SOME PROPERTIES OF AN EMPIRICAL MODE TYPE SIGNAL DECOMPOSITION ALGORITHM

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ABSTRACT

The empirical mode decomposition (EMD) has seen widespread use for analysis of nonlinear and nonstationary time-series. Despite some practical success, it lacks a firm theoretical foundation. This work addresses this. The original EMD algorithm is slightly modified, in a way that facilitates its analysis. We prove three theorems that give conditions for the convergence and time scale separation of the EMD for stationary, band-limited signals.

***Index Terms*—** Empirical Mode Decomposition, Time Scale Separation, Signal Representation

1. INTRODUCTION

There are many ways to represent a given signal, in terms of other component signals. The empirical mode decomposition (EMD) [1], is an adaptive (data based) signal decomposition method developed for use on nonstationary time-series data from nonlinear systems. The component signals are known as *intrinsic mode functions* (IMF). They are determined from the original signal, rather than selected from some pre-defined set. The EMD algorithm iteratively extracts the IMFs (described in detail below). At each iteration a new IMF is found, and the difference between the sum of the IMFs and the original signal yields a *residual* which is used to extract the next IMF.

One interpretation of what the EMD algorithm is doing is that it decomposes the original signal into component signals having successively slower time scales. In this work the EMD algorithm is slightly modified, to facilitate its analysis. The key modification is to use trigonometric interpolation, instead of cubic splines, in the extraction process. This modification allows comparisons to Fourier analysis. For a class of stationary, band-limited signals the convergence, the representation and time scale separation properties of the modified EMD algorithm are established.

2. NOTATION

The following notation will be used: let Z be the set of all zeros of the signal $x(t)$. Let $Z_c \subset Z$ be the set of purely complex zeros, $Z_r \subset Z$ be the set of purely real zeros, $h_i(t)$ be the i^{th} IMF, and $r_i(t)$ be the i^{th} residual. Let $Z(f_i)$, $Z_c(f_i)$, and $Z_r(f_i)$ be the sets of all zeros, complex zeros, and purely real zeros of function $f_i(t)$, respectively. Let the first IMF be zero ($h_0(t) = 0$), and let the first residual be the signal ($r_0(t) = x(t)$). Let the last residual be ($h_M(t) = r_{M-1}(t)$); it is not formally an IMF in general.

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3. BACKGROUND

3.1. The Empirical Mode Decomposition

The EMD was originally developed to represent oscillations present in a signal in a way amenable to spectral analysis using the Hilbert transform. The intrinsic mode functions were defined to meet requirements of Hilbert spectral analysis.

In Huang *et al* [1], the IMFs are defined as follows:

Definition 1 (Intrinsic Mode Function)**.** *An intrinsic mode function satisfies two conditions:*

1. *In the whole data set the number of extrema and the number of zero crossings differ by at most one.*
2. *The local mean is zero. Specifically, the average of the upper envelope (defined by the local maxima) and the lower envelope (defined by the local minima) is pointwise equal to zero.*

The upper and lower envelopes are derived by fitting a cubic spline through the local maxima and minima, respectively, with the added requirement that the signal lies between these envelopes. The first IMF is determined from envelopes of the original signal. The IMF is subtracted from the signal, yielding the residual. The residual is essentially a slowly varying portion of the signal. Then this process is repeated (using the residual instead of the original signal), to obtain the second IMF. The process is repeated until the residual meets some specified criterion.

The decomposition is thus an iterative algorithm (called sifting in [1]) that operates on a signal to refine and extract the IMF. In detail,

Definition 2 (EMD Sifting Procedure)**.** *Given a signal x(t), the procedure begins by letting $r_0(t) = x(t)$ and setting $i = 0$.*

