A STATISTICAL NOISE CONSTRAINED LEAST MEAN FOURTH ADAPTIVE ALGORITHM

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ABSTRACT

In this work, a statistical noise-constrained least mean fourth (SN CLMF) adaptive algorithm is proposed. Based on the fact that in many practical applications an accurate estimate of the fourth-order moment of the noise is available, or can be easily estimated, the learning speed of the LMF algorithm can be then increased considerably by adding a constraint to it. This noise constrained LMF algorithm can be seen as a variable step-size LMF algorithm. Moreover, the concept of energy conservation is used to carry out the rigorous steady-state analysis. Finally, a number of simulations are carried out to corroborate the theoretical findings, and as expected, improved performance is obtained through the use of this technique over the traditional LMF algorithm.

Index Terms — Adaptive filters, LMS, LMF, Constrained optimization, Noise constraints, SNCLMF algorithm.

1. INTRODUCTION

It is well known that the LMF algorithm [1] belongs to a class of stochastic gradient descent based algorithms and it seeks to minimize the mean fourth error, which is a convex (and thus unimodal) function of the adaptive weight vector. The power of the LMF algorithm, which has been used extensively in a variety of applications, lies in its faster initial convergence and lower steady state error relative to the LMS algorithm [2]-[4]. Still there continues to be an interest in improving the performance of the LMF algorithm.

It is also well established that the learning speed of any adaptive filtering algorithm can be increased by adding a constraint to it as in the case of the normalized LMS (NLMS) [5] and the normalized least-mean-fourth (NLMF) [6] algorithms. Recently an LMS-type algorithm that exploits the knowledge of channel noise variance for identification and tracking of FIR channels, called noise constrained LMS (NCLMS) algorithm, was proposed [7]. Better convergence rate was made possible due to the presence of three independent variables.

When partial knowledge of the channel is available we should try to use it to improve the performance of adaptive filters, provided it doesn't increase complexity and/or decrease robustness unduly. Here, we propose an LMF-based algorithm for identification of FIR channels which exploits assumed knowledge of the channel noise statistics. The main aim of this work is to derive the SNCLMF adaptive algorithm, analyze its convergence behaviour, and assess its performance in different noise environments. Moreover, the concept of energy conservation is used to carry out the rigorous steady-state analysis [8].

2. ALGORITHM DEVELOPMENT

To develop the proposed SNCLMF algorithm a time-invariant channel model is considered to be:

$$y_k = \sum_{i=0}^{N-1} c_i x_{k-i} + n_k = \mathbf{c}^T \mathbf{x_k} + n_k, \tag{1}$$

where $\{x_k\}$ is a stationary input process with mean zero and variance σ_x^2, $\{n_k\}$ is a stationary noise process with mean zero and variance σ_n^2 and $\mathbf{c}$ corresponds to a channel/impulse response with N taps. Under the above model, minimizing the mean-fourth error [1], that is,

$$\varepsilon(\mathbf{w}) = E[e_k^4] = E[(y_k - \mathbf{w}^T \mathbf{x}_k)^4], \tag{2}$$

over $\mathbf{w}$ gives the optimal weight value $\mathbf{w} = \mathbf{c}$. We can also notice that although the optimal weight does not depend on the channel noise statistics, this does not mean that a (partially) adaptive algorithm for estimating the optimum weight cannot exploit knowledge of noise statistics. In particular, away from the optimum, the knowledge of noise statistics might be useful in selecting search directions and/or step-size in an adaptive algorithm. This will be made clear as we proceed.

