OPTIMAL NOISE BENEFITS IN NEYMAN-PEARSON SIGNAL DETECTION

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ABSTRACT

We present an algorithm to find near-optimal "stochastic resonance" (SR) noise benefits for Neyman-Pearson (N-P) hypothesis testing or signal-detection problems. The optimal N-P SR noise is no more than two randomized noise realizations when the optimal noise exists. We give necessary and sufficient conditions for the existence of such optimal noise in fixed detectors. There exists a sequence of noise variables whose detection performance limit is optimal when such noise does not exist. An upper bound limits the number of iterations that the algorithm requires to find such near-optimal noise.

Index Terms— optimal noise, stochastic resonance, signal detection, Neyman-Pearson test, noise-finding algorithm

1. INTRODUCTION

Stochastic resonance (SR) occurs when noise benefits a nonlinear system [2, 6, 13]. The noise benefit can take many forms such as an increase in a bit count, a signal-to-noise ratio, a cross-correlation, or a decrease in a probability of error, or an increase in detection probability for a preset level of false-alarm probability [3, 4, 5, 7, 8, 9, 10, 14]. SR applications range from neural processing to physical devices and often involve some form of nonlinear signal detection. We focus here on the special case of SR in signal detection that uses Neyman-Pearson (N-P) hypothesis testing to decide between two simple alternatives. We define the noise as N-P SR noise if adding such noise increases the signal detection probability P_D while the false-alarm probability P_{FA} stays at or below a preset level α for *fixed* detection strategy.

We present three main SR results for Neyman-Pearson signal detection. The results do not require that the user knows the prior probabilities of the competing hypotheses as in Bayesian signal detection. The first SR result is that the existence of N-P SR noise does not itself imply the existence of *optimal* N-P SR noise. We state necessary and sufficient conditions for the existence of optimal N-P SR noise. There exists a sequence of noise variables whose detection performance limit is optimal when the optimal N-P SR noise does not exist. The second SR result is a sufficient condition to detect N-P SR noise benefits in simple threshold signal detection. The third SR result is an algorithm that finds near-optimal N-P SR noise from a finite set $\tilde{\mathcal{N}}$ of noise realizations. This noise is nearly optimal if the detection and false alarm probabilities in $\tilde{\mathcal{N}}$ and in the actual noise space $\mathcal{N} \supset \tilde{\mathcal{N}}$ are sufficiently close. Figure 1 shows how noise can improve N-P detection performance if the detector's receiver operating characteristic (ROC) curve is not concave.

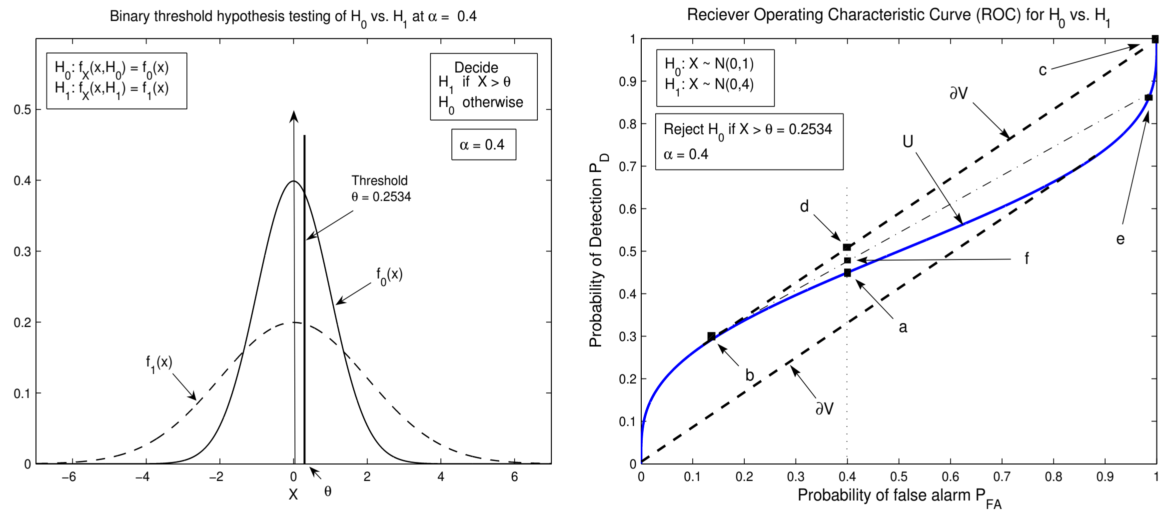
These SR results extend and correct prior work in "detector randomization" or adding noise in N-P signal detection. Tsitsiklis [12] explored the mechanism of detection-strategy randomization for a *finite* set of detection strategies (operating points) in decentralized detection. He first showed that there exists a randomized detection strategy that uses a proper convex or random combination of at most two existing detection strategies and that gives the optimal N-P detection performance. Such optimal detection strategies lie on the upper *boundary* of the convex hull of the ROC-curve operating points. Scott et al. [11] and Appadwedula et al. [1] later used the same optimization principle in classification systems and energy-efficient detection in sensor networks respectively. Then Chen et al. [3] used a *fixed* detector structure. They injected noise in the data samples to obtain a proper random combination of operating points on the ROC curve. They showed that the optimal N-P SR noise for fixed detectors is a proper randomization of no more than two noise realizations.