1. *Let $c_0(t) = r_i(t)$, set $j = 0$.*
2. *Identify the maxima and minima of $c_j(t)$, set $j = j + 1$.*
3. *Fit an upper envelope $u_j(t)$ through the maxima, and a lower envelope $l_j(t)$ through the minima.*
4. *Compute the envelope mean, $m_j(t) = \frac{1}{2}(u_j(t) + l_j(t))$.*
5. *Compute a candidate IMF. $c_j(t) = c_{j-1}(t) - m_j(t)$.*
    (a) *If $c_j(t)$ meets the definition of an IMF, set $i = i + 1$, let $h_i(t) = c_j(t)$, $r_i(t) = r_{i-1}(t) - h_i(t)$. Go to step 1.*
    (b) *Else go to step 2.*
6. *This procedure continues until no more IMFs can be extracted. For example, if $r_i(t)$ is a constant amplitude sinusoid then it is also an IMF; if not, it is a constant.*

3.2. Related Work

Since its inception the EMD has been applied to many and varied data sets. However work on the theoretical basis underlying the EMD has been relatively limited.

Flandrin and Rilling [2] have provide insight into the properties of the EMD. They address the affects of sampling on the decomposition, which is continuous time in its definition yet discrete time in its computer implementation.

Huang and coworkers have recently made several contributions regarding the theory of the EMD. In Chen *et al* [3] they develop the EMD using B-splines for the interpolation and analyze its properties under the Hilbert transform, deriving a recursive formula that may be computationally useful in this setting. In Kizhner *et al* [4] they provide partial answers to long standing questions regarding the convergence of the sifting procedure and the scale decomposition for the cubic spline based EMD.

Deléchelle *et al* [5] defines the EMD using a fourth order parabolic partial differential equation to find the envelope mean, rather than cubic spline interpolation [5]. This formalism still requires sifting, but it does not require definition of envelopes; the local mean is defined directly from the local extrema of the signal. The result still employs interpolation by piecewise cubic polynomials, however, which explains why their results are similar to the original cubic spline EMD.

Sharpley and Vatchev [6] provide some analysis of the IMFs. They assume that all of the IMFs are each expressable as a cosinusoid with time-varying amplitude and phase. Using this representation they analyze the properties of this class of functions and, in particular, their Hilbert transforms.

4. EMD WITH TRIGONOMETRIC INTERPOLATION

4.1. Definitions

Here we modify the EMD sifting procedure (definition 2) as defined by Huang *et al* by using trigonometric interpolation instead of cubic splines. The formula we use requires an odd number of interpolating points per period and provides a unique solution. The interpolating points (t_k, x_k) are determined from the extrema of the signal and $x_k = x(t_k)$ $k = 0, ..., 2n$.

$$f(t) = \sum_{k=0}^{2n} x_k \prod_{j=0, j\neq k}^{2n} \frac{\sin \frac{1}{2}(t - t_j)}{\sin \frac{1}{2}(t_k - t_j)} \tag{1}$$

To apply this formula to problems with even numbers of interpolating points (such as the example shown here) we add an average point to the set between actual points, *e.g.* $t_i = (t_{i+1} + t_{i-1})/2$ and $x_i = (x_{i+1} + x_{i-1})/2$; in practice this average point has a minor effect on the shape of the interpolating function. How to select this point, such that none of the conditions that we give for convergence are violated, is an open question.

This interpolation guarantees that the envelopes will pass through the required points, but does not require $u(t) \geq x(t) \geq l(t)$, or $u(t) > l(t)$ at any time t.

4.2. Results

Using this trigonometric interpolation method to obtain the envelopes, the following properties of the resulting EMD can be derived.

Lemma 1 (Frequency content of signal envelopes)**.** *If there are an odd number of interpolating points (N) to define the upper envelope using trigonometric interpolation, then the upper envelope will consist of a sum of integer frequency sinusoids and cosinusoids from* 0 *to* $\frac{N-1}{2}$.

Proof. It is easy to see by viewing Equation 1 in exponetial form (the equivalence is shown in [7]). The interpolation requires an odd number of points $N = 2n + 1$, where n is the maximum frequency of the trigonometric polynomial. □

The procedure we have described above guarantees an odd number of interpolating points, so the lemma holds for our analysis. The same argument holds for the lower envelope and the signal's minima. Based on the frequency limit of the upper and lower envelopes we have a similar bound on the envelope mean $m(t)$, because adding the two envelopes and dividing by two cannot add frequency content.

