BANDLIMITED SIGNAL RECONSTRUCTION FROM NOISY PERIODIC NONUNIFORM SAMPLES IN TIME-INTERLEAVED ADCS

Vijay Divi and Gregory Wornell

Dept. EECS, MIT
Cambridge, MA 02139
{vdivi,gww}@mit.edu

ABSTRACT

In this paper, we propose a time-domain method for uniform signal reconstruction from periodic nonuniform samples of a bandlimited signal when timing-skews are known. The algorithm computes a constrained least-squares estimate of the input via the pseudoinverse of a time-varying filter. The method exploits the oversampling in the system to provide increased performance in the presence of noise.

Index Terms— Converters, Signal sampling, Signal reconstruction, Estimation

1. INTRODUCTION

It is often possible to sample an analog signal at a higher rate by allowing nonuniform sampling. Time-interleaved analog-to-digital converters (TIADCs) are one such high-speed sampling circuit that can produce periodic nonuniform samples of an input. However, nonuniform sampling is generally avoided due to the added complexity and error to reconstruct uniform samples. In this paper, we introduce a method for reconstructing uniform samples from periodic nonuniform samples when the sample timings are known.

In periodic nonuniform sampling, the input is sampled via M sets of uniformly spaced samples, each set with period MT_s. The spacing between the sets is not uniform but is known. Although input reconstruction from these samples is nontrivial, signals bandlimited to π/T_s can be perfectly reconstructed because of an average period of T_s. This form of sampling is commonly addressed in the area of TIADCs, where timing between the ADCs may not be uniform.

Methods for reconstruction of uniform samples from nonuniform samples have been widely explored [1, 2]. The case of periodic nonuniform sampling has also been studied in detail since it allows for practical implementations. In [3], the added structure of the periodic nonuniform sampling is exploited to develop a filterbank which reconstructs bandlimited inputs perfectly. More computationally efficient methods for approximate reconstructions are given in [4, 5].

The reconstruction methods also operate in the presence of noise, which can be introduced through a variety of sources including quantization and thermal noise. This noise can seriously degrade performance when reconstructing uniform samples from the nonuniform samples [6]. The issue is addressed in the general setting [2, 7]; however, it is often cursorily treated in efficient periodic nonuniform reconstruction. In this paper, we present a time-domain reconstruction method for the noisy regime.

The paper is formatted as follows. In Section 2, we introduce the problem formulation. In Section 3, we present our reconstruction method, as well as its comparison to the naive reconstruction method. The simulated performance is given in Section 4.

This work was supported in part by MARCO C2S2 under Contract No. 2003-CT-888, by MIT Lincoln Laboratory, and by NEC Corp.

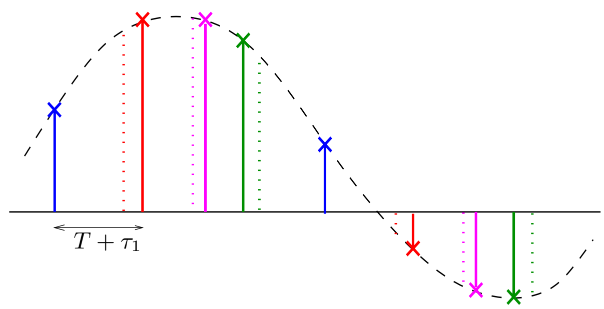

Fig. 1. Signal sampled with $M = 4$ sets of periodic samples. Dotted lines show uniform samples. Solid lines show nonuniform samples.

2. PROBLEM FORMULATION AND SIGNAL MODELS

The input signal $x(t)$ is modeled as a deterministic bandlimited signal with cutoff frequency Ω_c, i.e. the continuous time Fourier transform $X(j\Omega) = 0$ for $\Omega_c \leq |\Omega|$. The overall sampling period T_s of the system is chosen to ensure that the sampling rate strictly exceeds the Nyquist rate, i.e., $T_s < \pi/\Omega_c$, thus creating some amount of excess bandwidth. The signal recovery problem is to estimate $x[n] = x(nT_s)$, which is bandlimited to $\omega_c = \Omega_c T_s < \pi$.

