ESTIMATION OF THE FREQUENCY OF A COMPLEX EXPONENTIAL

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ABSTRACT

The estimation of the frequency of a complex exponential is relevant to many fields and has been the subject of a significant amount of research. In this paper, we present a novel complex exponential frequency estimation algorithm that is based on the iterative interpolation strategy of Aboutanios and Mulgrew. The A&M algorithm uses two Fourier coefficients and has been shown to reach, in two iterations, a variance that is 0.063dB above the Cramer-Rao Bound. It, however, requires the calculation of two additional DFT coefficients at each iteration. The new algorithm is computationally simpler as it exploits the standard DFT coefficients at the first iteration. Theoretical analysis and simulation results are presented that demonstrate that the new algorithm maintains the same performance as the A&M estimator.

***Index Terms*—** Frequency estimation, Iterative estimation, Interpolation, Complex exponential.

1. INTRODUCTION

The problem of estimating the frequency of a sinusoidal signal in noise is of prime importance in many applications such as radar, sonar and biomedical instrumentation to name a few. Consequently, this problem has received much attention in the literature, [1] and [2]. Of the characteristics that are desired for an estimation algorithm, small constant variance and low computational complexity are of prime importance. These two are usually in direct competition with one another. Many approaches have been proposed for the frequency estimation problem. Time-domain methods are computationally simple, but usually are efficient only at high signal to noise ratios (SNR) and suffer from a high SNR threshold below which the algorithm becomes useless as its variance increases rapidly. FFT-based algorithms such as those in [3, 4, 5, 6] enjoy a relatively low computational complexity, due to their reliance on the FFT, as well as good performance at low SNRs.

We consider the following signal model,

$$x[k] = s[k] + w[k],\ k = 0 \ldots N-1, \tag{1}$$

where the signal of interest $s[n]$ is an exponential of the form

$$s[k] = Ae^{j2\pi kf+\phi}. \tag{2}$$

Here A is the signal amplitude, θ the initial phase and f the signal frequency normalized with respect to the sampling frequency. Thus $f \in [-0.5, 0.5]$. The noise terms, $w[k]$, are assumed to be zero mean, complex additive white Gaussian noise with variance σ^2. The SNR is given by $\rho = \frac{A^2}{\sigma^2}$. The problem is then to estimate f from a block of N samples, whith the other quantities treated as nuisance parameters.

It is well known that the maximum likelihood estimator of the frequency is the maximiser of the periodogram, [7],

$$\hat{f}_{ML} = \arg\max_{\lambda}\{Y(\lambda)\} \tag{3}$$

where

$$Y(\lambda) = \left|\sum_{k=0}^{N-1} x[k]e^{-j2\pi k\lambda}\right|^2.$$

The Cramer-Rao Bound (CRB) of the frequency estimates was derived in [7] and is given by

$$\sigma_f^2 = \frac{6}{(2\pi)^2\rho N(N^2-1)}. \tag{4}$$

The heavy computational cost of the maximisation step of equation (3) has led to research into simple and efficient alternatives. These estimators usually employ a two-stage strategy consisting of a coarse followed by a fine search, [7] and [8]. The coarse search returns the bin number with the largest magnitude from the L-point FFT, [5], where the FFT length L is not necessarily equal to N as zero-padding may be employed.

The resolution of the coarse search is limited to $1/L$. Performance improvements can be obtained by increasing L at the expense of a heavier computational load. An alternative approach is to use a fine estimation stage. Many such strategies have been proposed in the literature. For instance, [9] and [5] propose algorithms that rely on a sequential binary search for the true maximum. Another method that has received significant attention is interpolation-based estimation, where a number of bins around the highest bin are used to interpolate for the true maximum, [7], [3], and [6]. Although most of these algorithms get quite close to the CRB, some do not have a constant variance as a function of the true signal frequency. This is certainly the case for Quinn's algorithms, and their performance is worst when f coincides with the bin centre, [6]. In contrast, the interpolators presented in [6], which we denote as the A&M estimators, are suitable for iterative implementation which then endows them with a constant performance that is independent of the true frequency and only 1.0147 times the asymptotic CRB (ACRB). This is, however, achieved at the expense of increased computational cost as a result of the iterative procedure. In this paper, we present a new method that reduces the computational complexity of the A&M algorithms while maintaining the same performance.