Minimize $\varepsilon(\mathbf{w})$ over $\mathbf{w}$ subject to the constraint $\varepsilon(\mathbf{w}) = J_{min}$, which is the fourth-order moment of the noise, and is defined as $J_{min} = 3\sigma_n^4$ for Gaussian noise $(0, \sigma_n^2)$, $J_{min} = 24b^4$ for Laplacian $(0, 2b^2)$ and $J_{min} = \Delta^4/5$ for uniform noise $(0, \Delta^2/3)$. Therefore, the lagrangian for this problem can be set up as:

$$\varepsilon_1(\mathbf{w}, \lambda) = \varepsilon(\mathbf{w}) + \lambda(\varepsilon(\mathbf{w}) - J_{min}). \tag{3}$$

The critical values of $\varepsilon_1(\mathbf{w}, \lambda)$ are $\mathbf{w} = \mathbf{c}$ and λ is arbitrary. Note that although there is no spurious critical $\mathbf{w}$, the fact that there is no unique (or even constrained) critical λ will present problems for an iterative/adaptive algorithm. To correct this, we subtract $\gamma\lambda^2$(where $\gamma > 0$) from $\varepsilon_1(\mathbf{w}, \lambda)$ to get the augmented Lagrangian:

$$\varepsilon_2(\mathbf{w}, \lambda) = \varepsilon(\mathbf{w}) + \gamma(\lambda(\varepsilon(\mathbf{w}) - J_{min}) - \lambda^2). \tag{4}$$

The solution of the above equation for the augmented Lagrangian is obtained using a procedure similar to that in [7], and can be shown to result in the SNCLMF adaptive algorithm:

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \alpha_k e_k^3 \mathbf{x}_k, \tag{5}$$

$$\alpha_k = \alpha(1 + \gamma\lambda_k), \tag{6}$$

$$\lambda_{k+1} = \lambda_k + \beta\Big\{\frac{1}{2}(e_k^4 - J_{min}) - \lambda_k\Big\}, \tag{7}$$

where α, β, and γ are positive parameters providing more control over the performance of the proposed algorithm. It can be observed from (6), that the LMF algorithm is recovered when $\gamma = 0$. moreover, as can be seen from (5)-(7), that the SNCLMF is a type of a variable step-size LMF algorithm.

3. CONVERGENCE ANALYSIS OF THE SNCLMF ALGORITHM

The weight error vector of SNCLMF is defined as $\mathbf{v}_k = \mathbf{c} - \mathbf{w}_k$. Therefore, equation (5) becomes:

$$\mathbf{v}_{k+1} = \mathbf{v}_k - \alpha_k e_k^3 \mathbf{x}_k. \tag{8}$$

We will now define two kinds of estimation errors known as *a-priori* given by $e_{ak} = \mathbf{x}_k^T \mathbf{v}_k$ and *a-posteriori* given by $e_{pk} = \mathbf{x}_k^T \mathbf{v}_{k+1}$. A new relation involving a-priori and a-posteriori estimation errors is given by:

$$e_{pk} = e_{ak} - \alpha_k ||\mathbf{x}_k||^2 e_k^3. \tag{9}$$

Let us define a new term $\bar{\mu_k} = \frac{1}{||\mathbf{x}_k||^2}$ and substitute expression (9) in equation (8), then the new equation becomes:

$$\mathbf{v}_{k+1} = \mathbf{v}_k - \bar{\mu_k} \mathbf{x}_k [e_{ak} - e_{pk}]. \tag{10}$$

By evaluating the energies on both sides, we obtain:

$$||\mathbf{v}_{k+1}||^2 + \bar{\mu_k}|e_{ak}|^2 = ||\mathbf{v}_k||^2 + \bar{\mu_k}|e_{pk}|^2. \tag{11}$$

This important fundamental energy relation developed previously in [8] will now be used to evaluate Excess MSE (EMSE) of the proposed statistical noise-constrained LMF at steady state. As we all know, an adaptive filter is said to operate in steady state iff:

$$\lim_{k\to\infty} E[||\mathbf{v}_{k+1}||^2] \quad = \quad \lim_{k\to\infty} E[||\mathbf{v}_k||^2]. \tag{12}$$

Let us take the expectation on both sides of equation (11). At steady state, we get:

$$E[\bar{\mu_k}|e_{ak}|^2] = E[\bar{\mu_k}|e_{ak} - \frac{\alpha_k}{\bar{\mu_k}} e_k^3|^2]. \tag{13}$$

