NYQUIST PULSE SHAPING CRITERION FOR TWO-DIMENSIONAL SYSTEMS

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ABSTRACT

We present Nyquist pulse shaping conditions for two-dimensional (2-D) digital communication systems to be free from inter-symbol interference, for general regular sampling grids. We provide examples of 2-D pulse functions satisfying these Nyquist conditions. In particular, we show that a family of 2-D pulse functions that we construct as separable pulse functions from one-dimensional Nyquist-1 pulse functions obey the 2-D Nyquist criterion and moreover include the one with minimum support area in the frequency domain.

***Index Terms*—** 2-D digital communication systems, 2-D inter-symbol interference, Nyquist pulse shaping criterion, 2-D Nyquist-1 pulse functions

1. INTRODUCTION

The Nyquist pulse shaping criterion in digital communications provides necessary and sufficient conditions on the pulse function such that no inter-symbol interference (ISI) results. In particular, for a one-dimensional (1-D) digital communications system, the Nyquist criterion [1] states that

Theorem 1 *The sufficient and necessary condition for the pulse function $x(t)$ to satisfy the zero ISI condition*

$$x(nT) = \delta(n) \tag{1}$$

is that its Fourier transform $X(f)$ satisfies

$$\sum_{m=-\infty}^{\infty} X(f + m/T) = T, \tag{2}$$

where $\delta(\cdot)$ is the Kronecker delta function and T is the sampling period.

The pulse functions that satisfy Theorem 1 are known as *1-D Nyquist-1* pulse functions. In practice, real-valued pulse functions $x(t)$ are of interest, for which $X^*(f) = X(-f)$. A particular example of a real-valued 1-D Nyquist-1 pulse function is the raised cosine function, given by

$$x_{\beta,T}(t) = \text{sinc}(\pi t/T)\frac{\cos(\pi\beta t/T)}{1 - 4\beta^2 t^2/T^2}, \tag{3}$$

where β, with $0 \le \beta \le 1$, is the roll-off factor. The Fourier transform of $x_{\beta,T}(t)$ is

$$X_{\beta,T}(f) = \begin{cases} T, & |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2} + \frac{T}{2}\cos\left(\frac{\pi T}{\beta}(|f| - \frac{1-\beta}{2T})\right), & \frac{1-\beta}{2T} < |f| \le \frac{1+\beta}{2T} \\ 0, & \text{elsewhere.} \end{cases} \tag{4}$$

Note that $\beta = 0$ results in the minimum bandwidth pulse function, which is $x_T(t) = \text{sinc}(\pi t/T)$, with Fourier transform $X_T(f) = T$, for $|f| \le \frac{1}{2T}$, and zero, elsewhere.

In this paper, we consider 2-D systems, and present counterpart results to Theorem 1. Interest in characterization of ISI and associated signal processing techniques has resulted from recent work on two-dimensional storage systems [2–5]. However, to our knowledge, a Nyquist pulse shaping criterion for such 2-D systems has not been addressed in literature. In this paper, we derive 2-D Nyquist pulse shaping conditions for regular 2-D sampling grid shapes and present families of pulse functions that satisfy this criterion. One construction is provided based on 1-D Nyquist-1 pulse functions, and we show that such 2-D Nyquist pulse functions have minimum support area in the frequency domain. We specialize our results to two commonly used sampling grids - rectangular and hexagonal. Hexagonal sampling, for instance, has attracted attention in applications in storage systems given the storage density improvement under this grid shape.

The rest of the paper is organized as follows. We develop the Nyquist ISI-free criterion for 2-D digital communication systems in Section 2. A general approach to construct a class of 2-D Nyquist-1 pulse functions is presented in Section 3 for general sampling grids. Section 4 concludes this paper. In dealing with 2-D signal analysis, we follow the notations from [6].

2. 2-D NYQUIST PULSE SHAPING CRITERION

In a 2-D digital communication system, transmitted symbols are arranged on an infinite plane, with their locations prescribed by a regular grid. At the receiver, each symbol results in a pulse centered at the symbol location. There is possible overlap among the pulses. The receiver in turn samples at the symbol locations.

A 2-D discrete signal is represented by samples $\{\tilde{y}(n_1, n_2)\}$, where $\tilde{y}(n_1, n_2)$ represents the sample value at the location described by the ordered integer pair (n_1, n_2), where $-\infty < n_1, n_2 < \infty$. These samples are obtained by regular sampling of a continuous 2-D signal $y(\mathbf{t})$ defined in the continuous (t_1, t_2) plane, with

$$\mathbf{t} = \mathbf{V}\mathbf{n},$$

where $\mathbf{t} = [t_1, t_2]^T$, $\mathbf{n} = [n_1, n_2]^T$, and $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2]$ is the sampling matrix, with $\mathbf{v}_1$ and $\mathbf{v}_2$ being two 2×1 linearly independent vectors. Among the numerous choices of the sampling matrix $\mathbf{V}$, two special cases corresponding to rectangular and hexagonal sampling are of common interest.