The M sets of received samples are ordered in time and we model the signal from the ith set of samples as:

$$y_i[n] = x(nMT_s + iT_s + \tau_i) + w_i[n] \tag{1}$$

where the τ_i model the timing skew from the uniform sample locations. Although timing of the M sets is not uniform, the skews are treated as a known parameters in our setup. In practice, it may be necessary to first estimate the skews; a variety of methods have been developed for doing so [8, 9]. The $w_i[n]$ in (1) represent the sampling noise which can be modeled as white Gaussian noise with variance σ^2. In quantized signals, the noise variance depends on the number of bits to which the input is quantized. Without loss of generality, we can choose an arbitrary time reference, thus we let $\tau_0 = 0$. Figure 1 shows an example sampling pattern.

We use the following notation to represent the signal obtained by multiplexing the M signals

$$y[n] \quad = \quad y_i\left[\frac{n-i}{M}\right] \qquad n \,(\mathrm{mod}\, M) = i. \tag{2}$$

We can rewrite $y[n]$ in terms of the uniform samples $x[n]$ as follows. First we note that since $x(t)$ is bandlimited and the sampling frequency is higher than the Nyquist frequency, the input can be written as

$$x(t) = \sum_m x[m]\mathrm{sinc}(t - mT_s). \quad (3)$$

Second, for $n \pmod{M} = i$,

$$\begin{aligned} y[n] &= x(nT_s + \tau_i) + w[n] & (4) \\ &= \sum_m x[m]\mathrm{sinc}((n-m)T_s + \tau_i) + w[n]. & (5) \end{aligned}$$

The received signal can be viewed as the output of a linear time-varying filter with noise added

$$y[n] = (f_i * x)[n] + w[n] \quad n \pmod{M} = i \quad (6)$$

where the $f_i[n]$ filters represent the fractional sample delays in (5).

The goal of the reconstruction is to estimate uniform samples of the input, i.e. $\hat{x}[n] = x[n]$, from the received signal $y[n]$ and timing skews τ_i.

3. SIGNAL RECONSTRUCTION

In this section, we develop methods for input signal reconstruction. The general approach for estimation is to invert the effects of the time-varying filters in (6) while handling the noise in a careful manner. Typically, the inverse of a time-varying filter is difficult to compute; however, due to the added structure of our formulation, there exists a closed form solution. In the absence of noise, the input $x[n]$ can be reconstructed from $y[n]$ through another time-varying filter $g_i[n]$ such that

$$x[n] = (g_i * y)[n] \quad n \pmod{M} = i. \quad (7)$$

The filters are explicitly given in [3].

When noise is introduced into the system, the reconstruction must be modified in order to reduce the effect of the noise on the estimate. Designing new filters to optimally handle the noise is intractable in the signal domain since it requires the inverse of more complicated time-varying filters. We switch to a matrix representation of the system to aid in the development of our reconstruction.

3.1. Matrix Formulation

For convenience, we rewrite our equations in matrix form. Specifically, we write the received signal (5)

$$\boldsymbol{y} = \mathbf{F}\boldsymbol{x} + \boldsymbol{w}. \quad (8)$$

with column vectors $\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{w}$ representing blocks of size N of the received ADC signal, the ideal uniform input signal, and the noise signal, respectively. The matrix

$$\mathbf{F}_{k,l} = \mathrm{sinc}((k-l)T_s + \tau_i) \quad i = k \pmod{M} \quad (9)$$

where $0 \leq k, l \leq N-1$. With the matrix approximation, we have treated filters (6) as finite length and neglected edge effects, as is appropriate for larger N. We consider these effects more carefully during simulation.

We now develop a method for signal reconstruction from noisy periodic nonuniform samples. We begin by introducing the naive estimate and then introduce a constrained least-squares formulation which allows for increased accuracy in the presence of noise.