But Chen et al. [3] assumed that the convex hull V of the set of ROC curve operating points $U \subseteq \mathbf{R}^2$ always contains its boundary ∂V and thus that the convex hull V is closed. This is not true in general. The topological problem is that the convex hull V need not be closed if U is not compact: the convex hull of U is open if U itself is open. Chen et al. argued correctly along the lines of the proof of Theorem 3 in [3] when they concluded that the "optimum pair can only exist on the *boundary*." But their later claim that "each z on the boundary can be expressed as the convex combination of only two elements of U" is not true in general because V may not include *all* of its boundary points. The optimal N-P SR noise need not exist at all in a *fixed* detector. Figure 1 shows a case where the N-P SR noise exists but the *optimal* N-P SR noise does not exist in the noise space $\mathcal{N} = \mathbf{R}$. We show below that if we restrict the noise space to the compact interval $[-5, 5]$ then the optimal SR noise does exist. Our algorithm finds a nearly optimal N-P SR noise from a discretized set of noise realizations $\tilde{\mathcal{N}}$ = [-5:0.0001:5] in just 9 iterations.

The next three sections present and illustrate these SR noise benefits. The first section presents the formal Neyman-Pearson framework and gives necessary and sufficient conditions for the existence of optimal N-P SR noise and gives the exact form of the optimal noise probability density (pdf) in the noise domain $\mathbf{R}^m$ if such noise exists. The second section presents the algorithm for finding the N-P SR noise from a finite set of noise realizations $\tilde{\mathcal{N}}$. The final section presents a detailed example of how these SR results can improve Neyman-Pearson signal detection by deliberately but judiciously adding noise.

2. OPTIMAL NOISE PDFS FOR N-P DETECTION

Consider a binary hypothesis test where we decide between $H_0 : f_{\mathbf{X}}(\mathbf{x}, H_0) = f_0(\mathbf{x})$ and $H_1 : f_{\mathbf{X}}(\mathbf{x}, H_1) = f_1(\mathbf{x})$ using an m-dimensional noisy observation vector $\mathbf{Y} = \mathbf{X} + \mathbf{N}$. Here $\mathbf{X} \in \mathbf{R}^m$ is the original observation data vector, $\mathbf{N} \in \mathcal{N} \subseteq \mathbf{R}^m$ is a noise vector with pdf f_N, and $\mathcal{N}$ is the noise space. We assume that the noise $\mathbf{N}$ is independent of $\mathbf{X}$. The noise vector $\mathbf{N}$ can



(a) (b)

