ANALYTICAL COMPUTATION OF FAST FREQUENCY WARPING

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ABSTRACT

In this work we introduce an analytical characterization of the frequency warping operator of arbitrary shaped non-smooth warping maps. The transformation matrix is decomposed in two additive terms: the first term represents its Nonuniform Fourier Transform approximation while the second term is imposed for aliasing suppression. The first transformation is known to be analytically characterized and fast computable by an interpolation approach. For the second transformation an analytical representation is introduced which allows a fast computation and a simple design. Finally, an example of a potential application is shown.

Index Terms— Frequency warping, fast transforms.

1. INTRODUCTION

In the last years time–frequency transformation techniques have acquired a leading role in signal processing. However, such transformations have some restrictive properties which make them not suitable in some applications. In particular, the possibility of generalizing and adaptively varying the time–frequency plane tiling, as shown in fig. 1, is a major demand.

In order to accomplish this task many strategies are possible, such as the application of a preliminary invertible transformation (warping) to reshape the frequency axis in an invertible and flexible way [1].

Frequency warping has been introduced some years ago [2] and then modeled as a projection on a set of frequency and amplitude modulated functions (FAM) [3]. The commonly adopted Laguerre transform approach to frequency warping allows a reduced set of mapping functions and is based on a recursive computationally expensive algorithm. The method proposed in [4] allows a fast computation for arbitrary maps, but it is based on numerical procedures and is not supported by an analytical model. Here we introduce an analytical model carrying a fast computation too.

The paper is organized as follows. In the next section we briefly review the discrete warping operator. In section 3 we present the analytical model for aliasing suppression. Finally, in section 4 and 5 we provide a design example and some conclusions.

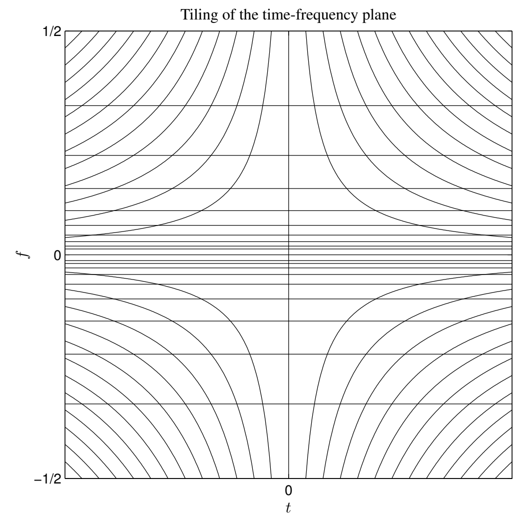


Fig. 1: Tiling of the time-frequency plane obtainable by the application of frequency warping. The frequency axis is splitted in a constant-Q fashion with $Q = 2/3$.

2. FREQUENCY WARPING IN BRIEF

Given a discrete-time signal, we want to introduce a deformation of the periodic frequency axis f with a proper warping function $w(f)$. In order to guarantee invertibility, $w(f)$ has to be chosen so that it maps f axis on itself, that is:

$$\dot{w}(f) > 0 \quad a.e. \quad \Rightarrow \quad \exists w^{-1},\ w^{-1}(w(f)) = 1 \qquad (1)$$

where $\dot{w}$ represents the first derivative of the map w. The warping function $w(f)$ is defined in the interval $[-1/2, 1/2]$ and extended as $w(f+k) = k+w(f)$, with $k \in \mathbb{Z}$. Moreover, it must be an odd function in order to guarantee that a real signal is transformed into a real signal. The frequency warping operator can be written as the composition of an inverse discrete Fourier transform $\mathbb{F}$ and a modified discrete Fourier transform $\mathbb{F}_w$:

$$\mathbb{W} = \mathbb{F}^{-1}\mathbb{F}_w \qquad (2)$$

where $\mathbb{F}_w$ is defined as follows:

$$[\mathbb{F}_w s](f) = \sqrt{\dot{w}(f)} \sum_{n\in\mathbb{Z}} s(n) e^{-j2\pi n w(f)}. \qquad (3)$$