This next theorem ensures that the result will have equal numbers of zero crossings and extrema, as required by the definition of IMF.

Theorem 1 (Effect on signal extrema and zero crossings)**.** *Assuming the EMD procedure defined above, if $u(t) > l(t)$ $\forall t$ then any local maxima (t_k) of the signal below the real line ($x(t_k) < 0$) will be moved such that they are above the real line (though located at a different time t_k) by the EMD sifting procedure.*

Proof. By definition the envelope mean $m(t)$ will always be halfway between the upper and lower envelopes. If $u(t) > l(t)$ we are guaranteed $l(t) < m(t) < u(t)$ and $m(t)$ will have to pass through $x(t)$ below the maxima and above the minima of the signal. This guarantees that $m(t)$ intersects the signal once on either side of an extremum. The intersections are the zero crossings (real zeros) of the candidate IMF and the portion of $x(t) > m(t)$ will be above the real line and similarly the portion $x(t) < m(t)$ will be below.

The frequency limit on $m(t)$ from Lemma 1 ensures that it is slowly varying relative to $x(t)$, so the subtraction of $m(t)$ from $x(t)$ will not result in the creation of additional extrema between the intersections, it will only shift the location in time of the extrema. Therefore after the sifting procedure all maximal lie above the real line. □

The same argument holds for the signal's minima above the real line being moved below it as well. Theorem 1 implies that the number of complex zeros of $r_i(t)$ will be greater than or equal to the number of complex zeros of $h_{i+1}(t)$, $|Z_c(r_i)| \geq |Z_c(h_{i+1})|$.

The following theorem shows that the sifting procedure will converge such that the result will meet the second part of the definition of an IMF.

Theorem 2 (Convergence of sifting procedure)**.** *The trigonometric EMD sifting procedure converges such that $m_i(t) \to 0$*

Proof. The proof is in two parts. First we show the locations of the extrema to change progressively less as i increases. Second we show how this implies that $m_i(t) \to 0$ pointwise.

Recall that the next candidate IMF in the procedure is given by:

$$h_i(t) = h_{i-1}(t) - m_i(t) \tag{2}$$

Take the derivative with respect to time and then expand each term of the right hand side in a Taylor series. Apply the Taylor remainder theorem to truncate the series at the quadratic term. Then apply the mean value theorem for the derivative in the remainder term to derive:

$$\begin{aligned} h'_{i-1}(t^i_k) &= \frac{1}{2}(h''_{i-1}(t^i_k) + h''_{i-1}(t^{i-1}_k))(t^i_k - t^{i-1}_k) \\ m'_i(t^i_k) &= m'_i(t^{i-1}_k) + \frac{1}{2}(m''_i(t^i_k) + m''_i(t^{i-1}_k))(t^i_k - t^{i-1}_k) \end{aligned} \tag{3}$$

Next we set the derivative of equation 2 to zero and solve for $t_k^i - t_k^{i-1}$ to derive:

$$\begin{aligned} t_k^i - t_k^{i-1} &= \frac{2m_i'(t_k^{i-1})}{h_i''(t_k^i) + h_i''(t_k^{i-1})} \\ &= \frac{u_i'(t_k^{i-1}) + l_i'(t_k^{i-1})}{h_i''(t_k^i) + h_i''(t_k^{i-1})} \\ &= \frac{-2h_i'(t_k^{i-1})}{h_i''(t_k^i) + h_i''(t_k^{i-1})} \end{aligned} \tag{4}$$

We show that this approaches zero by applying the definition of the derivative and using the uniform continuity of these functions.

$$\begin{aligned} \lim_{\delta \to 0} &= \frac{-2\delta(h_i(t_k^{i-1}) - h_i(t_k^{i-1} + \delta))}{\alpha} \\ \text{where } \alpha &= h_i(t_k^i) - 2h_i(t_k^i + \delta) + h_i(t_k^i + 2\delta) \\ &+ h_i(t_k^{i-1}) - 2h_i(t_k^{i-1} + \delta) + h_i(t_k^{i-1} + 2\delta) \end{aligned} \tag{5}$$

The numerator clearly goes to zero as $\delta \to 0$. The denominator can only be zero if between iterations the k^{th} extrema switched from being a maxima to a minima (or vice-versa) with essentially the same curvature. This cannot happen by subtraction of $m_i(t)$ as defined.

