

BOOTSTRAP TESTS FOR THE TIME CONSTANCY OF MULTIFRACTAL ATTRIBUTES

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ABSTRACT

On open and controversial issue in empirical data analysis is to decide whether scaling and multifractal properties observed in empirical data actually exist, or whether they are induced by intricate non stationarities. To contribute to answering this question, we propose a procedure aiming at testing the constancy along time of multifractal attributes estimated over adjacent non overlapping time windows. The procedure is based on non parametric bootstrap resampling and on wavelet Leader estimations for the multifractal parameters. It is shown, by means of numerical simulations on synthetic multifractal processes, that the proposed procedure is reliable and powerful for discriminating true scaling behavior against non stationarities. We end up with a practical procedure that can be applied to a single finite length observation of data with unknown statistical properties.

Index Terms— Multifractal analysis, Non parametric bootstrap, Stationarity test, Time constancy test, Wavelet Leaders

1. INTRODUCTION

Scale invariance. Scale invariance, or scaling, is a property that has been extensively observed in empirical data of very different nature, such as turbulence, network traffic or biomedical signals. In essence, scale invariance can be defined as the fact that moments of order q of some multiresolution quantities $T_X(a, t)$ (e.g., wavelet coefficients) are characterized by power law behaviors with respect to the analysis scale a . The exponents of such power law behaviors, labeled $\zeta(q)$, are referred to as the scaling exponents and are often involved in detection, identification or classification tasks. Empirical multifractal analysis, whose goal is to analyze scale invariance in real data and to provide measurements of the scaling attributes, has been successfully used in various applications and is currently becoming a standard tool in empirical time series analysis.

Scale invariance and non stationarity. In the practical multifractal analysis of empirical data, there has been, and there still is, an important controversy: Do scaling actually exist in data, or are they rather the consequence of non stationarities that conspire to mimic scaling behavior? To contribute to answering this question, let us first clarify the issue. There exist two major classes of stochastic processes used to model scale invariance: Self-similar and multifractal processes. Both classes consist of non stationary processes, and there is hence no contradiction between scale invariance and non stationarity in that respect. The controversy between scale invariance and non stationarity can in fact be cast in the following three categories: First, scaling actually exist but a smooth trend (in the mean or variance, for example), hence a non stationarity, is superimposed to the data and is likely to impair the analysis; Second, scaling exist in data but their parameters exhibit some form of variability with respect to time, for instance due to a change in experimental conditions; Third, scaling

are not present in data but a strong non stationary variability is confused with a scaling property. The first category has been addressed in a number of research papers (cf. [1] and the references therein) and will not be further considered here. The second and third categories are much more involved as a *non stationary variability* can correspond to many different realities. Nevertheless, their detection is of crucial practical importance, since the blind analysis of such time series is likely to produce misleading interpretations of scaling.

Constancy along time of scaling exponents. The discrimination of true scaling against various forms of non stationary variability can be addressed with the following heuristic: When data possess true scaling properties, scaling exponents estimated over the entire time series or over non overlapping adjacent windowed time series are statistically consistent. Conversely, when scaling exponents obtained over non overlapping adjacent subsets of the data are not statistically consistent, this can only be the signature of some form of non stationarity, whatever its precise and a priori unknown nature. Therefore, the issue of testing scale invariance against non stationarity can be meaningfully recast into a test of time constancy of scaling exponents estimated over adjacent non overlapping subsets of the analyzed time series. This is precisely the intuition developed in [2], where a time constancy test is proposed for the (wavelet coefficient based) estimation of the Hurst parameter of Gaussian self similar stationary increment (H -sssi) processes.

Goals of the present contribution: bootstrap and multifractal.

Our goal is to extend such an approach to testing for the time constancy of scaling exponents of *multifractal* process. This implies two major changes in methodology: First, the description of multifractal processes requires a whole collection of attributes, related to positive and negative statistical orders q , while the Gaussian nature of the self similar processes analyzed in [2] allows to concentrate on the second order ($q = 2$) only. Second, estimations are no longer based on wavelet coefficients but on wavelet Leaders [3]. While the former are obtained by a linear transform of the data, the latter stem from a non linear transform. Moreover, multifractal processes are necessarily non Gaussian, heavy tailed and strongly correlated [4]. For these reasons, the design of statistical tests is significantly more complicated for multifractal processes. In particular, the properties of the statistics underlying the test can no longer be obtained analytically, as opposed to the Gaussian H -sssi case. To address such changes in goals and methodology and to overcome the corresponding difficulties, a non parametric bootstrap based test procedure is proposed. Its performance are assessed by application to a large number of realizations of synthetic processes whose (multifractal and non stationarity) properties are known and controlled a priori. We end up with a practical and operational non parametric test procedure, that exhibits satisfactory statistical performance and that can be applied to a single observation of empirical data to assess the true existence of scaling.

2. EMPIRICAL MULTIFRACTAL ANALYSIS

The theory of multifractal analysis is not recalled here and the reader is referred to e.g., [3–5]. We concentrate here only on multifractal parameter estimation and on empirical multifractal analysis, referred to as multifractal formalism.

