AN INTERPOLATION ALGORITHM FOR HIGH RESOLUTION SPECTRAL ESTIMATION

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ABSTRACT

We will present an algorithm for a high resolution spectral estimation based on central solution of Nevanlinna-Pick interpolation. The approach is very well suited to short data records and shows superior performance, compared to traditional approaches, especially in detecting and resolving spectral lines buried in colored non-Gaussian noise.

***Index Terms*—** Spectral estimation, high resolution, parallel filtering, Nevanlinna-Pick interpolation

1. INTRODUCTION

In this paper, we investigate a rational power spectrum estimator initially considered in [1]. The approach is based on defining a complex function f of complex variable z, which its real part on the unit circle provides an estimate of power spectrum density. The value of this complex function can be obtained in some desired points in $|z| > 1$ using a bank of filters. Then, the value of $f(z)$ on the boundary $|z| = 1$ can be found through interpolation. A complex interpolation theory will be needed to find the proper interpolator. We are interested in rational interpolators to provide an approximate ARMA model $g(z)$ for the signal. Standard algorithms such as Thiele's continued fraction interpolating function [2] may be used at this stage to find an interpolator. However, the interpolation can be improved by constraining f to be analytical and to have positive real part in $|z| > 1$. The latter is called Nevanlinna-Pick interpolation (NPI) problem, which has also other applications such as in robust control, circuit theory, etc. [3].

A striking advantage of this approach is that it is very well suited to short observation record of data since it uses statistical estimates of only zeroth order covariance lags. This is as opposed to traditional spectral estimators which rely on high order lags where reliability of larger lags decrease with lag's order. Moreover, the method provides a superior resolution in distinguishing close spectral lines buried in colored noise, which is considered as a challenging problem [6]. However, a crucial matter in this approach is choosing interpolation points or equivalently, poles of the filters, which has a great impact on both the quality and complexity of estimation. We will discuss the theory behind the estimation method and propose an algorithm that leads to a proper selection of interpolation points.

The structure of the paper is as follows. In section 2 we introduce a complex function and relate it to the power spectral density. Then we introduce the concept of bank of filters and discuss how covariances of their outputs provide estimates of the power spectrum at the reflected pole positions. Section 4 presents the basic elements of Nevanlinna-Pick interpolation. In section 5, we will discuss about suitable selection of interpolation points considering statistical variations and propose an algorithm for it. We will use computer simulations for distinguishing and resolving spectral lines embedded in a highly colored noise. Section 6 concludes the paper.

2. EVALUATING FUNCTION

Let x_n be a scalar, real-valued, zero mean, stationary stochastic process. The problem is to find an estimate of its power spectral density $\Phi(e^{j\theta}), -\pi \le \theta \le \pi$, from a finite observation record $\mathcal{X} = \{x_0, x_1, \cdots, x_M\}$. Define

$$f(z) \triangleq \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k z^{-k} \quad , \quad |z| > 1 \tag{1}$$

where c_k's are covariance lags of input sequence x_k given by

$$c_k = E[x_n x_{n+k}] \tag{2}$$

where $E[\cdot]$ denotes statistical expectation.

The power spectral density of x_k, is found as Discrete Time Fourier Transform (DTFT) of the sequence c_k. Thus we can write

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(e^{j\theta}) e^{jk\theta} d\theta \tag{3}$$

The PSD of sequence x_k may be expressed in terms of f as

$$\Phi(e^{j\theta}) = \sum_{k=-\infty}^{\infty} c_k e^{-jk\theta} = c_0 + 2\sum_{k=1}^{\infty} c_k \cos k\theta$$
$$= [f(e^{j\theta}) + f(e^{-j\theta})]$$

yielding

$$\Phi(e^{j\theta}) = 2\Re[f(e^{j\theta})] \tag{4}$$

where $\Re[\cdot]$ denotes real part operator. Actually the above relation should be interpreted as

$$\Phi(e^{j\theta}) = 2\lim_{r\to 1}\Re[f(re^{j\theta})]$$

whereas "spectral lines" correspond to poles of f on the boundary $|z|=1$. Inversely, f may be expressed in terms of Φ by substituting (2) into (1):

$$f(z) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\Phi(e^{j\theta})d\theta + \sum_{k=1}^{\infty}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi(e^{j\theta})e^{jk\theta}d\theta\right)z^{-k}$$

and consequently,

$$f(z) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\Phi(e^{j\theta})\frac{z+e^{j\theta}}{z-e^{j\theta}}d\theta \tag{5}$$

$f(z)$ is analytic in $|z|>1$ and has a positive real part there. The method can be used to find an ARMA model $g(z)$ based on spectral factorization $f(z)+f(z^{-1}) = g(z)g(z^{-1})$ such that $|g(e^{j\theta})|^2$ provides an approximation to the true power spectrum $\Phi(e^{j\theta})$.