Assuming that the a-priori estimation error e_{ak} and the noise process n_k are independent and are related through: $e_k = e_{ak} + n_k$. It is obvious from this relation that the excess mean-square error EMSE, and hence can be defined as:

$$\text{EMSE} = \lim_{k\to\infty} E[|e_{ak}|^2] = J_* \tag{14}$$

At steady-state, the third and higher powers of e_{ak} become very small and can be ignored [8]. Let $E[n_k^m] = \delta_n^m$ and $E[\alpha_k] = \overline{\alpha}_k$. After some straight forward algebraic manipulations, equation (13) can be rewritten as:

$$6\sigma_n^2 \overline{\alpha}_k J_k = \overline{\alpha^2}_k \text{Tr}(\mathbf{R})[15\delta_n^4 J_k + \delta_n^6]. \tag{15}$$

The mean square multiplier $\overline{\lambda^2}_{k+1}$ can be obtained by taking square on both sides λ_{k+1} and then take expectation to get:

$$\begin{aligned} \overline{\lambda^2}_{k+1} \quad &= \quad (1-\beta)^2 \overline{\lambda^2}_k + \beta(1-\beta)[E[e_k^4] - J_{min}]\overline{\lambda}_k \\ &\quad + \frac{\beta^2}{4}[E[e_k^8] - 2J_{min}E[e_k^4] + J_{min}^2] \end{aligned} \tag{16}$$

where,

$$E[e_k^4] \quad = \quad \delta_n^4 + 6\sigma_n^2 J_* + (3+6\epsilon)J_*^2, \tag{17}$$

$$E[e_k^8] \quad = \quad 70J_*^2\delta_n^4 + 28J_*\delta_n^6 + \delta_n^8, \tag{18}$$

and ϵ is a variable bounded by $(1/N) \leq \epsilon \leq 1$. At steady state, equation (15) becomes:

$$6\sigma_n^2 \overline{\alpha}_* J_* = \overline{\alpha^2}_* \text{Tr}(\mathbf{R})[15\delta_n^4 J_* + \delta_n^6]. \tag{19}$$

Let $\overline{\mathbf{v}}_*$ denote the limiting value of $\overline{\mathbf{v}}_k$ (as $(k \to \infty)$) with a like notation for other sequences and their limits (which are henceforth assumed to exist). Therefore, we can write:

$$\overline{\mathbf{v}}_* \quad = \quad 0, \tag{20}$$

$$\overline{\lambda}_* \quad = \quad \frac{1}{2}(6\sigma_n^2 J_* + 9J_*^2), \tag{21}$$

$$\overline{\alpha}_* \quad = \quad \alpha\Big\{1 + \frac{\gamma(6\sigma_n^2 J_* + (3+6\epsilon)J_*^2)J_*}{2}\Big\}, \tag{22}$$

$$\overline{\alpha^2}_* \quad = \quad \alpha^2(1 + 2\gamma\overline{\lambda}_* + \gamma^2\overline{\lambda^2}_*). \tag{23}$$

Substituting from (20) to (23), and (16) with $(k \to \infty)$ in equation (19) and after some rigorous algebra we obtained a $5th$ order equation for the excess MSE J_* of the form,

$$AJ_*^5 + BJ_*^4 + CJ_*^3 + DJ_*^2 + EJ_* + F = 0, \tag{24}$$

where

$$A \quad = \quad \frac{-(3+6\epsilon)^2 \text{Tr}(\mathbf{R})\alpha\gamma^2}{2-\beta}\Big\{\frac{1-\beta}{2}\Big\}15\delta_n^4, \tag{25}$$

$$B \quad = \quad \frac{-12(3+6\epsilon)\text{Tr}(\mathbf{R})\alpha\gamma^2}{2-\beta}\Big\{\frac{1-\beta}{2}\Big\}15\sigma_n^4\delta_n^4, \tag{26}$$

