# LOW-COMPLEXITY RECEIVERS FOR MULTIUSER DETECTION WITH AN UNKNOWN NUMBER OF ACTIVE USERS


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## ABSTRACT

In multiuser detection, the set of users active at any time may be unknown to the receiver. A two-step detection procedure, in which multiuser detection is preceded by active-user identification, is suboptimum. The optimum solution consists of detecting simultaneously the set of active users and their data, problem that can be solved exactly by applying Random-Set Theory (RST). However, implementation of optimum detectors can be limited by their complexity, which grows exponentially with the number of potential users. In this paper we illustrate how the complexity of optimum can be reduced. In particular, Sphere Detection (SD) techniques (possibly in an approximate version) are examined.



***Index Terms*—** Signal detection, Complexity theory, Bayes procedures, Stochastic processes.


## 1. INTRODUCTION

In multiuser detection (MUD), an important issue is that the set of users that are active at a given time may be unknown to the receiver. A simple solution to the problem of detecting data in a multiuser system in which the number of active users is unknown consists of a two-step procedure, in which multiuser detection is preceded by active-user identification (see, e.g., [6, 8]). In [3], the optimum solution is described: instead of proceeding in two steps, the number of active users and their data are detected simultaneously. Several scenarios are examined in [3]: in the simplest among them, no a priori information about user activity is available, and maximum-likelihood detection is used. When a priori information is available in the form of the probability that a user is active, maximum a posteriori (MAP) detection can be performed. If, in addition, a dynamic model is available for users logging in and out of the system, the use of random-set theory (which we argue to be the most natural tool for the analysis of random-access systems: see, e.g., [3, 7]) allows one to describe the evolution of the a posteriori probability of the set of active users and their data. A further extension of the theory permits estimation of users parameters (e.g., their power) in addition to their data [2].

Applying the developments described above requires the derivation of detectors whose complexity allows practical implementation. This is precisely the goal of the present paper. Here, sphere detection techniques (possibly in an approximate version) are applied to the implementation of the multiuser detectors. This paper is organized as follows. Section 2 describes the essentials of the sphere detection algorithm in the form that is immediately applicable to the problem examined in this paper. Sections 3 examines sphere-detection-based algorithms for the ML and MAP estimations of the active users and their transmitted data, respectively, in an environment where the dynamic behavior of the users cannot be accounted for, and detection must be undertaken at each symbol interval. Section 4 describes simplified receivers accounting for a Markov model of the users logging in and out of the system. Numerical results are presented in Section 5, and conclusions are drawn in Section 6.

## 2. SPHERE DETECTION

Consider the minimization of a function $f(x_1, \dots, x_K)$ with respect to its $K$ arguments, all taking values in a discrete set with $M$ elements. While brute-force minimization involves the evaluation of all $M^K$ values of $f$, SD simplifies the problem under the assumption that $f$ can be written in the form of a sum of nonnegative functions with an increasing number of arguments:

$$f(x_1, \dots, x_K) = \sum_{k=0}^{K-1} f_{K-k,K-k+1,\dots,K}(x_{K-k}, x_{K-k+1}, \dots, x_K) \tag{1}$$

The minimization of (1) can be described graphically by using an $(K+1)$-level tree graph whose paths merge into a common uppermost node (*level* 0) to the $M^K$ leaves (*level* $K$). Each node at level $k$ emanates $M$ branches which join it to a node at level $k+1$, each one being associated with a value of $x_{K-k}$; hence, each node at level $k$ correspond to a value of the partial sum of the first $k$ terms of (1), and each terminal branch (or *leaf*) to a value of $f$.

Now, brute-force minimization of $f$ can be interpreted as the process of probing all the $M^K$ paths joining the root node to all terminal leaves. SD simplifies the process as follows. Start from the root node and proceed downwards; at level-$k$ node ($k = 0, \ldots, K-1$), only one branch stemming from it is chosen, that associated with the smallest value of $f_{K-k,K-k+1,\ldots,K}$. This leads to a single node at level $k+1$, from which only one branch is chosen according to the same criterion, etc. This is equivalent to computing sequentially, for $k = 0, \ldots, K-1$,

$$\hat{x}_{K-k} = \arg\min_{x_{K-k}} f_{K-k,K-k+1,\ldots,K}(x_{K-k}, \hat{x}_{K-k+1}, \ldots, \hat{x}_K) \tag{2}$$

where $\hat{x}_\ell$ denotes the value chosen for $x_\ell$. At the end of this process, we obtain a *preliminary estimate* of the minimum value of $f$, which we call $\bar{f}$. Next, we proceed to probe the branches that were left out, backtracking from the leaf associated with $\bar{f}$ *and excluding all the branches that will certainly end up into a leaf corresponding to a value of $f$ larger than $\bar{f}$)*. To do this, all branches emanating from a node are removed from the tree ("pruned out") whenever the value of the partial sum at that node is already greater than $\bar{f}$. Whenever a leaf is reached, if this is associated with a value $f < \bar{f}$, then this new value replaces $\bar{f}$, and the procedure is continued.