The distortion resulting from overlapping among different pulses at the sampling location is termed *2-D ISI*. The *2-D zero ISI criterion* corresponds to the condition that the received sampled signal contains only the component from one transmitted pulse, or equivalently, sampling a single pulse at a sampling location results in a 2-D Kronecker delta function. The 2-D Nyquist criterion for zero ISI thus is given by the condition:

$$\tilde{y}(\mathbf{n}) = \delta(\mathbf{n}), \tag{5}$$

where $\delta(\mathbf{n})$ is a 2-D Kronecker delta function.

Now, the respective discrete and continuous 2-D Fourier transforms, $\tilde{Y}()$ and $Y()$, of signals $\tilde{y}(\mathbf{n})$ and $y(\mathbf{t})$ are related by [6]

$$\tilde{Y}(\mathbf{V}^T \boldsymbol{f}) = \frac{1}{|\mathbf{V}|} \sum_{\mathbf{n}} Y(\boldsymbol{f} + \mathbf{V}^{-T}\mathbf{n}), \tag{6}$$

where $|\mathbf{V}|$, $\mathbf{V}^T$ and $\mathbf{V}^{-T}$ denote the determinant, transpose and inverse-transpose of matrix $\mathbf{V}$, respectively. Note that $\tilde{Y}(\mathbf{V}^T\boldsymbol{f})$ can be considered to be a periodic extension of $Y(\boldsymbol{f})$, with the periodicity specified by the matrix $\mathbf{V}^{-T}$. Taking the Fourier transforms of both sides of (5), we get that (5) is satisfied if and only if $\tilde{Y}(\boldsymbol{f}) = 1$ and equivalently $\tilde{Y}(\mathbf{V}^T\boldsymbol{f}) = 1$. Then, using (6), we get that

$$\tilde{Y}(\mathbf{V}^T \boldsymbol{f}) = \frac{1}{|\mathbf{V}|} \sum_{\mathbf{n}} Y(\boldsymbol{f} + \mathbf{V}^{-T}\mathbf{n}) = 1. \tag{7}$$

We thus have the following 2-D Nyquist pulse shaping criterion for zero 2-D ISI.

Theorem 2 *The sufficient and necessary condition for the pulse function $y(\mathbf{t})$ to achieve zero 2-D ISI on a regular grid defined by the matrix $\mathbf{V}$ is*

$$\sum_{\mathbf{n}} Y(\boldsymbol{f} + \mathbf{V}^{-T}\mathbf{n}) = |\mathbf{V}|. \tag{8}$$

This theorem shows that if the sum of all shifted versions of $Y(\boldsymbol{f})$ in a direction given by $\mathbf{V}^{-T}$ gives a flat spectrum, there is no ISI among the samples of the received signal. The pulse functions satisfying the condition in Theorem 2 are named *2-D Nyquist-1 pulse functions*.

2.1. 2-D Nyquist-1 Pulse Functions with Minimum Support Area

For 1-D digital communication systems, Nyquist-1 pulse functions with minimum bandwidth are of particular interest. As mentioned earlier in the introduction, $x_T(t) = \text{sinc}(\pi t/T)$ is the unique real-valued 1-D Nyquist-1 pulse function with minimum bandwidth equal to $1/T$. We consider the notion of minimum support area corresponding to a 2-D Nyquist-1 pulse function in order to extend the idea of minimum bandwidth to the 2-D case. Let $Y(\boldsymbol{f})$ be the Fourier transform of a pulse function $y(\mathbf{t})$. The support of $Y(\boldsymbol{f})$ is then defined as the set of (f_1, f_2) such that $Y(\boldsymbol{f})$ is non-zero; the support area is the area of this support. From Theorem 2, 2-D Nyquist-1 pulse functions with minimum support area are those for which there is no overlap among the regularly shifted versions, specified by $\mathbf{V}^{-T}$, of $Y(\boldsymbol{f})$. The minimum regular shift is determined by $\mathbf{V}^{-T}$, and hence the minimum support area is $|\mathbf{V}^{-T}| = 1/|\mathbf{V}|$. Unlike the 1-D case, the real-valued Nyquist-1 pulse functions with minimum support area for general grid shapes in the 2-D case may not be unique.

