UPPER BOUNDS ON ALIASING ERROR ENERGY FOR MULTIDIMENSIONAL SAMPLING OF NONBANDLIMITED SIGNALS

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ABSTRACT

We present upper bounds on the 2-norm of the aliasing error in multidimensional Shannon sampling. Our bounds complement the previously known 1-norm upper bounds for the peak aliasing error. The proposed bounds provide a good estimate of the total error energy for any signal, rather than just for certain pathological extremals, as is the case with the 1-norm bounds which, as a result, tend to be too conservative for practical applications. The sampling representation is general, possibly multiband, and not restricted to bandlimited signals. Error bounds are phrased in terms of the energy of signal components that lie outside the assumed band-region. Therefore, they are easy to interpret and compute as is demonstrated for two practical signal classes, namely signals with exponential or polynomial out-of-band decay.

***Index Terms*—** Shannon sampling, aliasing error, multidimensional, nonbandlimited, error energy.

1. INTRODUCTION

The classical Whitaker-Shannon-Kotelnikov sampling theorem has been extended to multidimensional signals bandlimited on sets in the n-dimensional Euclidean space [1, 2, 3]. However, in many signal processing and imaging applications, and in particular in multidimensional (M-D) cases where the signal typically has finite spatial/temporal support, the signal is only approximately bandlimited. This leads to aliasing error in the sampling series expansion. While bounds on the aliasing error in the one-dimensional (1-D) case have been extensively studied during the past three decades (see [3, 4, 5] and references therein), their extensions to n-dimensions are much more recent. In particular, Higgins [6] considered the most general case of a multidimensional passband and derived a uniform (1-norm) upper bound on the aliasing error.

The Higgins' result provides upper bounds for the point-wise peak error, which in most applications is too conservative to be practical. Therefore, there is great interest in deriving upper bounds that instead provide a bound on the total energy (or 2-norm) of aliasing error. The advantage of these bounds is that they are demonstrated to provide a good estimate of the error for any signal, rather than just for certain pathological extremals, as is the case with the peak-error (1-norm) upper bound.

Unfortunately, to date, such bounds do not exist for M-D sampling of nonbandlimited signals. The purpose of this work is to introduce, for the first time, aliasing error energy upper bounds that hold under relatively weak conditions, apply to nonbandlimited signals, and only depend on the "out-of-band" signal components, i.e., those lying outside the assumed band-region for the signal.

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In related work, Khurgin and Yakovlev [5] have introduced an aliasing error energy upper bound for the low-pass sampling of 1-D signals. The bounds we propose in this work, however, apply to M-D problems and the sampling representation is general, possibly multiband. Furthermore, we also show that our bounds can be arbitrarily tighter than Khurgin-Yakovlev's for a practical class of signals.

The presented results also complement alternative approaches proposed recently (for the 1-D case only) which involve conditions on the signal's "modulus of continuity" (a smoothness measure) in the sampling (spatial) domain [7, 8]. It is important to note that in signal processing, it is natural to express sampling restrictions based on the Fourier transform of the signal (which is the case for our bounds) whereas conditions on modulus of continuity or derivatives are, in most cases, of little practical significance. Furthermore, these bounds are generally more restrictive and apply to a smaller class of signals compared to our results. Specifically, the bound in [7] requires (at least) mean square differentiability in the spatial domain.

The proposed bounds can potentially be applied in most areas of signal processing that involve aliasing in the sampling scheme. The error bounds provide quantitative measures of performance limits and characterize fundamental limitations. More specifically, the introduced bounds can be applied in designing sampling patterns for signals with a known spectral estimate to satisfy a threshold on the tolerable aliasing error. Problems of this type are of interest in many areas of signal processing, for example, in multi-dimensional imaging such as dynamic 4-D MRI of the human heart [9].