The paper is organized as follows: In the following section we review the Quinn and A&M algorithms. In section 3 we present the new algorithm and verify its performance in 4. Finally, some conclusions are given in 5.

2. INTERPOLATION ALGORITHMS

Let the index returned by the maximum bin search (MBS) be denoted by $\hat{m}_N$, where the subscript N denotes the dependence of the estimate $\hat{m}$ on N. It was shown in [6] that, as $N \rightarrow \infty$,

$$\delta_N = \hat{m}_N - Nf \in [-0.5, 0.5] \quad \text{a.s.} \tag{5}$$

Thus, the MBS returns the true maximum index m_N almost surely (a.s.). In the following we drop the subscript N for notational simplicity. Now the task becomes that of estimating δ. Consider the DFT coefficients in the vicinity of the true frequency and let X_p be the coefficient corresponding to the index $\hat{m} + p$. From [6] we have that

$$X_p = b\frac{\delta}{\delta - p} + W_p + O(N^{-2}) \tag{6}$$

where W_p is the DFT coefficient of the noise and

$$b = -Ne^{j\theta}\frac{1 + e^{j2\pi\delta}}{j2\pi\delta}$$

A general iterative estimation algorithm is shown in table 1. The interpolation function $h(\delta)$ is implemented on a set of DFT coefficients $\mathcal{S} = \{X_p\}$. Specific interpolation algorithms differ in the their choice of $\mathcal{S}$, $h(\delta)$, and number of iterations Q. An iterative interpolation function satisfying the fixed point theorem will converge to its fixed point, δ_0, where δ_0 satisfies $h(\delta_0) = \delta_0$. The variance of the iterative estimates also converges to its value at δ_0, [6]. In the following subsections we review Quinn's algorithm and the A&M estimator.

2.1. Quinn's Estimator

Quinn, in [3], proposed estimating δ by interpolating on the maximum and either of the two coefficients adjacent to it. The

Table 1. Iterative Frequency Estimation by Interpolation on Fourier Coefficients Algorithm

Let	$\mathbf{X} = FFT(\mathbf{x})$ and $Y(n) = \lvert X(n)\rvert^2$
Let	$\hat{m} = \arg\max_n\{Y(n)\}$
Set	$\hat{\delta}_0 = 0$
Loop:	for each i from 1 to Q do
	$X_p = \sum_{k=0}^{N-1} x(k)e^{-j2\pi k\frac{m+\hat{\delta}_{i-1}+p}{N}}$
	$\hat{\delta}_i = \hat{\delta}_{i-1} + h(\hat{\delta}_{i-1})$
Finally	$\hat{f} = \frac{\hat{m}+\hat{\delta}_Q}{N}$

estimation function is given by

$$h_p(\delta) = -p\frac{\alpha_p}{1 - \alpha_p} \tag{7}$$

where

$$\alpha_p = \mathrm{Re}\left\{\frac{X_p}{X_0}\right\}, \text{ for } p = \pm 1. \tag{8}$$

Thus Quinn's algorithm obtains two estimates of δ and chooses the one with the better SNR. The ratio of the asymptotic variance to the ACRB of the resulting estimator has a maximum of $\pi^2/3 = 3.2899$ at $\delta = 0$ and a minimum of $\pi^4/96 = 1.0147$ at $\delta = \pm 1/2$. Since the function $h(\delta)$ has a fixed point at $\delta_0 = 0$, the point with the worst variance, an iterative application of the algorithm results in a degradation in estimation performance, [6].