For the second part of the proof we note that $t_k^i - t_k^{i-1} \to 0$ implies that $u_i'(t) + l_i'(t) \to 0$. Considering it an equality for $i \geq N$ we integrate to see that $u_i(t) = -l_i(t) + c$ where c is a constant. It is clear that for large enough i, $m_i(t) = \frac{c}{2^{i-N}} \to 0$ as $i \to \infty$. □

This next result shows the time scales of the signal are separated by the decomposition procedure.

Theorem 3 (Relation between residual zeros and IMF zeros)**.**

$$|Z(r_i)| = |Z(h_{i+1})| \tag{6}$$

$$|Z(r_i)| > |Z(r_{i+1})| \tag{7}$$

Proof. By Lemma 1 the Nyquist rates of the envelopes are limited. This implies a similar limit on the Nyquist rate of the envelope mean, because adding two signals cannot add frequency content that was not present in one of the component signals.

Likewise subtracting the envelope mean from the signal will not add frequency content to the result. Subtracting the envelope mean leaves frequencies not in $m(t)$ in the IMF and the rest of the frequency components in the residual.

Since the number of zeros (per period) in a signal is equal to the Nyquist rate, the IMF will have the same number of zeros as the signal (residual) it was derived from. Thus the resulting residual will have fewer zeros. □

5. EXAMPLE

Here we present an example using the trigonometric interpolation method. The signal $x(t)$ is a randomly selected seventh-order Fourier series, $x(t) = 1.7204 + 2.2878\cos(t) - 0.33213\cos(2t) + 3.7540 \times \cos(3t) - 3.6664\cos(4t) + 1.8433\cos(5t) + 1.9002\cos(6t) - 0.56439\cos(7t)$.

Note the signal, shown in Figure 1, is not an IMF itself. The local minima at approximately $0.5sec$ and $5.sec$ are above the real line, which is a violation of part one of the definition of an IMF. The signal is decomposed into three IMFs and a constant residual, as shown in Figure 2. The IMFs are shown with their upper and lower

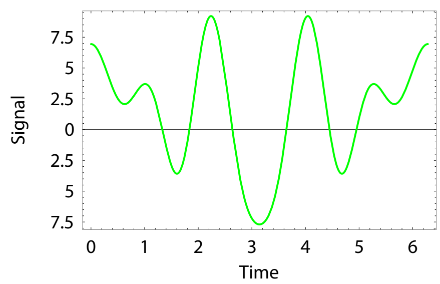

Fig. 1. Signal used in EMD comparison

envelops (dotted lines) and their envelope mean (dashed line). The intermediate residuals, which the second and third IMFs are derived from, are also shown.

The signal $(x(t))$ has also been decomposed using the cubic spline method as implemented by Rilling [8]. In Figure 3 the IMFs and residual derived using the spline method are compared with those of trigonometric interpolation method. The first IMF of both methods are very similar, but the subsequent IMFs from the spline based method have a consistently higher number of oscillations. This indicates that the limit on frequency content of the residuals (as shown in Lemma 1) is lower for the trigonometric interpolation than it is for splines.

Another interesting feature in the comparison is that the residuals differ. The residual of the spline method looks like a one Hz sinusoid, compared to the constant of the trigonometric method. A sinusoid has the capability to be an IMF so the decomposition procedure should not have stopped at this point given the periodic nature of $x(t)$. This mistake, when using the cubic spline based EMD, is due to the assumption that the signal is not stationary. Whereas, the spline EMD only looks at a window of the signal, the trigonometric EMD assumes a periodic signal of infinite extent. This suggests that the trigonometric EMD might be the preferred method for Fourier analyzable signals.