3.2. Naive Least-Squares Estimation

The least-squares (LS) estimate for the input signal is given by

$$\hat{\boldsymbol{x}}_{\mathrm{LS}} = \arg\min_{\boldsymbol{x}} ||\boldsymbol{y} - \mathbf{F}(\boldsymbol{\tau})\boldsymbol{x}||^2 \quad (10)$$

where $\boldsymbol{x}$ is bandlimited, a constraint we ignore for the moment. We explicitly indicate the dependence of $\mathbf{F}$ on the vector of timing skews $\boldsymbol{\tau}$. Because the noise $\boldsymbol{w}$ is modeled as white Gaussian, the LS estimate is equivalent to the maximum-likelihood (ML) estimate. The solution is equal to

$$\hat{\boldsymbol{x}}_{\mathrm{LS}} = (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T\boldsymbol{y} = \mathbf{F}^{-1}\boldsymbol{y} \quad (11)$$

since $\mathbf{F}$ is invertible when the τ_i are sufficiently small.

This estimation method is equivalent to the filters in [3]. The estimate performs well in high SNR situations. However, as noise power increases, it is clear to see that this estimate is suboptimal. A better estimate of $\boldsymbol{x}$ can be produced by enforcing the bandlimited constraint on the estimator $\hat{\boldsymbol{x}}$. We label this as the filtered least-squares (FLS) estimator:

$$\hat{\boldsymbol{x}}_{\mathrm{FLS}} = \mathbf{L}\mathbf{F}^{-1}\boldsymbol{y}. \quad (12)$$

where $\mathbf{L}$ is a matrix implementing a lowpass filter bandlimited to ω_c. This naive estimation technique uses a standard nonuniform reconstruction filter to estimate $\boldsymbol{x}$ from $\boldsymbol{y}$ and then lowpass filters the output to remove any noise in the out-of-band spectrum.

By substituting for $\boldsymbol{y}$ in (12), we find

$$\begin{aligned} \hat{\boldsymbol{x}}_{\mathrm{FLS}} &= \mathbf{L}\mathbf{F}^{-1}(\mathbf{F}\boldsymbol{x} + \boldsymbol{w}) & (13) \\ &= \boldsymbol{x} + \mathbf{L}\mathbf{F}^{-1}\boldsymbol{w}. & (14) \end{aligned}$$

Thus the error term equals $\mathbf{e}_{\mathrm{FLS}} = \mathbf{L}\mathbf{F}^{-1}\boldsymbol{w}$.

3.3. Constrained Least-Squares Estimation

A more accurate estimator results from imposing the bandlimited requirement directly into the least-squares estimation. The constrained least-squares (CLS) optimization problem is given by

$$\hat{\boldsymbol{x}}_{\mathrm{CLS}} = \arg\min_{\boldsymbol{x}\in\mathcal{S}} ||\boldsymbol{y} - \mathbf{F}(\boldsymbol{\tau})\boldsymbol{x}||^2 \quad (15)$$

where $\mathcal{S} = \{\boldsymbol{x} \mid \boldsymbol{x} \in \mathcal{R}^N,\ \mathbf{L}\boldsymbol{x} = \boldsymbol{x}\}$.

By introducing a secondary variable $\mathbf{z}$, where $\boldsymbol{x} = \mathbf{L}\mathbf{z}$, we can remove the constraint in the LS formulation

$$\begin{aligned} \hat{\boldsymbol{z}}_{\mathrm{CLS}} &= \arg\min_{\mathbf{z}} ||\boldsymbol{y} - \mathbf{F}\mathbf{L}\mathbf{z}||^2 & (16) \\ \hat{\boldsymbol{x}}_{\mathrm{CLS}} &= \mathbf{L}\hat{\boldsymbol{z}}_{\mathrm{CLS}}. & (17) \end{aligned}$$

First, we must compute $\hat{\boldsymbol{z}}_{\mathrm{CLS}}$. Because the matrix $\mathbf{L}$ is singular (high frequency vectors lie in the nullspace), the product matrix $\mathbf{FL}$ is also singular. Thus it is not possible to take the inverse of this matrix in order to compute the maximum likelihood estimate $\hat{\boldsymbol{z}}_{\mathrm{CLS}}$. Instead, we use the pseudoinverse

$$\hat{\boldsymbol{z}}_{\mathrm{CLS}} = (\mathbf{FL})^{\dagger}\boldsymbol{y} \quad (18)$$

where $(\cdot)^{\dagger}$ denotes the Moore-Penrose pseudoinverse [10]. The overall solution is given as

$$\hat{\boldsymbol{x}}_{\mathrm{CLS}} = \mathbf{L}(\mathbf{FL})^{\dagger}\boldsymbol{y}. \quad (19)$$