3. 2-D NYQUIST-1 PULSE FUNCTIONS

3.1. Rectangular Sampling Grid

The rectangular sampling grid is obtained by periodically sampling the (t_1, t_2) plane such that sample points are spaced T_1 and T_2 apart along the t_1 and t_2 axes respectively. For the rectangular grid, the sampling matrix is given by

$$\mathbf{V}_{\text{rec}} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}. \tag{9}$$

From (8), the Nyquist zero-ISI criterion can be written as

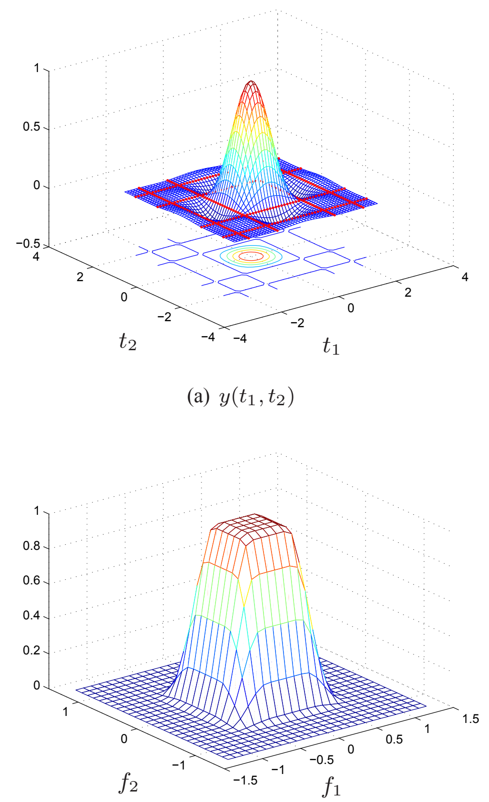
$$\sum_{\mathbf{n}} Y(\boldsymbol{f} + [\frac{n_1}{T_1}, \frac{n_2}{T_2}]^T) = T_1 T_2. \tag{10}$$

We now provide constructions of 2-D Nyquist-1 pulse functions satisfying the zero ISI criterion, using 1-D Nyquist-1 pulse functions.

Separable 2-D Nyquist-1 pulse functions: Consider the following separable pulse function

$$y(t_1, t_2) = y_1(t_1) y_2(t_2). \tag{11}$$

In this case, condition (10) is satisfied if $\sum_{n_1} Y_1(f + \frac{n_1}{T_1}) = T_1$ and $\sum_{n_2} Y_2(f + \frac{n_2}{T_2}) = T_2$, where $Y_1(f)$ and $Y_2(f)$ are the Fourier transforms of $y_1(t_1)$ and $y_2(t_2)$, respectively. Now



(a) $y(t_1, t_2)$

(b) $Y(f_1, f_2)$

Fig. 1. Example of Nyquist pulse functions for rectangular grid, for $T_1 = T_2 = 1$ and $\beta_1 = \beta_2 = 0.5$.

note that, if $y_i(t), i = 1, 2$, are 1-D Nyquist-1 pulse functions with zeros at $n_i T_i, n_i \neq 0$, then $y(t_1, t_2) = y_1(t_1)y_2(t_2)$ is a 2-D Nyquist pulse function since it satisfies (5). In fact, $y(t_1, t_2)$ is equal to zero along the lines $t_1 = n_1 T_1$ and $t_2 = n_2 T_2$, for $n_1 \neq 0$ and $n_2 \neq 0$. Also, the corresponding Fourier transform of $y(t_1, t_2)$ can be written as $Y(f_1, f_2) = Y_1(f_1)Y_2(f_2)$, where $Y_1(f_1)$ and $Y_2(f_2)$ are the Fourier transforms of $y_1(t_1)$ and $y_2(t_2)$, respectively. It is straightforward to see that (8) is satisfied for this $Y(f_1, f_2)$.

A particular choice that arises from 1-D raised cosine functions using (3) is $y(t_1, t_2) = x_{\beta_1, T_1}(t_1)x_{\beta_2, T_2}(t_2)$, where β_1 and β_2 represent roll-off factors. The plots of $y(t_1, t_2)$ and $Y(f_1, f_2)$ for $\beta_1 = \beta_2 = 0.5$ and $T_1 = T_2 = 1$ are shown in Fig. 1.