## 3. SPHERE DECODING FOR ML AND MAP MUD

In this section we review the MUD approach based on random-set theory (RST), as first advocated in [3], considering ML and maximum a posteriori (MAP) detectors in situation where decisions are made at each signaling interval.

### 3.1. Signal Model

We assume a random number of users transmitting digital data over a common channel. We denote by $K$ the maximum number of active users, and by $\mathbf{s}(\mathbf{x}_t^{(k)})$ the signal transmitted at discrete time $t$ by the $k$th user, if active. Each signal has in it a number of known parameters, reflected by a deterministic function $\mathbf{s}(\cdot)$, and a number of random parameters, summarized by $\mathbf{x}_t^{(k)}$. The observed signal at time $t$, denoted $\mathbf{y}_t$, includes $\mathbf{s}(\mathbf{x}_t^{(k)})$, the signals generated by the users active at $t$, which are in a random number, and stationary random noise $\mathbf{z}_t$. Thus,

$$\mathbf{y}_t = \sum_k \mathbf{s}(\mathbf{x}_t^{(k)}) + \mathbf{z}_t \tag{3}$$

Let $\mathcal{X}_t$ denote the random set encapsulating what is unknown about the active users. We write $\mathcal{X}_t = \bigcup_{k=1}^{K} \mathcal{X}_t^{(k)}$, where $\mathcal{X}_t^{(k)}$ is the singleton-or-empty set

$$\mathcal{X}_t^{(k)} = \begin{cases} \{\mathbf{x}_t^{(k)}\} = \{[k, x_t^{(k)}]^T\} & \text{if user } k \text{ is active at time } t \\ \emptyset & \text{otherwise} \end{cases} \tag{4}$$

Here, $\mathbf{x}_t^{(k)}$ is a singleton whose element is the vector containing the user index $k$ and an unknown (possibly random) parameter $x_t^{(k)}$. The latter takes values in the finite set $\mathcal{M}$, with cardinality $|\mathcal{M}| = M$, representing the digital data transmitted by user $k$ at time $t$.

A simple model [3] for the random-set sequence $\{\mathcal{X}_t\}_{t=1}^{\infty}$ assumes the sets as independent and identically distributed, with

$$f_{\mathcal{X}_t}(\mathcal{X}_t) = M^{-|\mathcal{X}_t|}\alpha^{|\mathcal{X}_t|}(1-\alpha)^{K-|\mathcal{X}_t|} \tag{5}$$

where $|\mathcal{X}_t|$ denotes the cardinality of $\mathcal{X}_t$. With this model, the users are independently active with the same probability $\alpha$, whereby the active users must be identified and possibly decoded at each signaling interval.

### 3.2. Observation model

Under the assumption of power-controlled, direct-sequence code-division multiple-access (DS-CDMA) with signature sequences of length $N \geq K$, and of zero-mean additive white Gaussian noise with power spectral density $N_0/2$, we can write, for the sufficient statistics of the received signal at time $t$,

$$\mathbf{y}_t = \mathbf{S}\mathbf{x}_t + \mathbf{z}_t \tag{6}$$

where $\mathbf{y}_t$ is the $N$-dimensional column vector of the observations, $\mathbf{S} \triangleq [\mathbf{s}_1, \ldots, \mathbf{s}_K]$ is a $N \times K$ matrix whose columns contain the signature sequences of all the potential $K$ users, and $\mathbf{x}_t = \mathbf{x}_t(\mathcal{X}_t)$ is a $K$-vector whose $k$th entry is defined as

$$x_t(k) = \begin{cases} 0 & \text{if } \mathcal{X}_t^{(k)} = \emptyset \\ x_t^{(k)} & \text{otherwise} \end{cases} \tag{7}$$

We have

$$f_{\mathbf{Y}_t|\mathcal{X}_t}(\mathbf{y}_t|\mathcal{X}_t) = \frac{1}{\sqrt{\pi N_0}} \exp\{-\|\mathbf{y}_t - \mathbf{S}\mathbf{x}_t\|^2/N_0\} \tag{8}$$