The term $\sqrt{\dot{w}(f)}$ has been introduced in order to make the operator be unitary, i.e. preserve orthogonality. By doing so, the operator kernel is a matrix of infinite dimensions whose elements are given by:

$$\mathbb{W}(m,n) = \int_0^1 \sqrt{\dot{w}(f)}\, e^{j2\pi(mf - nw(f))} df \quad m,n \in \mathbb{Z}. \quad (4)$$

An example of warping matrix is depicted in fig. 2(a). Since frequency warping is treated as a matrix, we need to limit the input sequence to N samples. To have invertibility, the number of rows M must be greater than N, then warping is not a unitary transformation, but a redundant transformation. When M is taken equal to infinite, the warping matrix has N orthogonal columns and represents a *tight* frame [5]. However, if M is properly truncated, frequency warping can still be inverted by applying the transpose operator $\mathbb{W}^T$ without significant loss of accuracy:

$$\mathbb{W}_{M,N} : s \mapsto \sum_{n \in \mathbb{Z}_N} \mathbb{W}(m,n)s(n) \quad m \in \mathbb{Z}_M. \quad (5)$$

where $\mathbb{Z}_N$ and $\mathbb{Z}_M$ are defined by:

$$\mathbb{Z}_N = \{-N/2, \ldots, N/2-1\}$$
$$\mathbb{Z}_M = \{-M/2, \ldots, M/2-1\}.$$

To preserve most of the input signal energy, we must consider:

$$M > 2\lceil N/2 \max \dot{w} \rceil$$

so, we are dealing with a *snug* frame [5]:

$$\mathbb{W}^\dagger_{M,N}\mathbb{W}_{M,N} \sim \mathbb{I}_N.$$

The implementation of (5) involves continuous operations along the frequency axis in (3) and consequently in (4), which are not achievable. So we introduce a sampling operation on M discrete frequencies $f_k = k/M$, $k \in \mathbb{Z}_M$:

$$\mathcal{F}_{w,M,N} : s \mapsto \sqrt{\dot{w}(f_k)} \sum_{n \in \mathbb{Z}_N} s(n)e^{-j2\pi n w(f_k)} \quad (6)$$

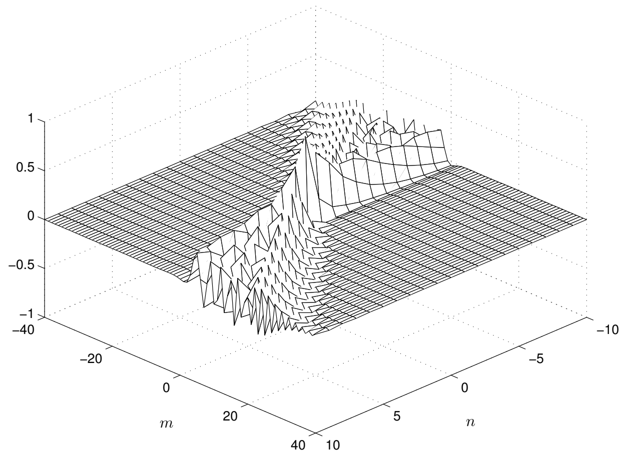
so, the frequency sampled warping operator is represented by:

$$\mathcal{W}_{M,N} = \mathcal{F}_M^{-1}\mathcal{F}_{w,M,N} \quad (7)$$

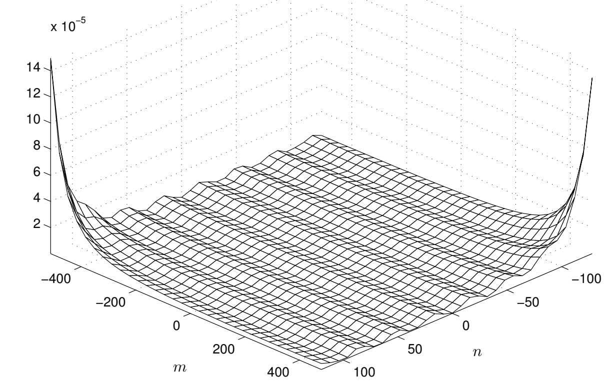
where $\mathcal{F}_M$ is the Discrete Fourier Transform of size $M \times M$. The sampling operation in the Fourier domain introduces aliasing in the warped signal domain. The effect of aliasing is merely represented by the transformation $\mathbb{A}_{M,N}$:

$$\mathbb{W}_{M,N} = \mathcal{W}_{M,N} - \mathbb{A}_{M,N}$$

Through this decomposition of the warping operator we can achieve a fast computation of frequency warping. In facts, the operator $\mathcal{W}_{M,N}$ can be opportunely factorized through a Nonuniform Fast Fourier Transform algorithm [6] [7] [8]



(a) Entries of a warping matrix



(b) Absolute value of the entries of the aliasing matrix referred to (a)

Fig. 2: In (a) the common structure of a warping matrix is shown: the most significant coefficients are enclosed between two lines whose slopes are given by the minimum and the maximus of $\dot{w}$ respectively. Entries of the aliasing matrix (b) are very correlated, so its rank is small.

while the operator $\mathbb{A}_{M,N}$ has a small rank, as can be observed in fig. 2(b). Neglecting the aliasing term would decrease the *snugness* of the frame of many orders of magnitude.

In the next section we investigate the existence of an analytic representation for $\mathbb{A}_{M,N}$ allowing a fast computation, since its representation by numerical approximation has already been treated in a previous work [4].

To this aim, we must introduce appropriate classes of function. It is worth to note that the warping map must be considered on the entire frequency axis, so also its behavior around the period edges must be taken into account. We assume that the first σ derivative of w are continuous function:

$$w \in \mathcal{C}^\sigma \quad (8)$$

that is, w is not a *smooth* function.

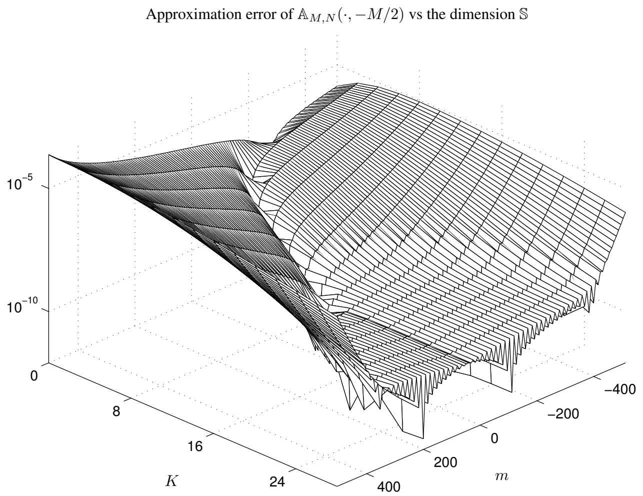


Fig. 3: Punctual error on the first column of $\mathbb{A}_{M,N}$ for increasing value of the dimension of matrix $\mathbb{S}$.

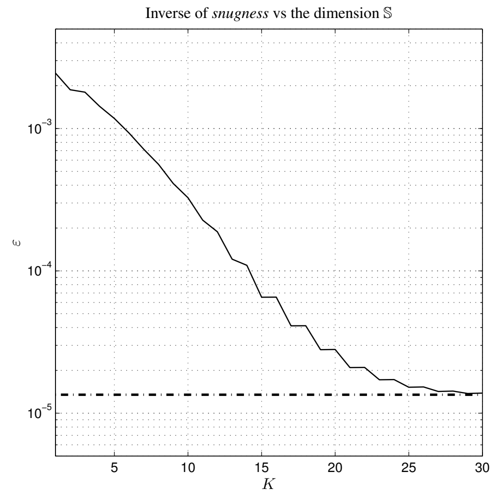


Fig. 4: Decrease of matrix norm ε (17) for increasing value of the dimension of matrix $\mathbb{S}$. The inverse of ε is actually a measure of the frame snugness.