The paper is organized as follows. Section 2 introduces the notation and provides a brief background of the previous work in this area. In Section 3, we derive upper bounds for the aliasing error energy in the M-D case. Next, Section 4 provides more intuition and new bounds by analyzing the 1-D problem. In Section 5, we compute closed-form bounds for two practical classes of signals and propose very simple "folk" rules for computing a bound on the error energy, which apply under certain decay conditions. Finally, in Section 6, we compare our bounds to the Khurgin-Yakovlev bound for signals with polynomially decaying spectra.

2. FORMULATION AND BACKGROUND

We denote a point lattice in $\mathbb{R}^n$ by $\Lambda = \{\mathbf{l} : \mathbf{l} = \mathbf{V}\mathbf{p}, \mathbf{p} \in \mathbb{Z}^n\}$ where $\mathbf{V} \in \mathbb{R}^{n \times n}$ is its nonsingular basis matrix. Recall that if a function is sampled on Λ, its Fourier transform is replicated on the polar lattice Λ^* of Λ that has basis matrix $\mathbf{V}^* = \mathbf{V}^{-\mathrm{T}}$ [1].

Denote the Fourier transform of a signal f defined on $\mathbb{R}^n$ by

$$\hat{f}(\mathbf{u}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi j \mathbf{u}\cdot\mathbf{x}} d\mathbf{x} \tag{1}$$

where $\mathbf{x}, \mathbf{u} \in \mathbb{R}^n$ are the spatial and frequency variables, respectively, and $\mathbf{u} \cdot \mathbf{x}$ is the Euclidean dot product. The bounds presented in this work apply to signals (possibly nonbandlimited) in the class

$$\mathcal{F} = \{f \in L_1(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n) : \hat{f} \in L_1(\mathbb{R}^n)\}$$

where $L_1(\mathbb{R}^n)$ and $\mathcal{C}(\mathbb{R}^n)$ denote the spaces of Lesbegue measurable functions absolutely integrable, and continuous on $\mathbb{R}^n$, respectively. These conditions are considered among the mildest in the context of aliasing error (see [10] for references).

Consider an *assumed band-region* $\mathcal{B}$ for f such that translates of $\mathcal{B}$ by the points in Λ^* do not overlap: $\mathcal{B} + \mathbf{k} \cap \mathcal{B} + \mathbf{l} = \phi, \forall \mathbf{k}, \mathbf{l} \in \Lambda^*, \mathbf{k} \neq \mathbf{l}$. Here, $(\Lambda^*, \mathcal{B})$ is called a "lattice packing." If in addition, the translates of Λ^* perfectly cover $\mathbb{R}^n$ then we have a "lattice tiling." Note that this formulation allows for general baseband and multiband spectral supports for f. The sampling expansion (or approximation) on the sampling lattice Λ of any f is defined as

$$(\mathcal{S}_{\Lambda,\mathcal{B}} f)(\mathbf{x}) = \sum_{\mathbf{l} \in \Lambda} f(\mathbf{l}) \varphi(\mathbf{x} - \mathbf{l}) \tag{2}$$

where $\hat{\varphi}(\mathbf{u}) = |\det(\mathbf{V})| \chi_{\mathcal{B}(\mathbf{u})}$ and $\chi_{\mathcal{B}}$ is the indicator function of $\mathcal{B}$ (i.e., ideal M-D bandpass filter).

Given a lattice packing $(\Lambda^*, \mathcal{B})$, sampling the signal f on lattice Λ yields the following aliasing error:

$$e(\mathbf{x}) = f(\mathbf{x}) - (\mathcal{S}_{\Lambda,\mathcal{B}} f)(\mathbf{x}). \tag{3}$$

Uniform (point-wise) bounds for the aliasing error $e(\mathbf{x})$ have been studied extensively (see [3, 6] and references therein). A generalized version of the M-D uniform bound is [6]

$$|e(\mathbf{x})| \leq 2 \int_{\mathbb{R}^n \setminus \mathcal{B}} |\hat{f}(\mathbf{u})| d\mathbf{u}. \tag{4}$$

If f is bandlimited to within $\mathcal{B}$, i.e., support of $\hat{f}$ is a subset of $\mathcal{B}$, the aliasing error is zero as can be inferred from the bound in (4). This generalizes the so called "lowpass" sampling theorem [1, 2] to almost arbitrary spectral supports. Extremal constructions that achieve equality in (4) are also known [6, 10].