3. ACQUIRING SAMPLE POINTS

3.1. Evaluating *f(z)* at a point

Consider a 1st order stable linear filter with transfer function

$$G(z) = \frac{z}{z-p} \quad , \quad |p|<1 \tag{6}$$

Assume that $x(n)$ is the input of this filter and its stationary output process denotes by $y(n)$. Then

$$y(n) = py(n-1) + x(n) \tag{7}$$

The output $y(n)$ can be written recursively in terms of $x(n)$ as

$$y(n) = \sum_{k=0}^{\infty}p^k x(n-k) \tag{8}$$

Consequently,

$$E[y(n)^2] = E[(\sum_{k=0}^{\infty}p^k x(n-k))^2]$$
$$= E[\sum_{k=0}^{\infty}p^{2k}x^2(n-k) + 2\sum_{k=0}^{\infty}\sum_{i=1}^{\infty}p^k x(n-k)\cdot p^{i+k}x(n-i-k)]$$
$$= c_0\sum_{k=0}^{\infty}p^{2k} + 2\sum_{k=0}^{\infty}p^{2k}\sum_{i=1}^{\infty}p^i c_i$$
$$= \frac{2}{1-p^2}f(\frac{1}{p})$$

Thus,

$$f(p^{-1}) = \frac{1}{2}(1-p^2)E[y(n)^2] \tag{9}$$

This is an interpolation condition for f. This means that we can have the value of $f(z)$ at arbitrary complex points in $|z|>1$. Note that $E[y(n)^2]$ is complex-valued and is different from traditional covariance which is defined as $E[y(n)y(n)^*]$.

Alternatively, we can use

$$E[y(n)x(n)] = E[x(n)\sum_{k=0}^{\infty}p^k x(n-k)]$$
$$= \sum_{k=0}^{\infty}p^k c_k = \frac{1}{2}c_0 + f(p^{-1}) \tag{10}$$

3.2. Bank of filters

We construct a parallel bank of filters as shown in Fig. 1. In this construction, each filter has a transfer function as

$$G_i(z) = \frac{z}{z-p_i} \quad 0\le i\le N \tag{11}$$

with its complex poles in the open unit disc. The output of each filter provides an interpolation condition for $f(z)$. As we will see in the next subsection the poles can not be arbitrarily closed to the unit circle as it would increase the uncertainty of estimated function. The second consideration is that the number of interpolation poles has direct effect on complexity and might not necessarily increase the quality of estimation. Having the value of $f(z)$ in some proper points, the idea is now to find the value of $f(z)$ on the unit circle boundary by interpolation. This can be done through a theory described in section 4. We now briefly take into account the statistical variability of the estimated points.

4. NEVANLINNA-PICK INTERPOLATION

Given a set of $N+1$ distinct points outside the unit circle

$$\mathcal{Z} = \{z_0, z_1, ..., z_N\} \quad |z_i|>1$$

and a set of $N+1$ values in the right half plane

$$\mathcal{W} = \{w_0, w_1, ..., w_N\} \quad \Re(w_i)\ge 0$$

we are seeking a function $\tilde{f}(z)$ which satisfies the conditions:

1) interpolation conditions $\tilde{f}(z_k) = w_k$, $k = 0,1,...,N$
2) $\tilde{f}(z)$ is analytic and has nonnegative real part in $|z|>1$.
3) $\tilde{f}(z)$ is rational of degree at most N.

We use a rational interpolator of the form:

$$\tilde{f}(z) = \frac{b(z)}{a(z)} = \frac{\sum_{i=0}^{N}b_i z^{-i}}{\sum_{i=0}^{N}a_i z^{-i}} \tag{12}$$

which provides an estimate of *f(z)* in (1) or (3). Requiring only condition 1 amounts to the standard Lagrange

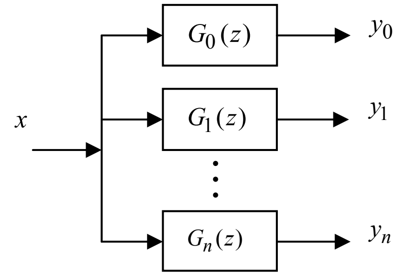

Fig. 1. The concept of bank of filters.