We now analyze this estimator. The lowpass filter matrix can be implemented by a frequency sampling filter matrix, i.e.

$$\mathbf{L} = \mathbf{D}^{-1}\mathbf{\Sigma}_L\mathbf{D} \tag{20}$$

where $\mathbf{D}$ is the $N \times N$ DFT matrix, and $\mathbf{\Sigma}_L$ is the frequency response of the filter. Since $\mathbf{L}$ represents a lowpass filter with cutoff frequency ω_c, this is equivalent to having the $(N-k)$ eigenvalues which correspond to the low frequency eigenvectors equal to one and the k eigenvalues which correspond to high frequency eigenvectors equal to zero, where $k = \lfloor N(\pi - \omega_c)/\pi \rfloor$. The $N \times N$ eigenvalue matrix is given by

$$\mathbf{\Sigma}_L = \begin{bmatrix} \mathbf{I}_{N-k} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{21}$$

The nullspace of $\mathbf{L}$ corresponds to linear combinations of high frequency vectors.

The pseudoinverse of $\mathbf{L}$ is equivalent to inverting all the non-zero eigenvalues. Because each of these eigenvalues is equal to one, their inverses are equal to one (and the zero eigenvalues remain zero). Thus we find that $\mathbf{L}^\dagger = \mathbf{L}$. Note that without the frequency sampling approximation (20), the matrix $\mathbf{L}$ may have very small non-zero eigenvalues. When the matrix is inverted, these eigenvalues become very large but can be negated with a power constraint on $\hat{\boldsymbol{x}}$.

Now, we analyze the properties of the product $\mathbf{FL}$. The singular value decomposition is given by

$$\mathbf{FL} = \mathbf{U}\mathbf{\Sigma}_{FL}\mathbf{V}^T \tag{22}$$

where $\mathbf{U}$ and $\mathbf{V}$ are $N \times N$ orthonormal matrices. It is easy to see that because $\mathbf{F}$ is full rank, the nullspace $\mathcal{N}(\mathbf{FL}) = \mathcal{N}(\mathbf{L})$ and has rank $N-k$. Thus $\mathbf{\Sigma}_{FL}$ can be decomposed as

$$\mathbf{\Sigma}_{FL} = \begin{bmatrix} \mathbf{\Sigma}_S & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{23}$$

where $\mathbf{\Sigma}_S$ is an $(n-k) \times (n-k)$ diagonal matrix.

The bottom k rows of $\mathbf{V}^T$ span the high frequency nullspace of $\mathbf{L}$, and the top rows span the low frequency space. Again, since $(\mathbf{FL})^\dagger$ inverts only the non-zero eigenvalues, we find that

$$(\mathbf{FL})^\dagger\mathbf{FL} = \mathbf{V}\mathbf{\Sigma}_L\mathbf{V}^T = \mathbf{L} \tag{24}$$

where the second equality holds because the nullspaces of $\mathbf{FL}$ and $\mathbf{L}$ are equal and the low frequency subspace has unity eigenvalues. This simplification does not imply anything about the product $\mathbf{FL}(\mathbf{FL})^\dagger$. For similar reasons, it is easy to see that $\mathbf{L}(\mathbf{FL})^\dagger = (\mathbf{FL})^\dagger$.

From the analysis above, we can better interpret the CLS estimator. First, the received signal is projected into the space of $\boldsymbol{\tau}$-spaced periodic nonuniform samples of any possible bandlimited signal, i.e. the space spanned by $\mathbf{FL}$. This produces the intermediate estimate $\mathbf{FL}(\mathbf{FL})^\dagger\boldsymbol{y}$. The nonuniform sampling is then inverted by applying $\mathbf{F}^{-1}$. Thus, the noise reduction occurs on the nonuniform samples before the signal reconstruction.