For signals $f \in \mathcal{F}$ with a spatial support $\mathcal{S} \in \mathbb{R}^n$ of finite Lebesgue measure $m(\mathcal{S})$, the 1-norm bound in (4) implies the following 2-norm one [10]

$$\|e\|_2 \leq 2\sqrt{m(\mathcal{S})} \int_{\mathbb{R}^n \setminus \mathcal{B}} |\hat{f}(\mathbf{u})| d\mathbf{u}.$$

This bound is not generally tight (in fact, can be arbitrarily loose) and will not be useful when $m(\mathcal{S})$ is relatively large.

3. NEW UPPER BOUNDS ON ALIASING ERROR ENERGY

3.1. Preliminaries

We start by considering the "bandlimiting" operators $P_{\mathcal{B}} : \mathcal{F} \to \mathcal{F}$ and $P_{\mathcal{B}^c} : \mathcal{F} \to \mathcal{F}$ defined by $(\widehat{P_{\mathcal{B}} f}) = \chi_{\mathcal{B}} \hat{f}$ and $(\widehat{P_{\mathcal{B}^c} f}) = \chi_{\mathcal{B}^c} \hat{f}$, respectively, where $\mathcal{B}^c = \mathbb{R}^n \setminus \mathcal{B}$ is the *out-of-band* region. The following lemma [10] establishes a useful property of the sampling expansion operator $\mathcal{S}_{\Lambda,\mathcal{B}}$.

Lemma 1 ([10]). *The Fourier transform of the sampling expansion in* (2) *is given by*

$$(\widehat{\mathcal{S}_{\Lambda,\mathcal{B}} f})(\mathbf{u}) = \chi_{\mathcal{B}}(\mathbf{u}) \sum_{\mathbf{k} \in \Lambda^*} \hat{f}(\mathbf{u} + \mathbf{k}). \tag{5}$$

Corollary 1 ([10]). $P_{\mathcal{B}^c} \mathcal{S}_{\Lambda,\mathcal{B}} f = \mathbf{0}$.

The following lemma, due to Bresler [10], expresses the 2-norm of aliasing error in terms of $P_{\mathcal{B}^c} f$, i.e., the out-of-band component of f.

Lemma 2 ([10]). *The aliasing error* $e(\mathbf{x})$ *defined in* (3) *satisfies*

$$\|e\|_2^2 = \|P_{\mathcal{B}^c} f\|_2^2 + \|\mathcal{S}_{\Lambda,\mathcal{B}} P_{\mathcal{B}^c} f\|_2^2.$$

The following lower bound for the energy of the aliasing error follows from Lemma 2.

Corollary 2 ([10]). $\|e\|_2^2 \geq \|P_{\mathcal{B}^c} f\|_2^2$.

3.2. Derivation of the Upper Bounds

From Lemma 2, we observe that the aliasing error energy has two components. The first,

$$\varepsilon^2 \triangleq \|P_{\mathcal{B}^c} f\|_2^2 = \int_{\mathbb{R}^n \setminus \mathcal{B}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u} \tag{6}$$

is the *out-of-band signal energy*. We turn next to upper bound the second component (in terms of the out-of-band signal).