interpolation. Requiring also condition 2 yields a classical problem in complex analysis called Nevanlinna-Pick interpolation [4]. This problem is solvable if and only if the Pick matrix

$$P = \left[\frac{w_k + w_l^*}{1 - z_k^{-1} z_l^{-1*}}\right]_{k,l=0}^{N} \tag{13}$$

is positive semidefinite. Moreover, solution is unique if and only if P is singular[4]. Among all rational solutions to the NPI problem, it is desirable to bind the degree of interpolator to some prescribed value. In general, even if the NPI problem is solvable, the set of interpolants of degree $K < N$ may be empty, and to determine whether this is the case is an open problem. But, the set of interpolants of degree at most N is always nonempty which prompts for condition 3. The problem now is called Nevanlinna-Pick interpolation with degree constraint (NPDC) and its solution is further considered in [3] and it is shown that the solution can be parameterized by *spectral zeros* which are the zeros of $g(z)$ or in other words, MA parameters of the model. In this scheme, there are two sets of design parameters to find an interpolator: the filterbank poles, and, the MA parameters or alternatively the spectral zeros of f.

The computational procedures come in two forms: For the default setting when the spectral zeros are chosen equal to the filterbank poles, a particularly simple algorithm, based on the so-called *central solution* of the classical interpolation theory, is available [3]-[5]. The algorithm handles linear equations and is computationally efficient. For any other setting, a convex optimization problem needs to be solved. An iterative algorithm to the solution can be found in [3] or more computationally efficient in [7].

5. LOCATING FILTERBANK POLES

The statistical averages in (9) or (10) are evaluated using sample averages and thus are with error. The variance of such an average is inversely proportional to the data length M and distance of pole to unit circle [1]. In general, choosing the filter poles too close to the unit circle increase the statistical variability and would not increase the quality of interpolation. Such strategy will also produce more accentuated transients which is more harmful when a short record of data is available and thus is not without cost, while the relevant trade offs must be studied. Moreover, choosing a large number of interpolation points may not lead to a more qualified estimation while unduly increasing the complexity of filters. Even choosing extra poles in a region would not yield local high resolution estimation [8]. However, the poles can be placed closer to the unit circle when noise power is low or even where spectral lines are present. Based on the discussed theory, an algorithm for high resolution estimation can be as follows. An initial AR model may be obtained through available parametric methods such as Burg or Yule-Walker where Burg acts better for short data sequences. Alternatively, NPI itself can be used to find an initial estimate, for example by using

$$p_1 = \{0, r, re^{j\pi 0.2}, re^{j\pi 0.4}, re^{j\pi 0.6}, re^{j\pi 0.8}, -r\} \tag{14}$$

The poles of the AR model by Burg or ARMA model by NPI can be used as interpolation points for NPI estimator. An initial estimate would give a pole close to the unit circle where a spectral line is present even though it can not resolve close lines. Such a pole contributes to resolving of lines as e.g. two peaks in the second stage of estimation, which in turn returns two close poles. Of course, each model may present extraneous peaks. These peaks may also be seen if we use NPI in the initial estimate due to statistical variability of interpolation points. However, right poles contribute to detecting true lines synergistically while others produce lines statistically that may be canceled out in the following stage. We may note that Burg is susceptible to initial phase of sinusoids. Such an effect is not seen in NPI and this effect of Burg in initial estimate is corrected.

6. CASE STUDIES

To investigate the ability of estimator, we will consider a case in which signal is comprised of three sinusoids embedded in a highly colored noise. Consider the following signal x_n :

$$\begin{cases} x_n = \sin(\omega_1 n + \varphi_1) + \sin(\omega_2 n + \varphi_2) + \sin(\omega_3 n + \varphi_3) + z_n \\ z_n = 0.7 z_{n-1} - 0.3 z_{n-4} + 0.5\nu_n + 2\nu_{n\text{-}2} + \nu_{n\text{-}4} \end{cases}$$

where $\varphi_1, \varphi_2, \varphi_3$ and ν_n are independent normal variables with zero mean and unit variance. The model has been used to generate several sets of 300 data points in separate runs. This is to investigate statistical variability of the estimates and the robustness of the estimation method. We have taken the spectral lines at normalized frequencies ω_1=0.30, ω_2=0.67 and ω_3=0.7. Lets preview Fig. 2(d) which shows the estimated PSD using pre-defined pole locations at $p_1 \cup p_2$ where

$$p_2 = \{\rho e^{j\pi 0.30}, \rho e^{j\pi 0.67}, \rho e^{j\pi 0.70}\} \tag{15}$$

with r=0.8 and ρ=0.99 and γ=0.97. p_1 provides a rough estimate of the overall spectrum and p_2 provides a good resolution in spectral lines and nulls. Choose of r and ρ is related to the statistical considerations [8]. The poles in the second line are selected to provide more resolution where spectral lines are present. Of course, this needs a prior knowledge of line frequencies.