2.2. The A&M Estimators

The drawback of Quinn's estimator is that its worst variance is at the fixed point of its estimation function. Two alternative interpolators were proposed in [6]. These employ a different set of DFT coefficients (precisely those at the edges of the maximum bin) and were shown to have their lowest variance at the fixed point $\delta_0 = 0$. Thus, they are amenable to iterative implementation. The estimation functions are given by

$$\hat{\delta} = h(\delta) = \frac{1}{2}\mathrm{Re}\left\{\frac{X_{0.5}+X_{-0.5}}{X_{0.5}-X_{-0.5}}\right\}, \quad \text{for algorithm 1}$$

and

$$\hat{\delta} = h(\delta) = \frac{1}{2}\frac{|X_{0.5}|-|X_{-0.5}|}{|X_{0.5}|+|X_{-0.5}|}, \quad \text{for algorithm 2}$$

Both of these estimators have the same ratio of the asymptotic variance to the ACRB given below:

$$R(\delta) = \frac{\pi^4}{6}\frac{\left(\delta^2 - 0.25\right)^2\left(4\delta^2 + 1\right)}{\cos^2(\pi\delta)} \tag{9}$$

The ratio $R(\delta)$ has its minimum of $\pi^4/96 = 1.0147$ for $\delta = 0$. Since $\delta_0 = 0$ is also the fixed point of the function, the iterative implementation of the estimators results in an improvement in the estimation variance. It was shown in [6] that $Q = 2$ is sufficient to achieve a variance of the same order as the CRB for all $\delta \in [-0.5, 0.5]$.

3. MODIFIED INTERPOLATION ALGORITHM

The A&M estimators achieve an improvement in performance over Quinn's interpolator at the expense of an increase in the computational cost. At each iteration, two additional DFT coefficients at the edges of the maximum bin must be calculated. In time-critical DSP applications, a reduction in the computational cost of the estimators becomes very important. Thus, we present here a modified algorithm that maintains the same performance as the A&M estimators but with a reduced computational cost.

The new estimator is shown in table 2. The algorithm is initialised using the MBS stage. Following this, the first iteration is extracted from the loop and modified to avoid the calculation of the two additional DFT coefficients required by the A&M algorithms. A new estimation function that uses the same set of coefficients as Quinn's algorithm is introduced. Like Quinn's estimator, two estimates for δ are obtained and a decision rule is needed to select the one with the better SNR. One possible rule is to compare the two coefficients either side of the maximum, but this has been shown in [4] to perform poorly as $|\delta| \to 0$. Thus, we consider instead

$$r_l = \mathrm{Re}\left\{X_l X_0^*\right\} \approx \mathrm{Re}\left\{b\frac{\delta}{\delta - l}b^*\right\} = |b|^2\frac{\delta}{\delta - l},\ l = \pm 1$$

Now we see that $r_1 \leq 0 \leq r_{-1}$ for $0 \leq \delta \leq 0.5$, while the reverse is true for $-0.5 \leq \delta \leq 0$. This justifies the decision rule used in table 2. Now looking at $h(\delta)$, consider the noiseless case and substitute the expressions for $X_{\hat{m}+p}$,

$$\begin{aligned} h(\delta) &= p\mathrm{Re}\left\{\frac{X_{\hat{m}+p}}{X_{\hat{m}+p} - X_{\hat{m}}}\right\} \\ &\approx p\mathrm{Re}\left\{\frac{b\frac{\delta}{\delta-p}}{b\frac{\delta}{\delta-p} - b}\right\} = \delta. \end{aligned}$$

Hence, $h(\delta)$ can be used as an estimator for δ. As in [6], we take the real part to ensure the result is real. This has the side benefit that the variance of the estimates is improved by 3dB.

3.1. Analysis

In this section we derive the asymptotic performance of the new estimator. The detailed analysis will not be given due to the lack of space. The interpolation function proposed here has a similar form to those in [3] and [6]. Thus, similar asymptotic properties would be expected. In fact its asymptotic performance holds under the relaxed noise conditions given in [6]. Now including the noise terms W_p in the interpolation function, we get after some manipulations,

$$\begin{aligned} \hat{\delta} &= p\mathrm{Re}\left\{\frac{X_{\hat{m}+p}}{X_{\hat{m}+p} - X_{\hat{m}}}\right\} \\ &= \mathrm{Re}\left\{\frac{\delta + (\delta - p)\frac{W_p}{b}}{1 + \frac{\delta-p}{p}\frac{W_p - W_0}{b}}\right\} \end{aligned}$$

Table 2. Modified Iterative Frequency Estimation by Interpolation on Fourier Coefficients Algorithm