Using the M-D version of Parseval's relation,

$$\|\mathcal{S}_{\Lambda,\mathcal{B}} P_{\mathcal{B}^c} f\|_2^2 = \|\widehat{\mathcal{S}_{\Lambda,\mathcal{B}} P_{\mathcal{B}^c}} f\|_2^2,$$

and Lemma 1 (substituting $P_{\mathcal{B}^c} f$ for f) yields

$$\begin{aligned} \|\mathcal{S}_{\Lambda,\mathcal{B}} P_{\mathcal{B}^c} f\|_2^2 &= \int_{\mathbb{R}^n} \left| \chi_{\mathcal{B}}(\mathbf{u}) \sum_{\mathbf{k} \in \Lambda^*} (\widehat{P_{\mathcal{B}^c} f})(\mathbf{u} + \mathbf{k}) \right|^2 d\mathbf{u} \\ &= \int_{\mathbb{R}^n} \left| \sum_{\mathbf{k} \in \Lambda^* \setminus \{\mathbf{0}\}} \chi_{\mathcal{B}}(\mathbf{u}) (\widehat{P_{\mathcal{B}^c} f})(\mathbf{u} + \mathbf{k}) \right|^2 d\mathbf{u} \end{aligned} \tag{7}$$

where (7) follows by noting that for $\mathbf{k} = \mathbf{0}$, the summand is zero since $\chi_{\mathcal{B}}(\mathbf{u}) \chi_{\mathcal{B}^c}(\mathbf{u}) \hat{f}(\mathbf{u}) = \mathbf{0}$. We need the following lemma to further simplify (7).

Lemma 3. *If* $(\Lambda^*, \mathcal{B})$ *is a lattice packing, then for all* $\mathbf{k} \in \Lambda^* \setminus \{\mathbf{0}\}$,

$$(\widehat{P_{\mathcal{B}^c} f})(\mathbf{u} + \mathbf{k}) = \hat{f}(\mathbf{u} + \mathbf{k})$$

Proof. The left hand side (LHS), according to the definition, can be expanded as $(\widehat{P_{\mathcal{B}^c} f})(\mathbf{u} + \mathbf{k}) = \chi_{\mathcal{B}^c}(\mathbf{u} + \mathbf{k}) \hat{f}(\mathbf{u} + \mathbf{k})$. Due to the lattice packing property (see Sec. 2), $(\mathcal{B} + \mathbf{k}) \cap \mathcal{B} = \phi$, that is, $\chi_{\mathcal{B}^c}(\mathbf{u} + \mathbf{k}) = 1$ for all $\mathbf{k} \in \Lambda^* \setminus \{\mathbf{0}\}$, which proves the claim. ☐

Applying Lemma 3 to (7), we have

$$\begin{aligned} \|\mathcal{S}_{\Lambda,\mathcal{B}} P_{\mathcal{B}^c} f\|_2^2 &= \int_{\mathbb{R}^n} |\chi_{\mathcal{B}}(\mathbf{u})|^2 \left| \sum_{\mathbf{k} \in \Lambda^* \setminus \{\mathbf{0}\}} \hat{f}(\mathbf{u} + \mathbf{k}) \right|^2 d\mathbf{u} \\ &= \int_{\mathcal{B}} \left| \sum_{\mathbf{k} \in \Lambda^* \setminus \{\mathbf{0}\}} \hat{f}(\mathbf{u} + \mathbf{k}) \right|^2 d\mathbf{u} \\ &= \int_{\mathcal{B}} \sum_{\mathbf{k}, \mathbf{l} \in \Lambda^* \setminus \{\mathbf{0}\}} \bar{\hat{f}}(\mathbf{u} + \mathbf{k}) \hat{f}(\mathbf{u} + \mathbf{l}) d\mathbf{u} \\ &= \sum_{\mathbf{k}, \mathbf{l} \in \Lambda^* \setminus \{\mathbf{0}\}} \int_{\mathcal{B}} \bar{\hat{f}}(\mathbf{u} + \mathbf{k}) \hat{f}(\mathbf{u} + \mathbf{l}) d\mathbf{u} \end{aligned} \tag{8}$$

where $\bar{\hat{f}}$ denotes the complex conjugate of $\hat{f}$. The switch of the order of the sum and integral in (8) holds under mild conditions. Next, we apply the Cauchy-Schwartz inequality (in the inner product space of $L_2(\mathbb{R}^n)$) to the summand in (8),

$$\int_{\mathcal{B}} \bar{\hat{f}}(\mathbf{u}+\mathbf{k})\hat{f}(\mathbf{u}+\mathbf{l})d\mathbf{u} \leq \sqrt{\int_{\mathcal{B}} |\hat{f}(\mathbf{u}+\mathbf{k})|^2 d\mathbf{u}}\sqrt{\int_{\mathcal{B}} |\hat{f}(\mathbf{u}+\mathbf{l})|^2 d\mathbf{u}} \tag{9}$$