This approach provides a more efficient method for noise reduction than the equivalent frequency-domain techniques for signal estimation, which are not as easily realizable in hardware. It also provides insight into optimal methods for treating noise when developing practical reconstruction filters. Fig. 2 shows a graphical representation of the operations of the estimators.

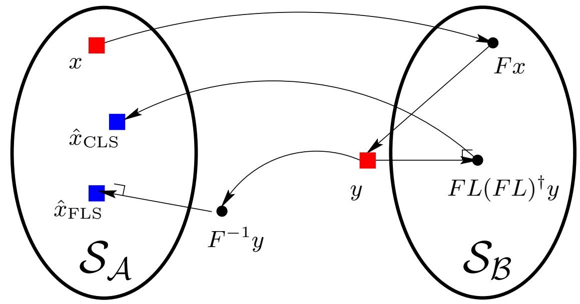

Fig. 2. Graphical representation of the estimator operations. Set $\mathcal{S}_\mathcal{A}$ represents the convex set of bandlimited signals. Set $\mathcal{S}_\mathcal{B}$ represents the convex set of signals which can be written as $\mathbf{F}s$, where $s \in \mathcal{S}_\mathcal{A}$.

To compute the CLS estimator error, we substitute for $\boldsymbol{y}$ in (18) and find

$$\begin{aligned} \hat{\boldsymbol{x}}_{\mathrm{CLS}} &= \mathbf{L}(\mathbf{FL})^\dagger(\mathbf{F}\boldsymbol{x} + \boldsymbol{w}) & (25) \\ &= \boldsymbol{x} + (\mathbf{FL})^\dagger\boldsymbol{w}. & (26) \end{aligned}$$

The error term equals $\mathbf{e}_{\mathrm{CLS}} = (\mathbf{FL})^\dagger\boldsymbol{w}$. In the following sections, we show the performance improvement of the CLS estimator over the naive FLS method.

3.4. Analysis of Estimators

In this section, we analyze the bias, variance, and efficiency of the estimators. From (14) and (26), it is clear that estimators FLS and CLS are unbiased, i.e., $E[\hat{\boldsymbol{x}}] = \boldsymbol{x}$. The unbiased property is due to fact that the expectation of any linear combination of $w[n]$ is zero.

The covariance matrices of the noise are given by

$$\begin{aligned} \Lambda_{\mathrm{FLS}} &= E[\mathbf{e}_{\mathrm{FLS}}\,\mathbf{e}_{\mathrm{FLS}}^T] = \sigma^2\mathbf{L}(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{L}^T & (27) \\ \Lambda_{\mathrm{CLS}} &= E[\mathbf{e}_{\mathrm{CLS}}\,\mathbf{e}_{\mathrm{CLS}}^T] = \sigma^2(\mathbf{FL})^\dagger(\mathbf{FL})^{\dagger T}. & (28) \end{aligned}$$

From these matrices, we can calculate the average variances

$$\begin{aligned} \sigma^2_{\mathrm{FLS}} &= \frac{1}{N}\sigma^2\,\mathrm{tr}(\mathbf{L}(\mathbf{F}^T\mathbf{F})^{-1}\mathbf{L}^T) & (29) \\ \sigma^2_{\mathrm{CLS}} &= \frac{1}{N}\sigma^2\,\mathrm{tr}((\mathbf{FL})^\dagger(\mathbf{FL})^{\dagger T}). & (30) \end{aligned}$$

Because comparing the errors is not analytically tractable, we numerically calculate them in Section 4 and verify that $\sigma^2_{\mathrm{CLS}} < \sigma^2_{\mathrm{FLS}}$.