Fig. 2(a) shows the estimated PSD of x_n through Burg algorithm of order 30. There are two peaks among others, one for 0.7 and 0.67 which are not resolved, and one for ω=0.3. Fig. 2(b) shows the NPI approach using the poles of

Burg estimate as filterbank poles to produce a 30th order ARMA model. The peak at 0.7 splits off and resolves two peaks with a superior resolution compared to traditional methods.

We put another step forward and used the poles of obtained model to produce another model. The result is presented in Fig. 2(c) showing sharper peaks. However, the results might be less reliable because the poles of previous model showing the lines are very close to the unit circle. For example, the second estimated model may create poles exactly on the unit circle at the estimated frequencies. Also spurious peaks may show up if data length is too short. Table 1 shows the estimated frequencies pertinent to Fig. 2(b) in three separate runs. The results show good statistical robustness. In Fig. 2(d) we have used the estimated frequency of sinusoids to find a reduced order (9th order) model for the signal with the poles in (14) and (15).

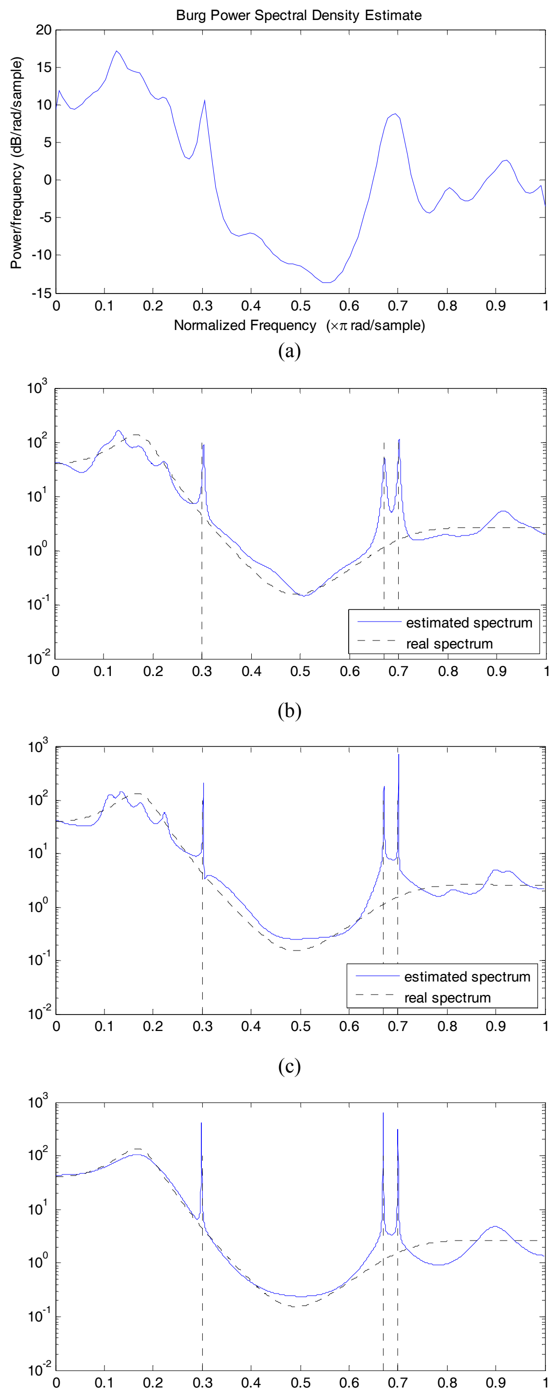

Fig. 2. (a) Initial estimation. (b) first and (c) second modification (d) order reduced estimation.

Table 1. Estimated frequencies in three separate runs.

(1)	0.6695	0.6988	0.3015
(2)	0.6689	0.6999	0.3000
(3)	0.6686	0.7002	0.3027

7. CONCLUSION

The concept of parallel filtering in conjunction with Nevanlinna-Pick interpolation provides a powerful method for spectral estimation and is especially suitable when a short record of data observation is available. A crucial point in this approach is the selection of filterbank poles which greatly impact both the quality and complexity of estimation. We presented an algorithm to locate the poles and debated why it works. Compared to traditional methods, the algorithm is very powerful in resolving closely spaced sinusoids in the presence of colored, non-Gaussian noise.

8. REFERENCES

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