Note that $\int_{\mathcal{B}} |\hat{f}(\mathbf{u}+\mathbf{k})|^2 d\mathbf{u} = \int_{\mathcal{B}+\mathbf{k}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u}$ (simply a change of integration variable). Next, define the sequence $\{a_{\mathbf{k}}\}_{\mathbf{k}\in\Lambda^*}$ as $a_{\mathbf{k}} \triangleq \sqrt{\int_{\mathcal{B}+\mathbf{k}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u}}$ for $\mathbf{k} \in \Lambda^* \backslash \{\mathbf{0}\}$ and $a_{\mathbf{0}} \triangleq 0$.

Combining (8) with (9), we have

$$\|\mathcal{S}_{\Lambda,\mathcal{B}} P_{\mathcal{B}^c} f\|_2^2 \leq \sum_{\mathbf{k},\mathbf{l}\in\Lambda^*} a_{\mathbf{k}} a_{\mathbf{l}} = \Big(\sum_{\mathbf{k}\in\Lambda^*} a_{\mathbf{k}}\Big)^2 \tag{10}$$

where the summation range need not exclude $\{\mathbf{0}\}$ since $a_{\mathbf{0}} = 0$. Adding the out-of-band signal energy ε^2 defined in (6) to both sides of (10) and applying Lemma 2 to the resulting inequality yields

$$\|e\|_2^2 \leq \varepsilon^2 + \Big(\sum_{\mathbf{k}\in\Lambda^*} a_{\mathbf{k}}\Big)^2 \tag{11}$$

which gives our first error energy upper bound, referred to as (B.1).

Lemma 4. *For a lattice packing* $(\Lambda^*, \mathcal{B})$*,*

$$\sum_{\mathbf{k}\in\Lambda^*\backslash\{\mathbf{0}\}} \int_{\mathcal{B}+\mathbf{k}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u} \leq \int_{\mathbb{R}^n\backslash\mathcal{B}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u} \tag{12}$$

with equality for a lattice tiling $(\Lambda^*, \mathcal{B})$.

Proof. Since $(\Lambda^*, \mathcal{B})$ is a lattice packing, we have that $\cup_{\mathbf{k}\in\Lambda^*}\mathcal{B} + \mathbf{k} \subseteq \mathbb{R}^n$ (equality holds for a lattice tiling). Therefore,

$$\sum_{\mathbf{k}\in\Lambda^*} \int_{\mathcal{B}+\mathbf{k}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u} = \int_{\bigcup_{\mathbf{k}\in\Lambda^*} \mathcal{B}+\mathbf{k}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u} \leq \int_{\mathbb{R}^n} |\hat{f}(\mathbf{u})|^2 d\mathbf{u}$$

Subtracting $\int_{\mathcal{B}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u}$ from both sides gives (12). □

Note that the LHS of (12) is equal to $\sum_{\mathbf{k}\in\Lambda^*} a_{\mathbf{k}}^2$ and its right hand side is ε^2. Applying Lemma 4 to an expanded version of (11) provides our second bound, referred to as (B.2).

Finally, denoting the L_∞ norm (function maximum) by $\|.\|_\infty$, it follows that

$$a_k = \sqrt{\int_{\mathcal{B}+\mathbf{k}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u}} \leq \sqrt{m(\mathcal{B})}\|\hat{f}\chi_{\mathcal{B}+\mathbf{k}}\|_\infty \tag{13}$$

where $m(.)$ denotes the Lebesgue measure. Combined with (B.1), (13) gives our third bound, (B.3). These bounds (together with Corollary 2) are summarized next.