An estimator is efficient if its error variance achieves the Cramer-Rao bound (CRB). In [11], the CRB is redefined for constrained parameter estimation problems. For the problem of

$$\hat{\boldsymbol{x}}_{\mathrm{LS}} = \arg\max_{\boldsymbol{x}, g(\boldsymbol{x})=0} p_{\boldsymbol{y}|\boldsymbol{x}}(\boldsymbol{y}|\boldsymbol{x}) \tag{31}$$

where the $g(\cdot)$ function may contain multiple nonlinear equations, the constrained CRB bound is given by

$$\mathbf{\Lambda}_{\hat{\boldsymbol{x}}} \geq \mathbf{J}^{-1} - \mathbf{J}^{-1}\mathbf{G}(\mathbf{G}^T\mathbf{J}^{-1}\mathbf{G})^{-1}\mathbf{G}^T\mathbf{J}^{-1} \tag{32}$$

where $\mathbf{J} = -E\left[\nabla_{\boldsymbol{x}} \ln p_{\boldsymbol{y}|\boldsymbol{x}}(\boldsymbol{y}|\boldsymbol{x}) \cdot \nabla_{\boldsymbol{x}}^T \ln p_{\boldsymbol{y}|\boldsymbol{x}}(\boldsymbol{y}|\boldsymbol{x})\right]$ is the Fisher information matrix and $\mathbf{G} = \nabla_{\boldsymbol{x}} g^T(\boldsymbol{x})$ is the $N \times N$ gradient matrix of the constraints. In the time interleaved converter setup, the constraints enforce that the estimate is bandlimited, $g(\boldsymbol{x}) = (\mathbf{I} - \mathbf{L})\boldsymbol{x}$. We find $\mathbf{J} = \mathbf{F}^T\mathbf{F}$ and $\mathbf{G} = (\mathbf{I} - \mathbf{L})^T$ yielding the bound on $\Lambda_{\hat{\boldsymbol{x}}}$ given in (32). It is clear that the CLS estimate will achieve the CRB for the constrained ML problem because the constraints are linear.

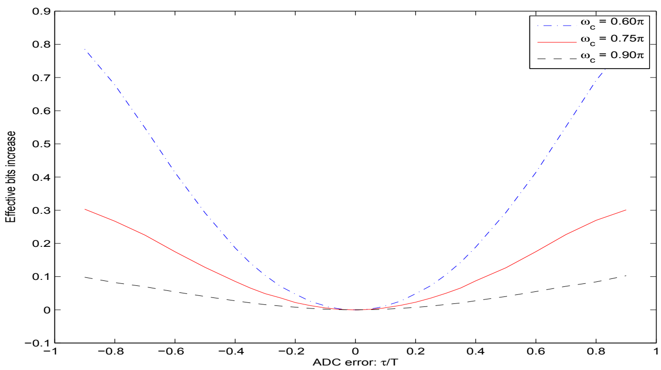

Fig. 3. Effective bit increase of estimator CLS over FLS in a 2-ADC system for varying amounts of skew. Average performance is plotted for different cutoff frequencies (amounts of oversampling).

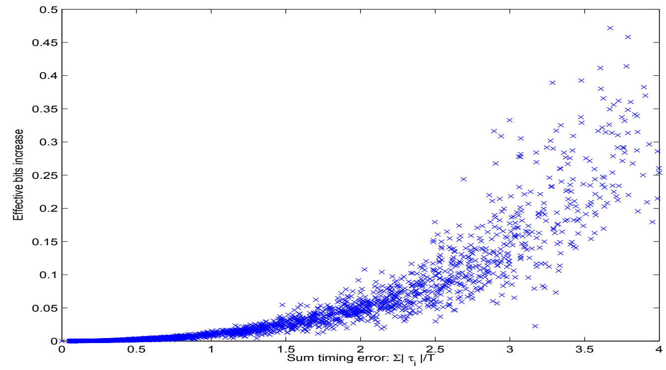

Fig. 4. Effective bit increase of estimator CLS over FLS in a 16-ADC system with $\omega_c = 0.75\pi$. System timing skew is measured by $\sum |\tau_i|/T$. Each x on the plot represents a random set of skews.