Fig. 1. SR noise benefits in N-P signal detection. (a) The thin solid line shows the probability density function (pdf) f_0 of signal X under the normal hypothesis H_0: $X \sim N(0,1)$ while the dashed line shows the pdf f_1 of X under the alternative normal hypothesis H_1: $X \sim N(0,4)$. The detector rejects H_0 if the noisy observation $X+N>\theta$. The thick vertical solid line shows the threshold θ. (b) The solid line shows the monotonic but nonconcave ROC curve $U=\{(p_{FA}(n),p_D(n))\colon n\in\mathbf{R}\}$ of the detector where n is the realization of the additive noise N in X, $p_{FA}(n)$ = 1-$\Phi(\theta$-$n)$, and $p_D(n)$ = 1-$\Phi(\frac{\theta-n}{2})$ for standard normal cumulative distribution function Φ. The detector operates at point a = $(p_{FA}(0),p_D(0))$ = (0.4, 0.4496) on the ROC curve in the absence of noise. Nonconcavity of the ROC curve U between the points b = $(p_{FA}(n_1),p_D(n_1))$ and c = (1,1) allows the N-P SR effect to occur. A proper convex or random combination of two operating points b and e = $(p_{FA}(n_2),p_D(n_2))$ gives a better detection performance (point f) than point a at the same false-alarm level $p_{FA}(0)$ = α = 0.4. Such a random combination of operating points results from adding a discrete noise N with pdf $f_N(n)$ = $\lambda\delta(n-n_1)$ + $(1-\lambda)\delta(n-n_2)$ to the data sample X where λ = $(p_{FA}(n_2)$-$\alpha)/(p_{FA}(n_2)$-$p_{FA}(n_1))$. Point d is on the upper boundary ∂V of the ROC curve's convex hull (dashed tangent line between b and c). So d is the supremum of detection performances that random or convex combination of operating points on the ROC can achieve so that α remains 0.4. Note that d is the convex combination of b and c but it is not realizable by adding only noise in the data sample X because point c = (1,1) is not on the ROC curve since there is no noise realization $n\in R$ such that 1-$\Phi(\theta$-$n)$ = 1 = 1-$\Phi(\frac{\theta-n}{2})$). Thus the optimal N-P SR noise does not exist in the noise space $\mathcal{N}$ = $\mathbf{R}$.

be random or even a deterministic constant such as $f_N(\mathbf{n})$ = $\delta(\mathbf{n}-\mathbf{n}_o)$. Here f_0 and f_1 are the pdfs of the observation $\mathbf{X}$ under the hypothesis H_0 and H_1. We assume that we do not know the prior probabilities $P(H_i)$ of the hypotheses H_i. Let $0\le\phi(T(\mathbf{Y}))\le 1$ be a critical (or test) function that decides between H_0 and H_1 using a test statistic $T(\mathbf{Y})$. Then the test chooses the hypothesis H_1 with probability $\phi(T(\mathbf{Y}))$ for any observation $\mathbf{x}$. Else it chooses the hypothesis H_1 with probability $1-\phi(T(\mathbf{Y}))$. We want to find the conditions for the existence of the optimal additive noise $\boldsymbol{N}_{\text{opt}}$ that gives the best achievable detection performance without sacrificing the test significance level α for the given *fixed* test function ϕ.

Define $P_D(\mathbf{n})$ and $P_{FA}(\mathbf{n})$ as the respective probabilities of detection and false detection (alarm) when the noise realization is $\mathbf{n}$. Define $P_D(f_N)$ = $\int_{\mathcal{N}}P_D(\mathbf{n})f_N(\mathbf{n})d\mathbf{n}$ and $P_{FA}(f_N)$ = $\int_{\mathcal{N}}P_{FA}(\mathbf{n})f_N(\mathbf{n})d\mathbf{n}$ as the respective probabilities of detection and false detection when the noise pdf is f_N. Let $f_{N_{\text{opt}}}$ be the pdf of the optimal SR noise $\boldsymbol{N}_{\text{opt}}$ that we add to the observed data $\mathbf{X}$ to maximize the probability of detection P_D when keeping $P_{FA}\le\alpha$. So we need to find

$$f_{N_{\text{opt}}} \;=\; \underset{f_N}{\arg\max}\int_{\mathcal{N}}P_D(\mathbf{n})f_N(\mathbf{n})d\mathbf{n} \qquad (1)$$

such that

$$f_{N_{\text{opt}}}(\mathbf{n}) \;\ge\; 0 \quad \text{for all } \mathbf{n}, \qquad (2)$$

$$\int_{\mathcal{N}}f_{N_{\text{opt}}}(\mathbf{n})d\mathbf{n} \;=\; 1, \quad \text{and} \qquad (3)$$

$$P_{FA}(f_{N_{\text{opt}}}) \;=\; \int_{\mathcal{N}}P_{FA}(\mathbf{n})f_{N_{\text{opt}}}(\mathbf{n})d\mathbf{n} \;\le\; \alpha. \qquad (4)$$