### 3.3. ML-based MUD

The (symbol-by-symbol) ML-based MUD is the optimum (minimum-error-probability) receiver if all of the outcomes of $\mathcal{X}_t$ are independent and equally likely. With this model, the ML receiver generates

$$\widehat{\mathcal{X}}_t = \arg\max_{\mathcal{X}_t} \ln f_{\mathbf{Y}_t|\mathcal{X}_t}(\mathbf{y}_t|\mathcal{X}_t) = \arg\min_{\mathcal{X}_t} \|\mathbf{y}_t - \mathbf{S}\mathbf{x}_t\|^2 \tag{9}$$

To reduce the complexity of this receiver, sphere detection can be applied. In fact, QR decomposition of $\mathbf{S}$ generates a metric with the form (1) [1]: for example, if the transmitted data are binary antipodal,[1] the assumption of an unknown number of users leads to three values possibly taken on by the components of $\mathbf{x}$, i.e., 0 and $\pm 1$.

[1] We make this assumption for simplicity's sake only: the extension to a more general case is straightforward.

### 3.4. MAP-based MUD

We further consider a case wherein the parameter $\mathcal{X}$ is random with the distribution (5). After QR decomposition of $\mathbf{S}$, MAP decision can be again performed using the SD algorithm [1].

## 4. MUD IN A DYNAMIC ENVIRONMENT: SPHERE DETECTION ALGORITHM

The dynamic model described in [3] has the transition density of the random set sequence $\mathcal{X}_t$ written in the form

$$\begin{aligned} f_{\mathcal{X}_t|\mathcal{X}_{t-1}}(\mathcal{C} \mid \mathcal{B}) &= f_{\mathcal{S}_t|\mathcal{X}_{t-1}}(\mathcal{C} \cap \mathcal{B}) f_{\mathcal{N}_t|\mathcal{X}_{t-1}}(\mathcal{C} \setminus (\mathcal{C} \cap \mathcal{B})) \\ &= M^{-|\mathcal{C}\cap\mathcal{B}|}\mu^{|\mathcal{C}\cap\mathcal{B}|}(1-\mu)^{|\mathcal{B}|-|\mathcal{C}\cap\mathcal{B}|} \\ &\quad M^{-|\mathcal{C}\setminus\mathcal{C}\cap\mathcal{B}|}\alpha^{|\mathcal{C}\setminus\mathcal{C}\cap\mathcal{B}|}(1-\alpha)^{K-|\mathcal{B}|-|\mathcal{C}\setminus\mathcal{C}\cap\mathcal{B}|} \end{aligned} \tag{10}$$

The above can be obtained by noticing that $\mathcal{X}_t = \mathcal{S}_t \cup \mathcal{N}_t$, with $\mathcal{S}_t$ the set of *surviving* users still active from $t-1$, and $\mathcal{N}_t$ the set of *new* users becoming active at $t$. The symbol $\mu$ denotes the probability of "persistence," i.e., the probability that a user survives from $t-1$ to $t$, and $\alpha$ denotes the probability that a new user arises. [2] Births and deaths of users are assumed conditionally independent given $\mathcal{X}_{t-1}$.

Given the above, the dynamics of the system are fully described by the probability density function $f(\mathbf{y}_t \mid \mathcal{X}_t)$ of the observation $\mathbf{y}_t$ given the realization of the random set $\mathcal{X}_t$ (eq. (8)) and by the transition probability $f(\mathcal{X}_t \mid \mathcal{X}_{t-1})$ (eq. (10)). These two functions can be used as the ingredients of Bayes recursions for countable sets: denoting $\mathbf{y}_{1:t} \triangleq (\mathbf{y}_1, \dots, \mathbf{y}_t)$ the channel-output observations from time 1 to time $t$, we have, for the conditional a posteriori densities,

$$f(\mathcal{X}_{t+1} \mid \mathbf{y}_{1:t}) = \sum_{\mathcal{X}_t} f(\mathcal{X}_{t+1} \mid \mathcal{X}_t) f(\mathcal{X}_t \mid \mathbf{y}_{1:t}) \tag{11}$$

$$f(\mathcal{X}_{t+1} \mid \mathbf{y}_{1:t+1}) \propto f(\mathcal{X}_{t+1} \mid \mathbf{y}_{1:t}) f(\mathbf{y}_{t+1} \mid \mathcal{X}_{t+1}) \tag{12}$$

Thus, the optimum causal detector for $\mathcal{X}_{t+1}$ is obtained through maximization of (12).

In [3], the above recursion is used to derive the optimal causal estimator of $\mathcal{X}_t$. The problem with (11)-(12) is that the calculation of (11), and the evaluation of the maximum in (12), have a complexity which grows exponentially with $K$. Moreover, taking the logarithm of both sides of (12), the problem is not amenable to SD since (11) does not admit a decomposition as in (1).