Note that it is also possible to construct 2-D Nyquist-1 functions that are non-separable. For instance, a family of 2-D non-separable functions can be written as

$$y(t_1, t_2) = t_1^2 y_1^2(t_1) + t_2^2 y_2^2(t_2) + y_1^2(t_1)y_2^2(t_2). \qquad (12)$$

However, we restrict our attention in this paper to separable pulse functions since they include the ones with minimum support area, as shown as follows.

3.1.1. 2-D Nyquist-1 Pulse Functions with Minimum Support Area for the Rectangular Grid

For the rectangular sampling grid, the pulse function $y(t_1, t_2) = x_{T_1}(t_1)x_{T_2}(t_2) = \mathrm{sinc}(\pi t_1/T_1)\mathrm{sinc}(\pi t_2/T_2)$ with Fourier transform $Y(f_1, f_2) = X_{T_1}(f_1)X_{T_2}(f_2)$ has a minimum support area of $1/(T_1 T_2) = 1/|\mathbf{V}_{\text{rec}}|$. More generally, it shows that a separable class of 2-D Nyquist-1 pulse functions exists which includes the one with minimum support area. Note that, unlike the 1-D case, this 2-D function with minimum support area is not unique.

The construction of 2-D Nyquist pulse functions for general regular sampling grids can be done based on 2-D Nyquist pulse functions on a rectangular grid. We now provide a construction of a family of such pulse functions for a general grid shape by using pulse functions that satisfy the zero ISI condition under rectangular sampling.

3.2. General Sampling Grids

Let $y_{\text{rec}}(\mathbf{t})$ be a Nyquist-1 pulse function that satisfies the zero ISI criterion under a rectangular grid. Therefore, we must have $y_{\text{rec}}(\mathbf{V}_{\text{rec}}\mathbf{n}) = \delta(\mathbf{n})$. For a general grid shape characterized by $\mathbf{V}_{\text{g}}$, define a pulse function $y_{\text{g}}(\mathbf{t})$ as follows

$$y_{\text{g}}(\mathbf{t}) = y_{\text{rec}}(\mathbf{V}_{\text{rec}}\mathbf{V}_{\text{g}}^{-1}\mathbf{t}). \qquad (13)$$

Now, we have

$$y_{\text{g}}(\mathbf{V}_{\text{g}}\mathbf{n}) = y_{\text{rec}}(\mathbf{V}_{\text{rec}}\mathbf{n}) = \delta(\mathbf{n}).$$

Thus $y_{\text{g}}(\mathbf{t})$ is a 2-D Nyquist-1 pulse function for the grid specified by $\mathbf{V}_{\text{g}}$.

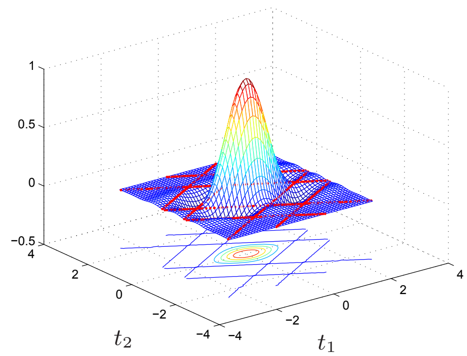
Moreover, if $y_{\text{rec}}(\mathbf{t})$ is a separable pulse function on the rectangular sampling grid, then $y_{\text{g}}(\mathbf{t})$ is also separable in the directions of the linearly independent vectors specified by $\mathbf{V}_{\text{g}}$. To see this, suppose $y_{\text{rec}}(\mathbf{t})$ is separable. Then $y_{\text{rec}}(\mathbf{t}) = y_1(t_1)y_2(t_2)$. Now, denote $\mathbf{V}_{\text{rec}}\mathbf{V}_{\text{g}}^{-1} = [\mathbf{u}_1, \mathbf{u}_2]^T$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are 2×1 linearly independent vectors. Then,

$$\begin{aligned} y_{\text{g}}(\mathbf{t}) &= y_{\text{rec}}(\mathbf{V}_{\text{rec}}\mathbf{V}_{\text{g}}^{-1}\mathbf{t}) \\ &= y_1(\mathbf{u}_1^T\mathbf{t})y_2(\mathbf{u}_2^T\mathbf{t}), \end{aligned} \qquad (14)$$

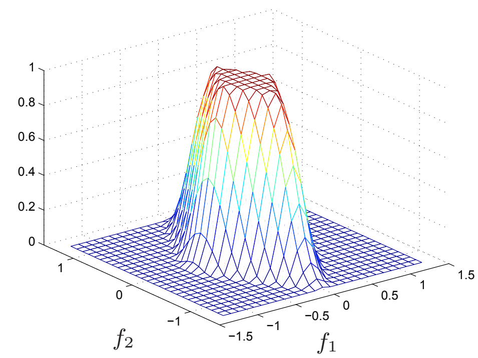
where the last equation holds since $y_{\text{rec}}(\mathbf{t})$ is separable.