Conditions (2) and (3) are defining properties of the pdf while (1) and (4) state the Neyman-Pearson criterion for the optimal SR noise pdf $f_{N_{\text{opt}}}$.

The next two theorems give necessary and sufficient conditions for the existence of the optimal N-P SR noise and the form of its pdf if it exists. We need the following definitions for Theorems 1 and 2. Define first the sets

$$D^+ \;=\; \{\mathbf{n}\in\mathcal{N}:(P_{FA}(\mathbf{n})-\alpha)\ge 0\} \quad \text{and} \qquad (5)$$

$$D^- \;=\; \{\mathbf{n}\in\mathcal{N}:(P_{FA}(\mathbf{n})-\alpha)\le 0\}\,. \qquad (6)$$

Assume $D^-\neq\emptyset$ to avoid a trivial nonexistence of $\boldsymbol{N}_{\text{opt}}$. Let $P_{D^+{}_{sup}}$, $P_{D^-{}_{sup}}$, and $P_{D_{sup}}$ be the respective supremum of $P_D(\mathbf{n})$ over the sets D^+, D^-, and $\mathcal{N}$. Next define a function

$$g(\mathbf{n},k) \;=\; P_D(\mathbf{n})-k\,(P_{FA}(\mathbf{n})-\alpha) \qquad (7)$$

and let $d^+(k)$, $d^-(k)$, and $d(k)$ be its respective supremum over the sets D^+, D^-, and $\mathcal{N}$. Finally let

$$G^+ \;=\; \{\mathbf{n}\in D^+:P_D(\mathbf{n})=P_{D^+{}_{sup}}\} \quad \text{and} \qquad (8)$$

$$G^- \;=\; \{\mathbf{n}\in D^-:P_D(\mathbf{n})=P_{D^-{}_{sup}}\}. \qquad (9)$$

Theorem 1: (a) Suppose that $P_{D^-{}_{sup}} \geq P_{D^+{}_{sup}}$. If G^- is nonempty then

$$f_{N_{\text{opt}}}(\mathbf{n}) \quad = \quad \delta(\mathbf{n} - \mathbf{n}_o) \tag{10}$$

for some $\mathbf{n}_o \in G^-$ is an optimal SR noise pdf for Neyman-Pearson detection and $P_{FA}(f_{N_{\text{opt}}}) \leq \alpha$. If G^- is empty then the Neyman-Pearson optimal SR noise does not exist for the given test level α. But there exists a noise pdf sequence $\{f_{N_r}\}_{r=1}^{\infty}$ of the form (10) such that

$$\lim_{r\to\infty} P_D(f_{N_r}) \quad = \quad P_{D_{sup}} \tag{11}$$

(b) Suppose that $P_{D^-{}_{sup}} < P_{D^+{}_{sup}}$. If the Neyman-Pearson optimal SR noise pdf $f_{N_{\text{opt}}}(\mathbf{n})$ exists then $P_{FA}(f_{N_{\text{opt}}}) = \alpha$.

Theorem 1 (a) gives the optimal N-P SR noise pdf $f_{N_{\text{opt}}}$ if it exists and if $P_{D^-{}_{sup}} \geq P_{D^+{}_{sup}}$. Theorem 2 gives necessary and sufficient conditions for the existence of $f_{N_{\text{opt}}}$ when $P_{D^-{}_{sup}} < P_{D^+{}_{sup}}$.

Theorem 2: Suppose that $P_{D^-{}_{sup}} < P_{D^+{}_{sup}}$.