A hint on how to proceed can be obtained by rewriting (11) in the form

$$\begin{aligned} \sum_{\mathcal{X}_t} f(\mathcal{X}_{t+1} \mid \mathcal{X}_t) f(\mathcal{X}_t \mid \mathbf{y}_{1:t}) &= f(\mathcal{X}_{t+1} \mid \widehat{\mathcal{X}}_t) f(\widehat{\mathcal{X}}_t \mid \mathbf{y}_{1:t}) \\ &+ \sum_{\mathcal{X}_t \neq \widehat{\mathcal{X}}_t} f(\mathcal{X}_{t+1} \mid \mathcal{X}_t) f(\mathcal{X}_t \mid \mathbf{y}_{1:t}) \end{aligned} \tag{13}$$

[2] With some abuse of notation, we retain the same symbol $\alpha$ for the probability of a user to be active and the probability of a new user's birth.

where $\widehat{\mathcal{X}}_t$ is the MAP estimate from the previous interval. On the other hand, for sufficiently large signal-to-noise ratio — which is the region where Sphere Decoding yields a significant complexity reduction over exhaustive search — the likelihood $f(\mathcal{X}_t \mid \mathbf{y}_{1:t})$ exhibits a sharp peak around its maximum $\widehat{\mathcal{X}}_t$, whereby the term $f(\mathcal{X}_{t+1} \mid \widehat{\mathcal{X}}_t) f(\widehat{\mathcal{X}}_t \mid \mathbf{y}_{1:t})$ may be conjectured to give the largest contribution (at least asymptotically). Thus, the Bayes recursions may be approximated as:

$$\begin{aligned} f(\mathcal{X}_{t+1} \mid \mathbf{y}_{1:t}) &\approx f(\mathcal{X}_{t+1} \mid \widehat{\mathcal{X}}_t) f(\widehat{\mathcal{X}}_t \mid \mathbf{y}_{1:t}) \\ f(\mathcal{X}_{t+1} \mid \mathbf{y}_{1:t+1}) &\approx f(\mathcal{X}_{t+1} \mid \widehat{\mathcal{X}}_t) f(\widehat{\mathcal{X}}_t \mid \mathbf{y}_{1:t}) f(\mathbf{y}_{t+1} \mid \mathcal{X}_{t+1}) \end{aligned} \tag{14}$$

Denoting

$$\begin{aligned} \Lambda(\mathcal{X}_{t+1} \mid \mathbf{y}_{1:t+1}) &\triangleq -N_0 \ln f(\mathcal{X}_{t+1} \mid \mathbf{y}_{1:t+1}) \approx \\ \Lambda(\widehat{\mathcal{X}}_t \mid \mathbf{y}_{1:t}) &+ \|\mathbf{y}_{t+1} - \mathbf{S}\mathbf{x}_{t+1}\|^2 - N_0 \ln f(\mathcal{X}_{t+1} \mid \widehat{\mathcal{X}}_t) \end{aligned} \tag{15}$$

we have, under the approximation (14), the MAP estimate:

$$\widehat{\mathcal{X}}_{t+1} = \arg\min_{\mathcal{X}_{t+1}} \left[\|\mathbf{y}_{t+1} - \mathbf{S}\mathbf{x}_{t+1}\|^2 - N_0 \ln f(\mathcal{X}_{t+1} \mid \widehat{\mathcal{X}}_t)\right] \tag{16}$$

Introducing the QR decomposition $\mathbf{S} = \mathbf{QR}$, with $(\mathbf{R})_{i,j} = 0$, $i > j$ and denoting $\widetilde{\mathbf{y}} = \mathbf{Q}^\dagger \mathbf{y}$, we have, after some algebra:

$$\begin{aligned} \Lambda(\mathcal{X}_{t+1} \mid \widetilde{\mathbf{y}}_{1:t+1}) &= \textstyle\sum_{i=1}^K g_i\big(x_{t+1}(K), \dots, x_{t+1}(i)\big) \\ &-N_0 \ln f(\pi(\mathcal{X}_{t+1}) \mid \pi(\widehat{\mathcal{X}}_t)) + N_0 |\mathcal{X}_{t+1}| \ln M \end{aligned} \tag{17}$$