3. ANALYTICAL SUPPRESSION OF ALIASING

We assumed that w is a non-smooth function (8). So, w must have at least one discontinuity on the $(\sigma+1)$-th derivative, and maybe other discontinuities on subsequent derivatives. The points of discontinuity will be represented by $\xi_i \in [0, 1/2]$. If these points of discontinuities stay on a discrete frequency:

$$M\xi_i \in \mathbb{N} \quad \forall i \tag{9}$$

it can be proved that the aliasing matrix is given by:

$$\tilde{\mathbb{A}}_{M,N} = 2\sum_i \Re[\mathbb{P}\,\mathbb{U}\,\mathbb{S}\,\mathbb{V}\,\mathbb{Q}](\xi_l) \tag{10}$$

where $\mathbb{P}$ and $\mathbb{Q}$ are diagonal matrix defined by:

$$\mathbb{P} = \text{diag}[e^{j2\pi m\xi_i}] \qquad m \in \mathbb{Z}_M \tag{11}$$

$$\mathbb{Q} = \text{diag}[e^{-j2\pi n w(\xi_i)}] \qquad n \in \mathbb{Z}_N \tag{12}$$

$\mathbb{V}$ is a $[K \times N]$ matrix whose entries are given by:

$$\mathbb{V}(k,n) = \frac{n^k}{(N/2)^k} \quad n \in \mathbb{Z}_N, k \in \mathbb{N} \tag{13}$$

$\mathbb{U}$ is a $[M \times K]$ matrix whose entries are given by:

$$\mathbb{U}(m,k) = \frac{(-1)^{k-1}}{2^k(k-1)!}\,\zeta_{k-1}\,(m/M) \quad m \in \mathbb{Z}_M, k \in \mathbb{N} \tag{14}$$

where ζ_k represents the k-th derivative of the following:

$$\zeta(z) = \pi\cot(\pi z) - \frac{1}{z}. \tag{15}$$

Finally, $\mathbb{S}$ is a $[K \times K]$ lower triangular matrix, whose entries depend on the warping map w and on N and M only. If ξ_l is a point of discontinuity of the $\sigma + 1$-th derivative, then the first $\sigma - 1$ diagonals are null. Generally, at most three non-zero diagonals are necessary. K should be equal to ∞, but the most significant non-zero diagonal has the following flow:

$$\mathbb{S}(k+\sigma, k) \propto \left(\frac{k}{M}\right)^{\sigma+1} \cdot \left(\frac{N\dot{w}(\xi_l)}{M}\right)^k \tag{16}$$

that is, it decays exponentially, so matrix $\mathbb{S}$ can be properly truncated allowing a fast computation. In fig. 3 we show the decrease of approximation error on the first column of $\mathbb{A}_{M,N}$ (which is actually the worst case) as K increases. Moreover, in fig. 4 we show the decrease of the matrix error norm after the application of the adjoint operator:

$$\varepsilon = \|(\mathcal{W}_{M,N} - \tilde{\mathbb{A}}_{M,N})^\dagger(\mathcal{W}_{M,N} - \tilde{\mathbb{A}}_{M,N}) - \mathbb{I}_N\| \tag{17}$$

from the upper to the lower limit, which are obtained by applying $\mathcal{W}_{M,N}$ (neglecting $\mathbb{A}_{M,N}$) and $\mathbb{W}_{M,N}$ respectively.

We point out that in the previous formulation all the information about the warping matrix is enclosed in matrixes $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{Q}$. The last ones are actually scaling matrixes, so the main term is $\mathbb{S}$. This structure allows a fast design, since the warping function can be easily changed by updating some multiplicative coefficients in the matrix $\mathbb{S}$.