6. CONCLUSION

Our results reveal the sensitivity of the EMD to interpretation of the definition of IMF. Specifically, the definition of IMF does not specify what is required of the upper and lower envelopes. We can meet the requirements of the IMF definition but produce significantly different results from the original spline version used in [1].

We do not require our envelopes to be greater or less than the signal, and we have proven that so long as the upper envelope is strictly greater than the lower envelope we still have an EMD procedure which meets the definitions. How to enforce this condition is an open question.

In recent work, Frei and Osorio [9] develop another EMD like algorithm. In this paper they point out that the EMD sifting procedure, distorts the IMFs by moving the locations of the extrema and reducing differences in the amplitudes of adjacent extrema. They argue against the use of the sifting procedure on these grounds and note that with their formulation they satisfy the first part of the definition of the IMF without sifting, and that the locations of the extrema do not change. The reason the extrema do not move is that their definition ensures that the derivative of the IMF is proportional to that of the signal, and therefore zero at the point of the extrema. We note

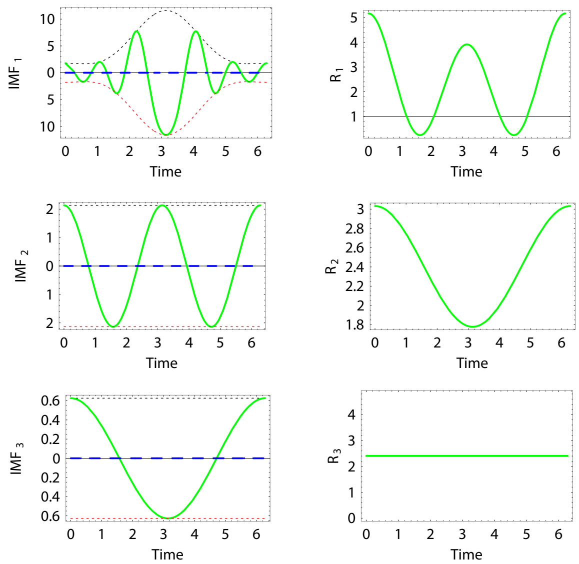

Fig. 2. IMF in solid green, m(t) in dashed blue, u(t) in dotted black, and l(t) in dotted red.

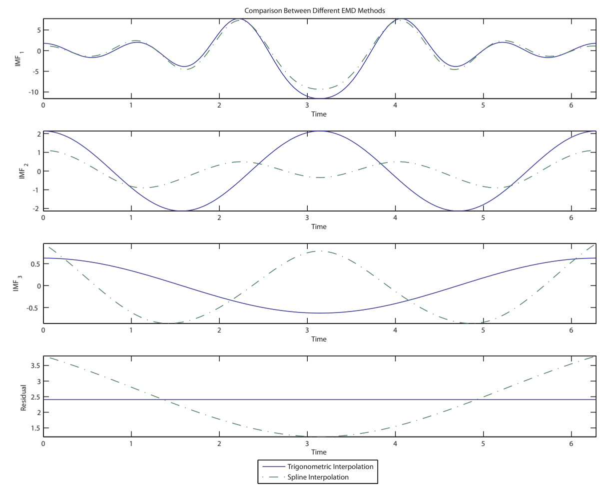

Fig. 3. Comparison between the IMFs and residual of the original spline based EMD (in dash dotted green) and the trigonometric interpolation based EMD (in solid blue)

that our formulation likewise meets the same definition and that we could modify the trigonometric interpolation formula to allow us to specify the value of the derivative at certain points[1]. Constraining the derivative of $m(t)$ to be equal to zero at t_k would prevent this distoration.

The principal contribution of this work is that the use of trigonometric interpolation in place of cubic splines offers improved potential for analysis. We provide conditions to guarantee the convergence of the sifting procedure, and that spurious creation of extrema in this procedure is not possible. We also describe the EMD in terms of its effect on the zeros of a signal [11, 12].

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