Sphere Decoding can be applied to maximize (17): decision-feedback may be employed to make the preliminary decision $\widehat{\mathcal{X}}_{t+1}^{(DF)}$, and $\overline{\Lambda} = \Lambda(\widehat{\mathcal{X}}_{t+1}^{(DF)} \mid \widetilde{\mathbf{y}}_{1:t+1})$ is the reference value to be adopted in the SD algorithm. For example, assuming that user $K$ belongs to $\mathcal{X}_{t+1}$ leads to an increment

$$g_K(x_{t+1}(K)) + N_0 \ln M - N_0 \begin{cases} \ln \mu & \text{if } K \in \widehat{\mathcal{X}}_t \\ \ln \alpha & \text{otherwise} \end{cases}$$

to be summed to the metric $\Lambda(\widehat{\mathcal{X}}_t \mid \mathbf{y}_{1:t})$ cumulated up to $t$ (see (15)): if the resulting metrics exceed $\overline{\Lambda}$, $\forall x_{t+1}(K) \neq 0$, and if $K \notin \widehat{\mathcal{X}}_{t+1}^{(DF)}$, then user $K$ can be immediately estimated as inactive. If this is not the case, a similar procedure can be re-applied to the pair $(K, K-1)$: this requires enumerating four possible situations, and evaluating the corresponding cumulated metrics to be compared against $\overline{\Lambda}$. This could allow pruning out further paths.

## 5. NUMERICAL RESULTS

Consider now the performance of the proposed SD algorithm for MUD in a dynamic environment. We examine error probability and algorithm complexity.

Assume the transmission of a known training sequence, and estimation of the active users. Let the spreading sequences be $m$-sequences. Moreover, assume a processing gain $N = 7$,

$K = 6$, $\mu = 0.8$, and $\alpha = 0.2$. The frames have length $T = 10$. The computational complexity of the SD-based receiver, measured as the average number of explored nodes per decision, is represented in Fig. 1 versus SNR. As expected,

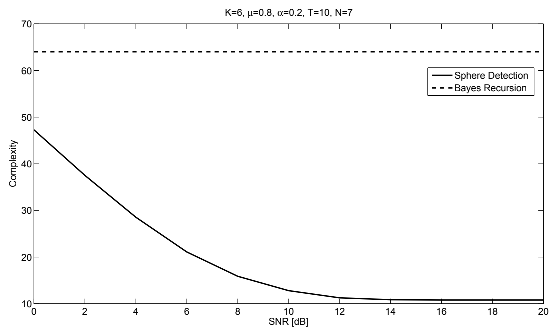


**Fig. 1**. Complexity of the SD receiver and a receiver based on exact recursions, for different values of SNR, in a training phase.

SD is less and less complex as the SNR increases, while exact Bayes recursions are exponentially complex, irrespective of SNR. It should be observed again that SD is "optimum" in the sense that it does not prune out any branch having an overall metric smaller than the one chosen at the algorithm termination, suboptimality being possibly due to our simplification of the algorithm. In order to demonstrate the validity of such an approximation, we refer to Fig. 2, which is derived under the same system parameters as Fig. 1. Fig. 2 shows the set-sequence error probability (SSEP), defined as

$$\text{SSEP} \triangleq \mathbb{P}\big(\mathcal{X}_{1:T} \neq \widehat{\mathcal{X}}_{1:T}\big) \tag{18}$$

for varying SNR. Inspecting Fig. 2 shows that the loss caused by our approximation with respect to exact Bayes recursions is irrelevant on the whole range of examined SNR.

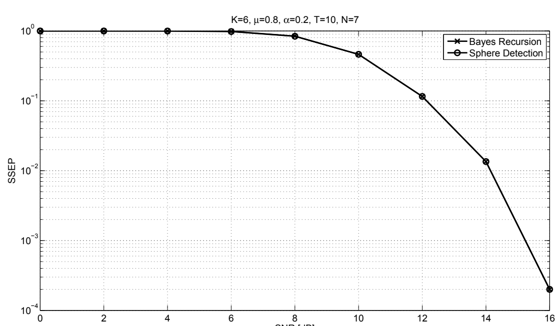


**Fig. 2**. Set-sequence error probability (SSEP) for the SD receiver and a receiver based on exact recursions, for different values of SNR, in a training phase.

## 6. CONCLUSIONS

We have examined multiuser detectors operating without information as of the number of active users, and hence detecting simultaneously the set of active users and their data. Since implementation of optimum detectors can be limited by their complexity, which grows exponentially with the number of potential users, we have derived techniques for the reduction of this complexity. In particular, sphere-detection (SD) techniques (possibly in an approximate version) were examined. SD algorithms have been derived for ML and MAP detectors in a static environment, while for a dynamic environment a simplification can be introduced which enables the application of SD with a marginal loss in performance. The performance of these detectors has been assessed in terms of set-sequence error probability and of complexity.

## 7. ACKNOWLEDGMENT


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