^{**} \triangleq \max_{\theta \in G} \min_k t_k^*(\theta_k)$. Then $\lceil t^{**} \rceil \leqslant r^*$.*

4. NUMERICAL RESULTS

To solve the SDP problems, we used the software package YALMIP [12] to parse them into a standard form with sparse matrices, and computed the optimum using SeDuMi [13].

We conducted numerical experiments with randomly generated matrices following the Gaussian distribution. We compared the obtained lower bounds with the exact results obtained by exhaustive search. Note that it is infeasible to use exhaustive search when n and q are large. The cost of computing $\min_{|J|=q} \lambda_{\min}(B_{\bullet,J})$ is $O\left(pq^2\binom{n}{q}\right)$, becoming impractical on current desktop machines already for $n > 30$ and comparable p and q. To enable comparison of the bound with the actual value, in this paper we restricted the numerical experiments to small, tractable sizes.

$q_L^* \setminus \lceil s_L^{**} \rceil$	1	2	3
1	**3**	0	0
2	**8**	**36**	0
3	**1**	**41**	**9**
4	0	0	**2**

(a)

\	1	2	3	4
2	**1**	**5**	0	0
3	**1**	**19**	**16**	0
4	0	**9**	**43**	**1**
5	0	0	**2**	**3**

(b)

Table 1. Joint histogram of $(q_L^*(\eta^2), \lceil s_L^*(\eta^2) \rceil)$ for Gaussian random matrices ($n = 16, p = 12$). (a) $\eta^2 = 0.1$ (b) $\eta^2 = 0.05$.

$q^* \setminus \lceil s^{**} \rceil$	1	2	3
1	**46**	0	0
2	**27**	**26**	0
3	0	**1**	0
4	0	0	0

(a)

\	1	2	3
1	**1**	0	0
2	**10**	**17**	0
3	**1**	**31**	**22**
4	0	**2**	**16**

(b)

Table 2. Joint histogram of $(q^*, \lceil s^{**} \rceil)$ for Gaussian random matrices ($n = 8, p = 6$). (a) $\xi = 5$ (b) $\xi = 20$.

Table. 1 compares $q_L^*(\eta^2)$ and its lower bound $\lceil s_L^*(\eta^2) \rceil$ for 100 trial matrices B. The results show that our lower bound for $\text{Spark}_\eta - 1$ coincides with with the actual value in many cases, or underestimates it by a small number. Table. 2 compares the maximum q^* satisfying the stability condition on $\overline{\kappa}(B; q)$ at level ξ, with the lower bound on q^*, providing similar favorable conclusions.

These results suggest that our lower bounds are tight, and will provide both provable and practically effective performance guarantees in compressed sensing.

5. REFERENCES

[1] P. Feng and Y. Bresler, *Proc. ICASSP*, pp. 1688–1691, 1996.

[2] R. Venkataramani and Y. Bresler, *Proc. ICIP*, pp. 752–756, 1998.

[3] D. L. Donoho, , *IEEE Trans. Inf. Theory*, vol. 52, no. 4, pp. 1289–1306, 2006.

[4] E. Candes and T. Tao, *IEEE Trans. Inf. Theory*, vol. 52, no. 12, pp. 5406–5425, Dec. 2006.

[5] Y. Bresler and P. Feng, *Proc. ICIP*, pp. 701–704, 1996.

[6] D. L. Donoho and M. Elad, *Proc. Nat. Aca. Sci.*, vol. 100, no. 5, pp. 2197–2202, 2003.

[7] E. J. Candes and T. Tao, *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4203–4215, 2005.

[8] D. L. Donoho, M. Elad, and V. N. Temlyakov, *IEEE Trans. Inf. Theory*, vol. 52, no. 1, pp. 6–18, 2006.

[9] E. Candes, J. Romberg, and T. Tao, *Comm. Pure Appl. Math*, vol. 59, no. 8, pp. 1207–1223, 2006.

[10] E. Candes, J. Romberg, and T. Tao, *IEEE Trans. Inf. Theory*, vol. 52, no. 2, pp. 489–509, Feb. 2006.

[11] T. Tao, "Open question: deterministic UUP matrices," `http://terrytao.wordpress.com`.

[12] J. Lofberg, "Yalmip : A toolbox for modeling and optimization in MATLAB." in *Proc. CACSD*, Taipei, Taiwan, 2004.

[13] J. F. Sturm, "SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones." *Optimization Methods and Software*, vol. 11–12, pp. 625–653, 1999.