Theorem 1. *For a lattice packing* $(\Lambda^*, \mathcal{B})$ *and* $f \in \mathcal{F}$*, the aliasing error energy satisfies the following bounds for* $i = 1, 2, 3$

$$\int_{\mathbb{R}^n\backslash\mathcal{B}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u} \leq \|e\|_2^2 \leq (c_i + Q_i) \int_{\mathbb{R}^n\backslash\mathcal{B}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u}$$

where

$$c_1 = 1, \quad Q_1 = \frac{1}{\int_{\mathbb{R}^n\backslash\mathcal{B}} |\hat{f}(\mathbf{u})|^2} \left(\sum_{\mathbf{k}\in\Lambda^*\backslash\{\mathbf{0}\}} \sqrt{\int_{\mathcal{B}+\mathbf{k}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u}}\right)^2$$

$$c_2 = 2,$$

$$Q_2 = \frac{1}{\int_{\mathbb{R}^n\backslash\mathcal{B}} |\hat{f}(\mathbf{u})|^2} \sum_{\substack{\mathbf{k},\mathbf{l}\in\Lambda^*\backslash\{\mathbf{0}\} \\ \mathbf{k}\neq\mathbf{l}}} \sqrt{\int_{\mathcal{B}+\mathbf{k}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u} \int_{\mathcal{B}+\mathbf{l}} |\hat{f}(\mathbf{u})|^2 d\mathbf{u}}$$

$$c_3 = 1, \quad Q_3 = \frac{m(\mathcal{B})}{\int_{\mathbb{R}^n\backslash\mathcal{B}} |\hat{f}(\mathbf{u})|^2} \left(\sum_{\mathbf{k}\in\Lambda^*\backslash\{\mathbf{0}\}} \|\hat{f}\chi_{\mathcal{B}+\mathbf{k}}\|_\infty\right)^2$$

and are referred to as (B.1), (B.2), (B.3), respectively.

Remarks. The above bounds have the following properties:

- All are only a function of out-of-band signal components.
- Equality conditions for all of the bounds involve the famous Cauchy-Schwartz equality condition in (9). For (B.2), an additional condition exists (lattice tilting) as noted in Lemma 4. The equality condition for (B.3) is evident from (13). Hence, (B.1) is the tightest bound among the three.
- In practice, the lattice packing $(\Lambda^*, \mathcal{B})$ is usually quite close to a lattice tiling. In that case, (B.1) and (B.2) would be close, and since $c_2 + Q_2 \geq 2$, all bounds are larger than or equal to $2\varepsilon^2$. In Sec. 5, we compute Q_1 for specific cases.

4. SIMPLIFIED BOUNDS FOR PRACTICAL CASES IN 1-D

Consider a 1-D function f with an assumed band-region $\mathcal{B} = [-\sigma, \sigma]$ for $\hat{f}(u)$. The (B.1) bound, assuming Nyquist rate sampling (with respect to $\mathcal{B}$), can be written as

$$\|e\|_2^2 \leq \varepsilon^2 + \left(\sum_{|p|=1}^{\infty} \sqrt{\int_{I_p} |\hat{f}(u)|^2 du}\right)^2 \tag{14}$$

where $I_p = [\sigma(2p-1), \sigma(2p+1)]$ and ε^2 is defined in (6). In this section, we first show that (14) can be reduced to a finite-term sum which can be advantageous for computational purposes.

In a practical setting, we expect that $|\hat{f}(u)|$ would drop sharply for out-of-band frequencies. Assume the following "decay" condition

$$\sqrt{\int_{I_{p+M}} |\hat{f}(u)|^2 du} \leq \int_{I_p} |\hat{f}(u)| du \quad \text{for } p \geq 1 \tag{15}$$

for some $M \in \mathbb{N}$ and a similar inequality for $p \leq -1$ with LHS replaced by integral over I_{p-M}. Considering that we assumed a low-pass model for $\mathcal{B}$ in this section, the condition arises naturally since the signal content at higher frequency intervals (with L_2-norm) will eventually be smaller than that of the lower frequency intervals (with L_1-norm). In fact, since $\hat{f} \in L_1(\mathbb{R})$, it can be shown that there always exists such $M \in \mathbb{N}$.