(a) There exists $k^* \in \mathbf{R}$ such that $d^+(k^*) = d^-(k^*) = d(k^*)$ and $\min\{d^+(k), d^-(k)\} \leq d(k^*) \leq \max\{d^+(k), d^-(k)\}$ for any $k \in \mathbf{R}$.

(b) Suppose noise pdf f_N satisfies $P_D(f_N) = d(k^*) > P_D(0)$ and $P_{FA}(f_N) = \alpha$. Then f_N is a Neyman-Pearson optimal noise pdf. So the optimal N-P SR detection probability $P_{D_{\text{opt}}}$ is $d(k^*)$.

(c) Suppose that there exist $\mathbf{n}_1 \in D^-$ and $\mathbf{n}_2 \in D^+$ such that $g(\mathbf{n}_1, k^*) = d^-(k^*) = d(k^*) = g(\mathbf{n}_2, k^*) = d^+(k^*)$. Then

$$f_{N_{\text{opt}}}(\mathbf{n}) \quad = \quad \lambda\delta(\mathbf{n} - \mathbf{n}_1) \; + \; (1-\lambda)\delta(\mathbf{n} - \mathbf{n}_2) \tag{12}$$

$$\text{with} \quad \lambda \quad = \quad \frac{P_{FA}(\mathbf{n}_2) - \alpha}{P_{FA}(\mathbf{n}_2) - P_{FA}(\mathbf{n}_1)} \tag{13}$$

is the optimal Neyman-Pearson SR noise pdf if $d(k^*) > P_D(0)$.

(d) Neyman-Pearson optimal SR noise does not exist if condition (c) does not hold. But there exists a noise pdf sequence $\{f_{N_r}\}_{r=1}^{\infty}$ of the form (12)-(13) such that

$$\lim_{r\to\infty} P_D(f_{N_r}) \quad = \quad d(k^*) \tag{14}$$

Theorem 3 below gives a sufficient condition to detect an N-P SR noise benefit in detectors that use a *single* noisy observation $Y \in \mathbf{R}$ to decide between H_0 and H_1.

Theorem 3: Suppose that the P_D and P_{FA} of a detector are second-order continuously differentiable in $\mathbf{R}$ and $P_{FA}(0) = \alpha$. Suppose also that P_{FA} is not locally minimum at 0 and P_D is not locally maximum at 0. Then an N-P SR noise exists if

$$P''_D(0)P'_{FA}(0) \quad > \quad P''_{FA}(0)P'_D(0). \tag{15}$$

Theorem 3 implies a following Corollary that gives a sufficient condition to detect an N-P SR noise benefit in threshold detectors that use a *single* noisy observation $Y \in \mathbf{R}$.

Corollary: Suppose that a set of thresholds $\Theta = \{\theta_1, ..., \theta_k\}$ partitions the real line $\mathbf{R}$ into acceptance and rejection regions. Suppose also that the hypothesized pdfs f_i are continuously differentiable in $\mathbf{R}$. Define $s(j) = 1$ if θ_j is a left endpoint of any rejection interval. Else let $s(j) = -1$. Then a proper additive noise can improve the N-P detection of such a detector if
$\left(\sum_j s(j)f'_0(\theta_j)\right)\left(\sum_j f_1(\theta_j)\right) > \left(\sum_j s(j)f'_1(\theta_j)\right)\left(\sum_j f_0(\theta_j)\right)$.