$$\begin{aligned} C \quad = \quad & 27\gamma\sigma_n^2 - \Bigg\{135\alpha\gamma\delta_n^4\text{Tr}(\mathbf{R}) + \frac{15\alpha\gamma^2\delta_n^4\text{Tr}(\mathbf{R})}{(2-\beta)} \\ & \Big\{\frac{36\sigma_n^4(1-\beta)}{2} - \frac{2(3+6\epsilon\beta J_{min}}{4}\Big\} \\ & + \frac{(1-\beta)12(3+6\epsilon)\alpha\gamma^2\sigma_n^4\text{Tr}(\mathbf{R})\delta_n^6}{2(2-\beta)}\Bigg\}, \end{aligned} \tag{27}$$

$$\begin{aligned} D \quad = \quad & 18\sigma_n^4\gamma - \Bigg\{90\alpha\gamma\delta_n^4\text{Tr}(\mathbf{R}) + \frac{15\alpha\gamma^2\beta\delta_n^4\text{Tr}(\mathbf{R})}{4}(28\delta_n^6 \\ & -12\sigma_n^2 J_{min}) + 9\alpha\gamma\delta_n^6\text{Tr}(\mathbf{R}) + \frac{\alpha\delta_n^6\text{Tr}(\mathbf{R})\gamma^2}{2-\beta} \\ & \Big\{\frac{36\sigma_n^4(1-\beta)}{2} - \frac{2(3+6\epsilon)\beta\delta_n^4}{4}\Big\}\Bigg\}, \end{aligned} \tag{28}$$

$$\begin{aligned} E \quad = \quad & 6\sigma_n^2 - \Bigg\{15\alpha\delta_n^4\text{Tr}(\mathbf{R}) + \frac{15\alpha\beta\gamma^2\delta_n^4\text{Tr}(\mathbf{R})}{4(2-\beta)}(\delta_n^8 \\ & -J_{min}^2) + 6\alpha\gamma\sigma_n^2\text{Tr}(\mathbf{R})\delta_n^6 + \frac{\alpha\gamma^2\text{Tr}(\mathbf{R})\delta_n^6}{2-\beta} \\ & (28\delta_n^6 - 12\sigma_n^2 J_{min})\Bigg\}, \end{aligned} \tag{29}$$

$$F \quad = \quad -\text{Tr}(\mathbf{R})\Bigg\{\alpha\delta_n^6 + \frac{\alpha\gamma^2\beta\delta_n^6}{4(2-\beta)}\Big\{\delta_n^8 - J_{min}^2\Big\}\Bigg\}. \tag{30}$$

Assuming $\alpha\text{Tr}(\mathbf{R}) << 1$, it can be proved that (24) has a root around 0, i.e. , $J_* << 1$. Therefore, higher power of J_* are ignored and we are left with $J_* = -\frac{F}{E}$. Hence, an asymptotic approximation for the excess MSE of the SNCLMF in the presence of AWGN can be written as:

$$J_{SNCLMF} \approx J_* \approx \frac{\alpha \mathrm{Tr}(\mathbf{R}) 15\sigma_n^6}{6\sigma_n^2}\left[1+\frac{\gamma^2\beta}{2(2-\beta)}(48\sigma_n^8)\right]. \tag{31}$$

The proposed SNCLMF algorithm has three tuning parameters namely α, β, and γ which can be used to control the speed of convergence whereas in case of LMF we have only one tuning parameter namely the step-size. It should also be noted that if we put $\gamma = 0$ in the above expression, then, we end up getting the EMSE of the LMF algorithm [1].