By combining (14) and (15), it can be shown that

$$\|e\|_2^2 \leq \varepsilon^2 + \left(R + \int_{|u|\geq\sigma} |\hat{f}(u)| du\right)^2 \tag{16}$$

where $R = \sum_{|p|=1}^{M} \sqrt{\int_{I_p} |\hat{f}(u)|^2 du}$.

In most practical cases, there is sufficient drop (or high enough decay rate) in the out-of-band energy such that (15) holds for $M = 1$

and $R^2 \leq \varepsilon^2$. Under these conditions, using the inequality $(R + a)^2 \leq 2R^2 + 2a^2$, we can further upper bound (16) as

$$\|e\|_2^2 \leq 3 \int_{|u|\geq\sigma} |\hat{f}(u)|^2 du + 2\Big(\int_{|u|\geq\sigma} |\hat{f}(u)| du\Big)^2 \qquad (17)$$

which only involves 1- and 2-norms of the total out-of-band signal (but is less tight than (14) or (16)). This bound, although not as tight, provides more intuition into the behavior of the aliasing error. For instance, one can readily infer the convergence of $\|e\|_2^2$ to zero as $\sigma \to \infty$. This proves a 2-norm version of the so called "approximate sampling theorem" for nonbandlimited signals [8, 11].

5. BOUNDS FOR TWO PRACTICAL SIGNAL CLASSES

In this section, we consider two large (but not disjoint) classes of signals that are frequently encountered in practice, namely signals with (at least) exponential or polynomial out-of-band decay:

$$|\hat{f}_{\exp}(u)| \leq c\, e^{-\gamma|u|},\ |u| > \sigma \quad |\hat{f}_{\text{poly}}(u)| \leq \frac{c}{|u|^\gamma},\ |u| > \sigma \quad (18)$$

for constants $c, \gamma \in \mathbb{R}^+$. Note that since $\hat{f} \in L_1(\mathbb{R})$, for $\hat{f}_{\text{poly}}$, decay rate is further restricted to $\gamma > 1$. The bound (B.1) in (14) (denoted by $\text{UB}_{\text{B.1}}$) for each of the signal classes above gives

$$\text{UB}_{\text{B.1}}^{\exp} = \varepsilon_{\exp}^2 \left(3 + 4e^{-2\sigma\gamma}\right)$$

$$\text{UB}_{\text{B.1}}^{\text{poly}} = \varepsilon_{\text{poly}}^2 \left(1 + 2\Big(\sum_{p=1}^{\infty} \frac{\sqrt{(2p+1)^{2\gamma-1} - (2p-1)^{2\gamma-1}}}{(4p^2-1)^{\gamma-1/2}}\Big)^2\right)$$

where $\varepsilon_{\exp}^2 = c^2 e^{-2\sigma\gamma}/\gamma$ and $\varepsilon_{\text{poly}}^2 = 2\sigma^{1-2\gamma}c^2/(2\gamma-1)$. Note that $\text{UB}_{\text{B.1}}^{\exp}$ is an approximation of the exact bound assuming $\exp(-4\sigma\gamma) \ll 1$ which is typically satisfied.

Figure 1 shows plots of Q_1 of the (B.1) bound as a function of the decay rate γ (recall that $\|e\|_2^2 \leq (1+Q_1)\varepsilon^2$). For the exponential class σ is taken to be 0.5. For polynomial decay, Q_1 is independent of σ. (Note that ε^2 decreases sharply as σ increases for both classes.) As depicted in Fig. 1, for polynomial decay with $\gamma \geq 2.4$, we have $Q_1 \leq 3$, hence $\|e\|_2^2 \leq 4\varepsilon^2$. Also for $\gamma \geq 2.4$, the exponential class has $Q_1 \leq 2.4$, i.e., $\|e\|_2^2 \leq 3.4\varepsilon^2$. For both classes with $\gamma \geq 4$, $Q_1 \leq 2.1$, i.e., $\|e\|_2^2 \leq 3.1\varepsilon^2$. Based on such observations, we can arrive at a series of useful "folk" results, which provide estimates of Q_1 under certain decay conditions, e.g., the following proposition. The tightness of these folk bounds depends on the choice of c and γ.