Let	$\mathbf{X} = FFT(\mathbf{x})$ and $Y(n) = \lvert X(n)\rvert^2$
Let	$\hat{m} = \arg\max_n\{Y(n)\}$
Let	$r_l = \mathrm{Re}\left\{X_l X_0^*\right\},\ l = \pm 1$
	if $r_1 < r_{-1}$
	then $p = 1, \alpha = -0.5$
	else $p = -1, \alpha = 0.5$
Set	$\hat{\delta}_1 = p\mathrm{Re}\left\{\frac{X_{\hat{m}+p}}{X_{\hat{m}+p} - X_{\hat{m}}}\right\}$
Loop:	for each i from 2 to Q do
	$X_l = \sum_{k=0}^{N-1} x(k)e^{-j2\pi k\frac{\hat{m}+\hat{\delta}_{i-1}+l+\alpha}{N}},\ l = 0, p$
	$\hat{\delta}_i = \hat{\delta}_{i-1} + p\mathrm{Re}\left\{\frac{X_{\hat{m}+p}}{X_{\hat{m}+p} - X_{\hat{m}}}\right\}$
Finally	$\hat{f} = \frac{\hat{m}+\hat{\delta}_Q+\alpha}{N}$

The terms W_p are $O\left(\sqrt{N\ln N}\right)$ whereas b is $O(N)$, [6]. Thus, $W_p/b = O\left(N^{-\frac{1}{2}}\sqrt{\ln N}\right)$. Using the fact that, for large N, $\frac{1}{1+x} \approx 1 - x + o(x^2)$ yields

$$\begin{aligned} \hat{\delta} &= \mathrm{Re}\left\{\left[\delta + (\delta - p)\frac{W_p}{b}\right]\left[1 - \frac{\delta - p}{p}\frac{W_p - W_0}{b}\right]\right\} \\ &= \delta + (\delta - p)\mathrm{Re}\left\{\frac{W_p}{b}\right\} - \delta\frac{\delta - p}{p}\mathrm{Re}\left\{\frac{W_p - W_0}{b}\right\} \\ &\quad + O(N^{-1}\ln N). \end{aligned}$$

Now taking the variance of the error $(\hat{\delta} - \delta)$ we get

$$\begin{aligned} \mathrm{var}\left[\hat{\delta} - \delta\right] &= \frac{(\delta - p)^2}{|b|^2}\mathrm{var}\left[\mathrm{Re}\left\{W_p\right\}\right] \\ &\quad + \frac{\delta^2(\delta - p)^2}{|b|^2}\mathrm{var}\left[\mathrm{Re}\left\{W_p - W_0\right\}\right] \end{aligned}$$

A convenient metric to use is the ratio of the estimator variance to the ACRB. Noting that $\mathrm{var}\left[\mathrm{Re}\left\{W_p\right\}\right] = \frac{N\sigma^2}{2}$, and substituting the expression for b, This ratio becomes

$$R(\delta) = \frac{\pi^4\delta^2}{3\sin^2(\pi\delta)}(1 - |\delta|)^2\left\{\delta^2 + (1 - |\delta|)^2\right\}$$

where in the last expression, the relationship between the sign of δ and the value of p was used.

The above expression is in fact identical to that of Quinn's algorithm. This is not surprising as the two interpolation expressions are quite similar. Now, recall that Quinn's algorithm cannot be implemented iteratively due to the function having maximum variance at its fixed point. Therefore, the proposed algorithm shifts the fixed point of the procedure to the point where the minimum variance occurs by incorporating the parameter α into the DFT coefficients, see table 2. In fact it is straightforward to show, as was done above, that this makes the asymptotic properties of the second iteration identical to those of the A&M algorithm and therefore the new iterative procedure converges in the same way as the A&M estimator.

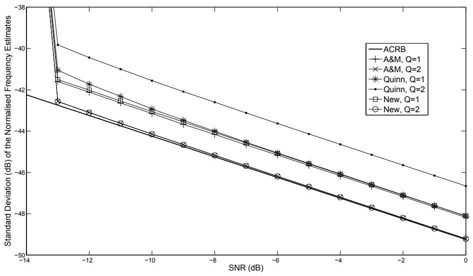

Fig. 1. Plot of the standard deviation of the frequency error as a function of SNR. 10000 Monte Carlo runs were used.