•Mean-square convergence of the step-size:
From [4], it is well known that the LMF converges in the mean square sense if the step-size α satisfies:

$$0 < \alpha < \frac{\sigma_n^2}{9\delta_n^4 \mathrm{Tr}(\mathbf{R})}. \tag{32}$$

Now assuming the mean step-size $\overline{\alpha}_k$ and the mean square step-size $\overline{\alpha^2}_k$ converge, it can be shown that the SNCLMF converges in the mean square sense if:

$$\frac{\overline{\alpha^2}_k}{\overline{\alpha}_k} < \frac{\sigma_n^2}{9\delta_n^4 \mathrm{Tr}(\mathbf{R})}. \tag{33}$$

•Adaptation Time Constants and Comparisons:
Form [1], the largest time constant for LMF is:

$$\tau_{\mathrm{LMF}} \approx \frac{1}{6\alpha\sigma_n^2\rho_{min}}, \tag{34}$$

where ρ_{min} is the smallest eigenvalue of $\mathbf{R}$. To get the time constant for the SNCLMF, we will assume that the step-size α_k adapts much faster than the weights $\mathbf{v}_k$ (this makes sense otherwise, the variable step-size would not be useful). In fact, we will assume that α_k is actually converged in the mean at each time instant so that we can get the time constants by replacing α by $\overline{\alpha}_k$ in (34). Hence, from (6) and using the fact that $E[e_k^2] >> \sigma_n^2$, the largest time constant for SNCLMF can be shown to be:

$$\tau_{\mathrm{NCLMF}} \approx \frac{1}{6\sigma_n^2\alpha\rho_{min}\left\{1+\frac{\gamma E[e_k^2]}{2}\right\}}. \tag{35}$$

We now observe that by fixing α, increasing γ, and decreasing β such that the product $\gamma^2\beta$ remains constant, τ_{SNCLMF} can be decreased to a small value without increasing the misadjustment. On the other hand, τ_{LMF} can be decreased only by increasing the step-size α or we can say by increasing the misadjustment of the LMF algorithm.

Furthermore, τ_{SNCLMF} can be decreased only to some fixed value without increasing the misadjustment, no matter how the parameters were chosen. This demonstrates analytically within the limitations of the approximations employed how the SNCLMF algorithm can exploit the knowledge of the noise statistics to increase the learning rate over the LMF algorithm.

4. SIMULATION RESULTS

In this section, the performance analysis of the LMF and the SNCLMF algorithms is investigated in an unknown system identification problem with $\mathbf{c} = [0.227, 0.460, 0.688, 0.460, 0.227]^T$. The objective of designing a constrained algorithm is to achieve a faster convergence rate which is a major performance measure. Three different noise environments have been considered namely Gaussian, Uniform and Laplacian for an SNR=20 dB.

In AWGN environment, as shown in Fig. 1, the proposed algorithm achieved the same steady-state in approximately 7000 iterations earlier than the LMF algorithm. This is also assessed in Fig. 2 where the third-tap of the SNCLMF converges faster than that of the LMF algorithm. This was expected due to the fact that the SNCLMF algorithm can be considered as a variable-step-size algorithm; Fig. 3 depicts this bahavior.

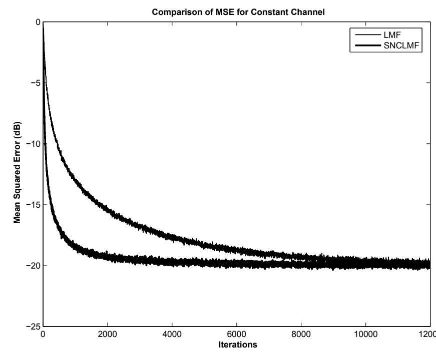

Fig. 1. Comparison of the convergence speed of the LMF and the proposed SNCLMF in AWGN environment.

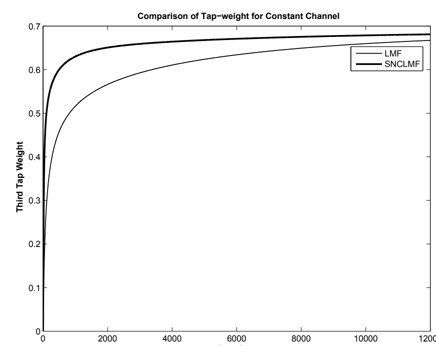

Fig. 2. Comparison of the learning curves for the third-tap of the LMF and the proposed SNCLMF in AWGN environment.