3. N-P SR NOISE FINDING ALGORITHM

Theorem 1 and 2 give the exact form of the optimal N-P SR noise pdf but such noise may not be easy to find in a given noise space $\mathcal{N}$. So we present an algorithm based on successive approximations that can find the N-P SR noise from a finite set of noise realizations $\tilde{\mathcal{N}} \subseteq \mathcal{N}$. The algorithm takes as input ϵ, α, $\tilde{\mathcal{N}}$ (in (5)-(9)), and the respective detection and false alarm probabilities P_D and P_{FA} on $\tilde{\mathcal{N}}$. The algorithm first searches for a constant noise from the set G^- if $P_{D^-{}_{sup}} \geq P_{D^+{}_{sup}}$. If the inequality does not hold then the algorithm finds a number $k(i)$ at every iteration i such that $|d^-(k(i)) - d(k^*)| \leq 2^{-i+1}$ which gives $|d^+(k(i)) - d^-(k(i))| < \epsilon$ in at most $i_{\max} = \lceil \log_2(2/\epsilon) \rceil + 1$ iterations. The algorithm then defines a noise $\tilde{N}'$ as a proper random combination of $\tilde{n}_1 \in D^-$ and $\tilde{n}_2 \in D^+$ so that $g(\tilde{n}_1, k(i_{\max})) = d^-(k(i_{\max}))$, $g(\tilde{n}_2, k(i_{\max})) = d^+(k(i_{\max}))$, and $P_{FA}(f_{\tilde{N}'}) = \alpha$. Theorem 4(a) shows that if $\tilde{N}_{\text{opt}}$ is the optimal N-P SR noise in $\tilde{\mathcal{N}}$ and if $f_{\tilde{N}_{\text{opt}}}$ is the pdf of $\tilde{N}_{\text{opt}}$ then $0 \leq P_D(f_{\tilde{N}_{\text{opt}}}) - P_D(f_{\tilde{N}'}) \leq \epsilon$.

N-P SR Noise Finding Algorithm

<u>If</u> $P_{D^-{}_{sup}} \;\geq\; P_{D^+{}_{sup}}$
  $f_{\tilde{N}_{\text{opt}}}(n) \;=\; \delta(n - \tilde{n}_0)$ for any $\tilde{n}_0 \in G^-$
<u>Else</u>
  Let $k(0) = 1$, $i = 2$, and Find $k(1)$: $d^-(k(1)) = [d^-(k(0)) + d^+(k(0))]/2$
  <u>For</u> $|d^-(k(i\text{-}1)) - d^+(k(i\text{-}1))| \;>\; \epsilon$ and $i \leq \lceil \log_2(2/\epsilon) \rceil$
    Let $r = \operatorname{sgn}[d^-(k(i\text{-}1)) - d^-(k(i\text{-}2))]$ and Find $k(i)$ so that
    $d^-(k(i)) = [d^-(k(i\text{-}1)) + r\max\{rd^-(k(i\text{-}2)), rd^+(k(i\text{-}1))\}]/2$
    $i = i + 1$
  <u>End For</u>
  <u>If</u> $|d^+(k(i\text{-}1)) - d^-(k(i\text{-}1))| \;>\; \epsilon$
    Find $k(i)$: $d^+(k(i)) = d^-(k(i\text{-}1)) + \operatorname{sgn}[d^+(k(i\text{-}1)) - d^-(k(i\text{-}1))]\epsilon$
  <u>Else</u>
    $k(i) = k(i\text{-}1)$
  <u>End If</u>
  $f_{\tilde{N}'}(n) \;=\; \lambda\delta(n - \tilde{n}_1) \;+\; (1-\lambda)\delta(n - \tilde{n}_2)$ where
  $\tilde{n}_1 \in D^-$ so that $P_D(\tilde{n}_1) - k(i)(P_{FA}(\tilde{n}_1) - \alpha) \;=\; d^-(k(i))$,
  $\tilde{n}_2 \in D^+$ so that $P_D(\tilde{n}_2) - k(i)(P_{FA}(\tilde{n}_2) - \alpha) \;=\; d^+(k(i))$,
  and $\lambda = \frac{P_{FA}(\tilde{n}_2) - \alpha}{P_{FA}(\tilde{n}_2) - P_{FA}(\tilde{n}_1)}$.
<u>End If</u>

Theorem 4(b) shows that if for each $n \in \mathcal{N}$ there exists $\tilde{n} \in \tilde{\mathcal{N}}$ so that

$$|P_D(n) - P_D(\tilde{n})| \quad \leq \quad \tau \quad \text{and} \tag{16}$$

$$P_{FA}(\tilde{n}) \quad \leq \quad P_{FA}(n) \tag{17}$$

and if N_{opt} is the optimal N-P SR noise in $\mathcal{N}$ with $f_{N_{\text{opt}}}$ as its pdf then $0 \leq P_D(f_{N_{\text{opt}}}) - P_D(f_{\tilde{N}'}) \leq (\tau + \epsilon)$. Thus the algorithm will find a near-optimal noise $\tilde{N}'$ for any small ϵ if we choose $\tilde{\mathcal{N}}$ such that τ is sufficiently small.