Proposition. *If, for a given* $c \in \mathbb{R}^+$, *a signal* $f \in \mathcal{F}$ *with* $\mathcal{B} = [-\sigma, \sigma]$ *belongs to the polynomial signal class in* (18) *with* $\gamma \geq 4$ *or the exponential signal class with* $\gamma\sigma \geq 2$ *then* $\varepsilon^2 \leq \|e\|_2^2 \leq 3.1\,\varepsilon^2$.

6. COMPARISON TO THE KHURGIN-YAKOVLEV BOUND

Khurgin et al. [5] provide the following bound for the 1-D problem for "most interesting cases of behavior" (not specified in [5]),

$$\|e\|_2^2 \leq 3\varepsilon^2 + 2\sqrt{2\varepsilon^2} \sum_{p=1}^{\infty} \sqrt{\int_{(2p+1)\sigma}^{\infty} |\hat{f}(u)|^2 du} \qquad (19)$$

where we assumed $|\hat{f}(-u)|^2 = |\hat{f}(u)|^2$ to shorten the presentation.

Let us consider a signal that has (at least) a polynomial decay for $|u| > \sigma$, i.e., $\hat{f}_{\text{poly}}(u)$ defined in (18). Computing the bound in (19) (denoted by $\text{UB}_{\text{K-Y}}$) for this signal yields

$$\text{UB}_{\text{K-Y}}^{\text{poly}} = \varepsilon_{\text{poly}}^2 \left(3 + 2\sum_{p=1}^{\infty} \frac{1}{(2p+1)^{\gamma-1/2}}\right)$$

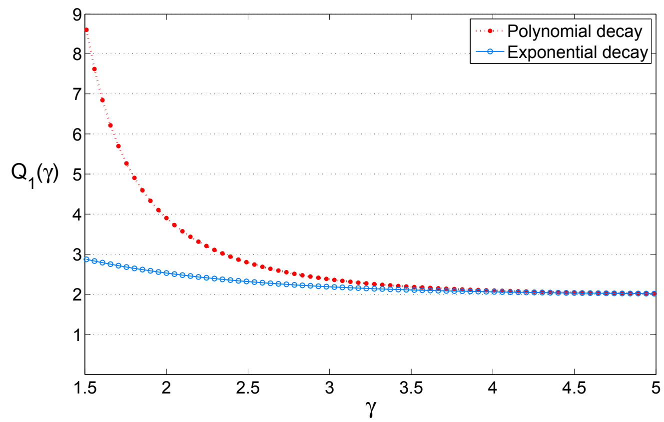

Fig. 1. Plot of Q_1 of our upper bound (B.1) as a function of the decay rate γ for both polynomially and exponentially decaying signals (with $\sigma = 0.5$).

It can be shown that for $1 < \gamma \leq 1.5$, $\text{UB}_{\text{K-Y}}^{\text{poly}} \to \infty$ whereas $\text{UB}_{\text{B.1}}^{\text{poly}}$ converges for any $\gamma > 1$, i.e., for all $f \in F$. This can be more clearly seen by plugging $\gamma = 1.5$ in the bounds computed above. It turns out that $\text{UB}_{\text{K-Y}}^{\text{poly}} \propto \sum_{p=1}^{\infty} p^{-1}$ which fails to converge. In contrast, $\text{UB}_{\text{B.1}}^{\text{poly}} \propto \left(\sum_{p=1}^{\infty} p^{-1.5}\right)^2$ which converges. Thus, we have shown that for a practical class of signals, the Khurgin-Yakovlev bound can exceed ours by an arbitrary factor.

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