Figure 4 depicts the effect of an inaccurate estimate of the noise statistic on the performance behavior on the f the SNCLMF algorithm. As can be seen for this figure, no deterioration occurred. Furthermore, an important advantage of the SNCLMF algorithm is demonstrated in Fig. 5 where it is observed that the convergence speed can be increased considerably by increasing γ and decreasing β (for a constant $\gamma \times \beta$) while the steady-state misadjustment is kept constant.

Finally, the effect of the noise environment on the performance of the SNCLMF is depicted in Fig. 6. A better performance is obtained when the noise statistics are uniform.

5. CONCLUSION

In this work, we proposed a new constrained LMF-type algorithm (SNCLMF) for wireless environments and studied both analytically and by simulations its performance. Our study included a thorough comparison of the proposed algorithm (SNCLMF) with

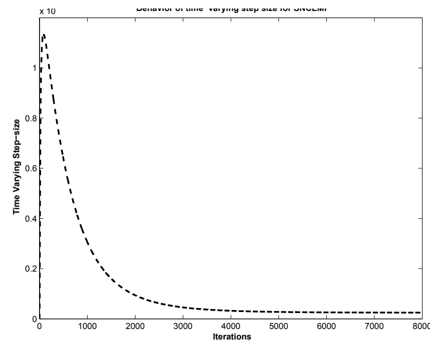

Fig. 3. Behavior of time-varying step-size of SNCLMF algorithm in AWGN environment.

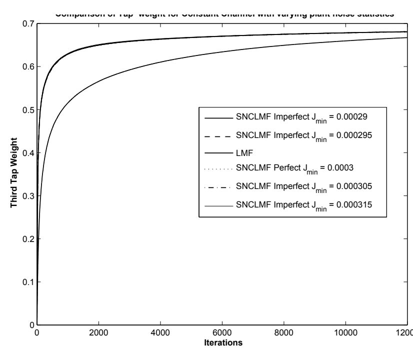

Fig. 4. The effect of an inaccurate estimate of noise statistics on the SNCLMF algorithm in AWGN environment.

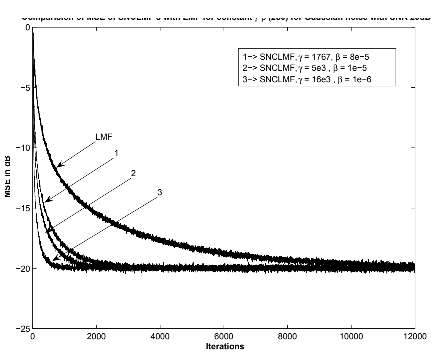

Fig. 5. The effect of γ and β on the convergence behavior of the SNCLMF algorithm in AWGN environment.

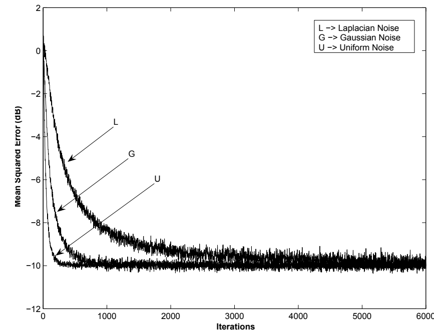

Fig. 6. Convergence behavior of the SNCLMF algorithm in presence of Gaussian, Uniform and Laplacian environments with an SNR=10 dB.

the well-established LMF algorithm and showed that, overall, the SNCLMF enjoys a superior performance in both transient and steady-state regimes for different environments. This superior performance was achieved with only a slight increase in computational complexity. Also, similar behavior is obtained for other noise environments, e.g., uniform and Laplacian, but due to space limitations these are reported here. Finally, a zero noise constrained LMF (ZNCLMF) algorithm with $J_{min} = 0$ in (7) can also be derived.

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