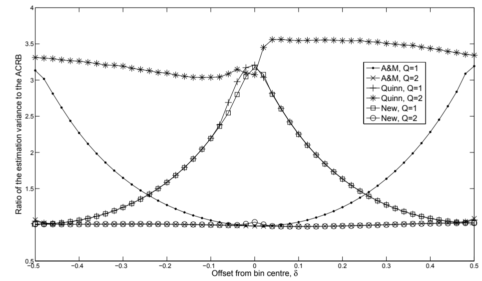

Fig. 2. Plot of the ratio of the estimation variance to the ACRB. 5000 Monte Carlo runs were used.

4. SIMULATION RESULTS

The new algorithm was simulated alongside the A&M and Quinn estimators and the results for one and two iterations are shown in figs. 1 and 2. $N = 1024$ samples were used. Fig. 1 shows the performance as a function of the SNR. The normalised frequency was selected randomly following a uniform distribution over the range $[-0.5, 0.5]$. Firstly, note that the proposed algorithm maintains an identical performance as the A&M estimator after two iterations. Fruthermore, although all three algorithms have similar performances at the first iteration (with minor differences at low SNR and the A&M performing best), the proposed and A&M algorithms improve at the second iteration whereas Quinn's clearly deteriorates. In fact we see a difference of about 6dB between the curves at $Q = 2$. Fig.2 sheds some light on the algorithms' behaviour as it shows the peformance against the offset from the bin centre. We see that whereas all of the interpolators start with a frequency dependent performance, the variance is lowest at the fixed point for the proposed and A&M algorithms, whereas it is worst for Quinn's. The curves for the second iteration confirm the expected performance.

Despite having an identical peformance to the A&M estimator, the proposed algorithm saves on the calculation of two DFT coefficients. This amounts to a saving of $2N$ complex multiplications and $2N - 2$ complex additions per estimate which, for $N = 1024$, is about 4096 operations. This is a significant saving in real-time and time critical applications.

5. CONCLUSIONS

In this paper we presented a new iterative frequency estimation algorithm that maintains the same performance as the A&M estimator while saving on the required computational load. This is achieved by avoiding the calculation of two new DFT coefficients in the first iteration and redesigning the algorithm to take advantage of the already available FFT bins. Analysis and simulation results were presented to verify the algorithm performance.

6. REFERENCES

[1] B. Boashash, "Estimating and interpreting the instantaneous frequency of a signal. I. fundamentals," *Proceedings of the IEEE*, vol. 80, no. 4, pp. 520 – 538, 1992.

[2] B. Boashash, "Estimating and interpreting the instantaneous frequency of a signal. II. algorithms and applications," *Proceedings of the IEEE*, vol. 80, no. 4, pp. 540 – 568, 1992.

[3] B. G. Quinn, "Estimating frequency by interpolation using Fourier coefficients," *IEEE Transactions on Signal Processing*, vol. 42, no. 5, pp. 1264–8, 1994.

[4] Barry G. Quinn and E. J. Hannan, *The estimation and tracking of frequency*, Cambridge University Press, New York, 2001.

[5] E. Aboutanios, "A modified dichotomous search frequency estimator," *IEEE Signal Processing Letters*, vol. 11, no. 2, pp. 186–188, 2004.

[6] E. Aboutanios, "Iterative frequency estimation by interpolation on fourier coefficients," *IEEE Transactions on Signal Processing*, vol. 53, no. 4, pp. 1237–1242, 2005.

[7] D. C. Rife and R. R. Boorstyn, "Single tone parameter estimation from discrete-time observations," *IEEE Transactions on Information Theory*, vol. IT-20, no. 5, pp. 591–8, 1974.

[8] M. D. Macleod, "Fast nearly ML estimation of the parameters of real or complex single tones or resolved multiple tones," *IEEE Transactions on Signal Processing*, vol. 46, no. 1, pp. 141–8, 1998.

[9] Y. V. Zakharov and T. C. Tozer, "Frequency estimator with dichotomous search of periodogram peak," *Electronics Letters*, vol. 35, no. 19, pp. 1608–1609, 1999.