4. EXAMPLE OF APPLICATION

Now we would like to exploit frequency warping to obtain a constant-Q splitting of the frequency plane, corresponding to the tiling of time-frequency plane depicted in fig. 1. In particular, we would like to have Q, actually the ratio between two contiguous bands, equal to $2/3$. In order to do this, first the input signal is frequency warped, then a L-band uniform

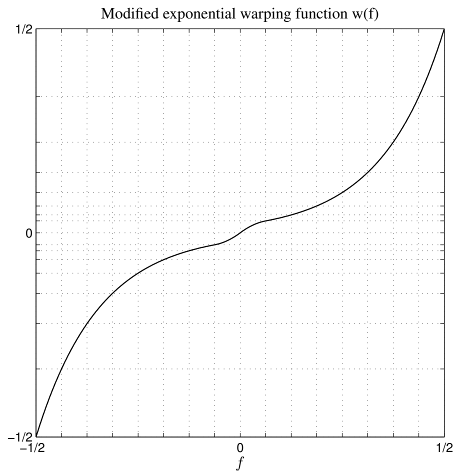


Fig. 5: An exponential splitting (vertical axis, dotted lines) is converted in a uniform splitting (horizontal axis), so that a uniform filter bank in the warped domain is equivalent to a nonuniform one in the unwarped domain.

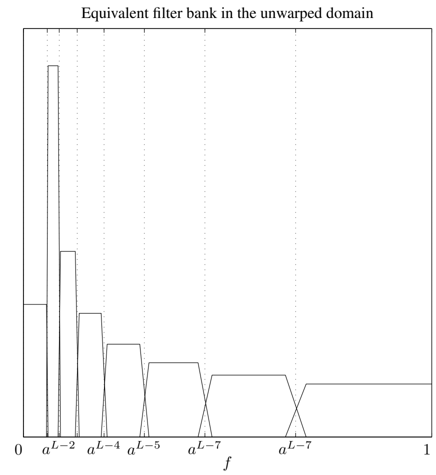


Fig. 6: Schematic representation of the equivalent filter bank in the unwarped domain, normalized axis, corresponding to a L-band uniform filter bank in the warped domain with $L = 8$, according to the map depicted in fig. 5.

filter bank is applied. The warping function should be:

$$w(f) = \frac{1}{2}\text{sign}(f)a^{L(2|f|-1)} \qquad f \in [-1/2, 1/2] \tag{18}$$

but this function is not allowed since $w(0) \neq 0$. So the map has to be modified in order make it be invertible. We notice that (18) in the neighborhood of $f = 1/2$ is $\mathcal{C}^1$, so we modify it to become $\mathcal{C}^1$ over the whole frequency axis:

$$w(f) = \begin{cases} \frac{1}{2}(c_1 2f + c_2(2f)^2) & f \in [0, 1/2L] \\ \frac{1}{2}a^{L(2f-1)} & f \in [1/2L, 1/2] \end{cases} \tag{19}$$

where $a = 3/2$. For the interval $[-1/2, 0]$ we consider $w(f) = -w(-f)$. The coefficients c_1, c_2 are obtained by imposing the continuity of w and $\dot{w}$:

$$c_1 = L(\log a - 1)a^{2-L} \tag{20}$$

$$c_2 = -L^2(\log a - 1)a^{1-L} \tag{21}$$

and the resulting warping function is depicted in fig. 5 for $L = 8$. This map has 3 points of discontinuities of the 2nd derivative in $f = 0, 1/L, 1/2$ respectively. It can be observed that the map has been modified so that, after the application of a L-band filter bank, the last $L-1$ bands achieve a constant-Q analysis with Q=2/3. A schematic structure of the equivalent filter bank in the unwarped domain is depicted in fig. 6

5. CONCLUSIONS

In this work we introduced an analytical model of the frequency warping operator of arbitrary shaped non-smooth warping maps. The warping matrix has been represented by its Nonuniform Fourier Transform approximation and an aliasing term. An analytical representation of the aliasing term has been introduced, allowing a fast computation and a simple design. Finally we have shown an example of application.

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