4. ANALYSIS AND SIMULATIONS

In this section, we present the performance of the FLS and CLS estimators in TIADC systems. To do this, we compute the traces of the covariance matrices of the estimators. As expected, in all cases, the noise power is lower for the CLS estimator. The results are given in terms of the increase in effective bits of the CLS estimator over the FLS estimator (assuming a 6.02 dB increase in SNR corresponds to one effective bit). The results in Fig. 3 were simulated with a 12-bit 2-ADC system with block lengths of 512 samples. We present the performance for varying oversampling ratios and timing skew size.

The plot shows that for small amounts of timing skew, the CLS estimate only provides marginal improvements in effective bits. However, the benefits of the estimator are more visible for higher levels of timing skew and oversampling. The effective bit increase is independent of the starting number of effective bits because the signal power remains the same. Thus, the 0.1 bit increase obtained for the 2-ADC system with cutoff $\omega_c = 0.75\pi$ (33% oversampling) and 40% timing skew holds even for low-resolution converters.

Fig. 4 shows performance in a 16-ADC system for tests where the set of timing skews is chosen at random. The system timing error is measured by the sum of the magnitudes of the timing skews $\sum |\tau_i|/T$. In this case, reconstruction for an average timing skew of $\sim 16\%$ yields a 0.1 bit increase in resolution.

5. CONCLUDING REMARKS

Using a least-squares formulation, we presented two methods for reconstructing the input from the noisy periodic nonuniform samples when the timing skews are known (or have been estimated). In the naive FLS approach, we enforce the bandlimited constraint by computing the LS estimate and then lowpass filtering it. By imposing the bandlimited requirement directly on the least-squares estimate, the CLS estimate provides an increase in performance.

The CLS method is useful within systems with large timing-skews and high levels of oversampling. The SNR gain is fixed and independent of the input SNR. Thus, the performance increase is most clearly seen in high noise systems (low resolution).

Future research must be conducted to determine the performance of the estimator when estimates of the timing skew are used. Also for practical implementation, it is necessary to reduce the complexity to a level which can be implemented easily in hardware. By combining the reconstruction method with a method for skew estimation, it may be possible to form an efficient method for joint timing-skew estimation and signal reconstruction in the presence of noise.

6. REFERENCES

[1] K. Yao and J. B. Thomas, "On some stability and interpolatory properties of nonuniform sampling expansions," *IEEE Trans. Circuit Theory*, pp. 404–408, Dec. 1967.

[2] A. Aldroubi and K. Gröchenig, "Nonuniform sampling and reconstruction in shift-invariant spaces," *SIAM Rev.*, vol. 43, pp. 585–620, 2001.

[3] Y. C. Eldar and A. V. Oppenheim, "Filterbank reconstruction of bandlimited signals from nonuniform and generalized samples," *IEEE Trans. Signal Processing*, vol. 48, no. 10, pp. 2864–2875, Oct. 2000.

[4] H. Johansson, P. Löwenborg, and K. Vengattaramane, "Least-squares and minimax design of polynomial impulse response fir filters for reconstruction of two-periodic nonuniformly sampled signals," *IEEE Trans. Circuits and Systems I*, vol. 54, pp. 877–888, Apr. 2007.

[5] T. Strohmer and J. Tanner, "Fast reconstruction algorithms for periodic nonuniform sampling with applications to time-interleaved ADCs," *Proc. IEEE ICASSP*, Apr. 2007.

[6] D. Seidner and M. Feder, "Noise amplification of periodic nonuniform sampling," *IEEE Trans. Signal Processing*, pp. 275–277, Jan. 2000.

[7] F. A. Marvasti, *Nonuniform Sampling: Theory and Practice*, Springer, 2001.

[8] V. Divi and G. Wornell, "Scalable blind calibration of timing skew in high-resolution time-interleaved ADCs," *Proc. IEEE ISCAS*, May 2006.

[9] S. Huang and B. C. Levy, "Blind calibration of timing offsets for four-channel time-interleaved ADCs," *IEEE Trans. Circuits and Systems I*, vol. 54, pp. 863–876, Apr. 2007.

[10] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins, 1996.

[11] J. D. Gorman and A. O. Hero, "Lower bounds for parametric estimation with constraints," *IEEE Trans. Info Theory*, vol. 36, pp. 1285–1301, Nov. 1990.