Theorem 4:

(a) For every $\epsilon > 0$ the above algorithm finds an N-P SR noise $\tilde{N}'$ from $\tilde{\mathcal{N}}$ in at most $i_{\max} = \lceil \log_2(2/\epsilon) \rceil + 1$ iterations so that

$$P_D(f_{\tilde{N}_{\text{opt}}}) \;\geq\; P_D(f_{\tilde{N}'}) \quad \geq \quad P_D(f_{\tilde{N}_{\text{opt}}}) - \epsilon \quad \text{and} \tag{18}$$

$$P_{FA}(f_{\tilde{N}'}) \quad \leq \quad \alpha. \tag{19}$$

(b) If $\tilde{\mathcal{N}}$ satisfies (16)-(17) then the detection performance with noise $\tilde{N}'$ is at most $\tau + \epsilon$ less than the optimal SR detection with noise N_{opt}: $P_D(f_{N_{\text{opt}}}) \;\geq\; P_D(f_{\tilde{N}'}) \;\geq\; P_D(f_{N_{\text{opt}}}) - (\tau + \epsilon)$.

4. APPLICATION OF SR NOISE ALGORITHM

Consider a hypothesis test for the variance between two zero-mean Gaussian distributions H_0: $f_0(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{x^2}{2}}$ vs. H_1: $f_1(x) = \frac{1}{\sqrt{2\pi}2}e^{\frac{x^2}{2(2)^2}}$ where we want to decide between H_0 and H_1 using only a *single* observation of X at the significance α = 0.4. Figure 1(a) shows both f_0 and f_1. Note that the optimal N-P test function at level α is a chi-square test that rejects H_0 if $X^2 > \chi^2_\alpha(1)$ because X^2 is a chi-square random variable with 1 degree of freedom. If the detection system uses only X (neither $|X|$ nor X^2) then it requires two thresholds $-\sqrt{\chi^2_\alpha(1)}$ and $\sqrt{\chi^2_\alpha(1)}$ for the optimal decision making.

Suppose that the detection system can use only one threshold θ due to resource limits or some design constraint. If we reject H_0 when $X > \theta$ then $P_D(n)$ = 1-$\Phi(\frac{\theta-n}{2})$ and $P_{FA}(n)$ = 1-$\Phi(\theta$-$n)$ for standard normal cumulative distribution function $\Phi(z) = \int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-w^2/2}dw$. Suppose we want $\alpha = P_{FA}(0)$ = 0.4. Then θ = 0.2534 and the noiseless detection probability $P_D(0)$ = 0.4496. Note that $P_D(n)$ and $P_{FA}(n)$ are monotonic increasing on $\mathbf{R}$ so that $P_{D^-_{sup}} = P_D(0) < P_{D^+_{sup}}$ = 1. Theorem 2(d) implies that *optimal* N-P SR noise does not exist if the noise space is $\mathbf{R}$. But the condition of Theorem 2(c) does hold if we we restrict the noise space to a compact interval (say [-5,5]) because $P_D(n)$ and $P_{FA}(n)$ are continuous functions of n. The hypothesis of the Corollary does not hold at θ = 0.2534 because the ROC curve is not locally convex at point a in Figure 1(b). But it does hold for all $\theta < 0$ because then the ROC curve is locally convex for $p_{FA} > 0.5$.