3.2.1. 2-D Nyquist-1 Pulse Functions with Minimum Support Area for General Grids

We know that the separable function, $y_{\text{rec}}(\mathbf{t}) = x_{T_1}(t_1)x_{T_2}(t_2)$, is the one with minimum support area, i.e. $1/|\mathbf{V}_{\text{rec}}|$, for the rectangular grid. We now show that $y_{\text{g}}(\mathbf{t}) = y_{\text{rec}}(\mathbf{V}_{\text{rec}}\mathbf{V}_{\text{g}}^{-1}\mathbf{t})$ also has minimum support area, i.e. $1/|\mathbf{V}_{\text{g}}|$, for the regular grid characterized by $\mathbf{V}_{\text{g}}$. The Fourier transforms $Y_g(\boldsymbol{f})$ and $Y_{\text{rec}}(\boldsymbol{f})$ of $y_{\text{g}}(\mathbf{t})$ and $y_{\text{rec}}(\mathbf{t})$ respectively are related as $Y_g(\boldsymbol{f}) = \frac{1}{|\mathbf{V}_{\text{rec}}\mathbf{V}_{\text{g}}^{-1}|}Y_{\text{rec}}((\mathbf{V}_{\text{rec}}\mathbf{V}_{\text{g}}^{-1})^{-T}\boldsymbol{f})$.



(a) $y(t_1, t_2)$



(b) $Y(f_1, f_2)$

Fig. 2. Example of Nyquist-1 pulse functions for hexagonal grid, for $T_1 = T_2 = T = 1$ and $\beta_1 = \beta_2 = 0.5$. The zeros of $y(t_1, t_2)$ are along the directions defined by $\mathbf{V}_{\text{hex}}$.

Thus the support area of $Y_g(\boldsymbol{f})$ is

$$\begin{aligned} &= \frac{1}{|(\mathbf{V}_{\text{rec}}\mathbf{V}_{\text{g}}^{-1})^{-T}|}.\text{minimum support area of } Y_{\text{rec}}(\boldsymbol{f}) \\ &= \frac{|\mathbf{V}_{\text{rec}}|}{|\mathbf{V}_{\text{g}}|}.\frac{1}{|\mathbf{V}_{\text{rec}}|} = \frac{1}{|\mathbf{V}_{\text{g}}|}. \end{aligned} \tag{15}$$

3.2.2. *Hexagonal Sampling Grid*

As an example of a general grid, we consider the hexagonal grid, which is characterized by the matrix $\mathbf{V}_{\text{hex}}$,

$$\mathbf{V}_{\text{hex}} = \begin{bmatrix} \sqrt{3}T/2 & \sqrt{3}T/2 \\ T/2 & -T/2 \end{bmatrix}, \tag{16}$$

where T is the distance between neighboring node on the hexagonal grid.

We follow the approach introduced in Section 3.2 to construct a 2-D Nyquist-1 pulse function for a hexagonal grid. Using (14), a 2-D Nyquist-1 pulse function for a hexagonal grid is given by

$$y(t_1, t_2) = y_1(\frac{T_1}{\sqrt{3}T}t_1 + \frac{T_1}{T}t_2)y_2(\frac{T_2}{\sqrt{3}T}t_1 - \frac{T_2}{T}t_2). \tag{17}$$

A particular case with $y_1(t_1) = x_{\beta_1,T_1}(t_1)$, and $y_2(t_2) = x_{\beta_2,T_2}(t_2)$, with $\beta_1 = \beta_2 = 0.5$ and $T_1 = T_2 = T = 1$ is shown in Fig. 2. Note that $\beta_1 = \beta_2 = 0$ yields a real-valued 2-D Nyquist pulse function with minimum support area.

4. CONCLUDING REMARKS

We presented a Nyquist pulse shaping criterion to achieve zero 2-D ISI in 2-D digital communication systems. We provided families of 2-D Nyquist-1 pulse functions for the rectangular sampling grid and extended them to general sampling grids. In particular, a family of separable 2-D Nyquist-1 pulse functions was constructed from 1-D Nyquist-1 pulse functions, and was shown to include the one with minimum support area. Thus, it is sufficient to restrict attention to separable pulse functions to achieve zero 2-D ISI, while having minimum support area. An interesting next step would be to develop pulse functions that satisfy the 2-D Nyquist criterion and have other desirable properties specific to practical applications such as storage systems.

5. REFERENCES

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