We apply the algorithm to find near optimal noise in $\mathcal{N} = [-5, 5]$ for $\epsilon = 2^{-20}$. Consider the discretized set $\tilde{\mathcal{N}}$ of noise realizations starting from -5 up to 5 with an increment of 0.0001 ($\tilde{\mathcal{N}}$ = [-5:0.0001:5]). Such $\tilde{\mathcal{N}}$ satisfies (16)-(17) for τ = 0.00004 because 0.4 bounds f_0 and f_1. Figure 2 shows the plots of $g(\tilde{n}, k(i)) = P_D(\tilde{n})$ - $k(i)(P_{FA}(\tilde{n})$-$\alpha)$ before the first iteration (i = 0) and after the 9^{th} iteration (i = 9) where $k(0)$ = 1. The noise finding algorithm finds a value of $k(9)$ = 0.8098 in just 9 ($< i_{max}$ = 22) iterations such that $|d^+(k(9))$ - $d^-(k(9))| < \epsilon = 2^{-20}$. Note that $g(\tilde{n}_1, k(9)) = d^-(k(9))$ at $\tilde{n}_1$ = -0.8805 $\in D^-$ and $g(\tilde{n}_2, k(9)) = d^+(k(9))$ at $\tilde{n}_2 = 5 \in D^+$. Then

$$f_{\tilde{N}'}(n) \;=\; \lambda\delta(n{+}0.8805) \;+\; (1{-}\lambda)\delta(n{-}5) \quad \text{with} \tag{20}$$

$$\lambda \;=\; (P_{FA}(5){-}0.4)/(P_{FA}(5){-}P_{FA}({-}0.8805)) \;=\; 0.6884 \tag{21}$$

is the pdf of a near-optimal N-P additive SR noise $\tilde{N}'$ because $P_D(f_{\tilde{N}'})$ = 0.5052 while the detection probability $P_D(f_{N_{opt}})$ for the optimal N-P SR noise N_{opt} in $\mathcal{N} = [-5, 5]$ will be at most 0.00004 + 2^{-20} more than $P_D(f_{\tilde{N}'})$ by Theorem 4(b). So the algorithm finds a near-optimal N-P SR noise that gives a 12% increase in the probability of detection from 0.4496 to 0.5052.

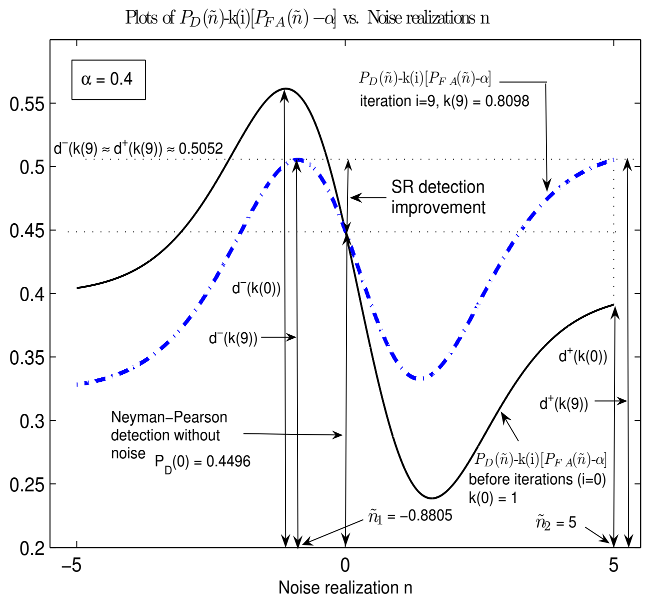


Fig. 2. Finding near-optimal N-P SR noise. Plots of $g(\tilde{n}, k(i))$ = $P_D(\tilde{n})$ - $k(i)(P_{FA}(\tilde{n}) - \alpha)$ before the first iteration (i = 0) and after the 9^{th} iteration (i = 9) where $k(0)$ = 1. The detection probability in the absence of additive noise is $P_D(0)$ = 0.4496. The noise finding algorithm finds a value of $k(9)$ = 0.8098 in just 9 iterations such that $|d^+(k(9)) - d^-(k(9))| < \epsilon = 2^{-20}$. Note that $g(\tilde{n}_1, k(9)) = d^-(k(9))$ at $\tilde{n}_1$ = -0.8805 $\in D^-$ and $g(\tilde{n}_2, k(9)) = d^+(k(9))$ at $\tilde{n}_2$ = 5 $\in D^+$. Then equations (20)-(21) give the pdf of a near-optimal N-P SR noise $\tilde{N}'$ and $P_D(f_{\tilde{N}'})$ = 0.5052. Thus the N-P SR noise $\tilde{N}'$ increases the detection probability P_D from 0.4496 to 0.5052.

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