Multiresolution quantities. Let $X(t)$ denote the time series to be analyzed. Let $d_X(j, k) = \int_{\mathbb{R}} X(t) 2^{-j} \psi_0(2^{-j}t - k) dt$ denote its discrete wavelet transform (DWT) coefficients. The mother-wavelet $\psi_0(t)$ consists of a reference pattern with a compact time support, characterized by its *number of vanishing moments* $N_\psi \geq 1$: $\forall k = 0, 1, \dots, N_\psi - 1$, $\int_{\mathbb{R}} t^k \psi_0(t) dt \equiv 0$ and $\int_{\mathbb{R}} t^{N_\psi} \psi_0(t) dt \neq 0$. Also, it is such that the collection $\{2^{-j/2} \psi_0(2^{-j}t - k), j \in \mathbb{Z}, k \in \mathbb{Z}\}$ forms an orthonormal basis of $L^2(\mathbb{R})$. Let us moreover define dyadic intervals as $\lambda = \lambda_{j,k} = [k2^j, (k+1)2^j]$, and let 3λ denote the union of the interval λ with its 2 adjacent dyadic intervals: $3\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}$. Following [3, 5], wavelet Leaders are defined as: $L_X(j, k) \equiv L_\lambda = \sup_{\lambda' \subset 3\lambda} |d_{X,\lambda'}|$. In other words, the wavelet Leader $L_X(j, k)$ consists of the largest wavelet coefficient $|d_X(j', k')|$ at all finer scales $2^{j'} \leq 2^j$ in a narrow time neighborhood. It has been shown theoretically that the multifractal formalism is more efficient when wavelet Leaders, rather than wavelet coefficients, are chosen as multiresolution quantities (cf. [3, 5]).

Multifractal attributes. The existence of scaling in data and the measurements of the corresponding parameters are commonly based on structure functions. The structure functions mostly consist of the sample moment estimates of order q of the $L_X(j, k)$, or of the sample cumulant estimates of order p of the $\ln L_X(j, k)$, labeled $S(2^j, q)$ and $C(2^j, p)$, respectively. Scaling properties are practically and operationally defined via the following equations:

$$S(2^j, q) = F_q 2^{j\zeta(q)}, \quad (1)$$

$$C(2^j, p) = c_p^0 + c_p \ln 2^j. \quad (2)$$

The scaling exponents $\zeta(q)$ and the so-called log-cumulants c_p practically represent the multifractal attributes of X . They are related together as $\zeta(q) = \sum_{p \geq 1} c_p q^p / p!$ [6].

Estimation procedures. Based on Eqs. (1) and (2), estimations of the $\zeta(q)$ and c_p are obtained by weighted linear regressions, from scales 2^{j_1} to 2^{j_2} , of $\log_2 S(2^j, q)$ and of $\log_2 e \cdot C(2^j, p)$ versus $\log_2 2^j = j$. This has been extensively assessed elsewhere [3] and is not further detailed here.

3. TIME CONSTANCY TEST

In [2], a uniformly most powerful invariant test for the time constancy of the Hurst parameter H of Gaussian H -sssi processes is devised and analyzed. The test is constructed from wavelet coefficient based estimates $\hat{H}_{(m)}$, obtained from adjacent non overlapping subsets $X_{(m)}$ of X , and relies on Gaussianity, independence and (analytically) known statistics of the estimates. Notably, the variance of the estimates is known a priori and independent of the true H . The test statistic reads:

$$T_H = \sum_{m=1}^M \frac{1}{\sigma_{(m)}^2} \left(\hat{H}_{(m)} - \frac{\sum_{n=1}^M \hat{H}_{(n)}}{\sum_{n=1}^M \frac{1}{\sigma_{(n)}^2}} \right)^2. \quad (3)$$

Under the null hypothesis (H constant), its distribution is known exactly, which enables the formulation of the test.

To adapt the test to multifractal processes, we have to extend it to any multifractal attributes $\theta \in \{\zeta(q), c_p\}$, whose estimations are

based on wavelet Leaders [3]. This induces two major difficulties: i) Variances $\sigma_{(m)}^2$ for the $\hat{\theta}_{(m)}$ are no longer known a priori and are likely to depend on the parameter values; ii) The null distribution of the test statistics T_θ is no longer known a priori.

4. BOOTSTRAP TEST

To overcome such severe difficulties in the test formulation and following investigations reported in [3], we propose a non parametric bootstrap based test procedure [7, 8]. Let θ denote the multifractal attribute under test. From the time series X to be analyzed, M wavelet Leader based subset estimates $\hat{\theta}_{(m)}$ of θ are obtained from adjacent non overlapping subsets $X_{(m)}$. Assessing the time constancy of θ then amounts to testing the hypothesis that the random variables $\{\hat{\theta}_{(m)}, m = 1, \dots, M\}$ have identical mean:

$$H_0 : \theta_{(1)} = \theta_{(2)} = \dots = \theta_{(M)}. \quad (4)$$

The test makes use of a bootstrap procedure on the wavelet Leaders, as sketched in Table 1. It is fully specified in e.g. [3] and not further detailed here. We only recall that it consists of a moving time-blocks bootstrap, accounting for dependence amongst wavelet Leaders.

Bootstrap test statistic. The test statistic consists of a modified version of Eq. (3). It is based on bootstrap variance estimates for the unknown variances, and on the Graybill Deal estimator instead of the maximum likelihood estimator of the consensus mean:

$$T_\theta = \sum_{m=1}^M \frac{1}{\hat{\sigma}_{(m)}^{2*}} \left(\hat{\theta}_{(m)} - \frac{\sum_{n=1}^M \frac{\hat{\theta}_{(n)}}{\hat{\sigma}_{(n)}^{2*}}}{\sum_{n=1}^M \frac{1}{\hat{\sigma}_{(n)}^{2*}}} \right)^2. \quad (5)$$

First, the $\{L_X(j, k)\}$ are cut into M subsets $\{L_{X_{(m)}}(j, k)\}$, corresponding to the subsets $X_{(m)}$. The subset estimates $\hat{\theta}_{(m)}$ are computed by applying Eqs. (1) and (2) to the $\{L_{X_{(m)}}(j, k)\}$.

Second, the variance estimates $\hat{\sigma}_{(m)}^{2*}$ for each $\hat{\theta}_{(m)}$ are obtained by applying the bootstrap as in Table 1 to each subset $\{L_{X_{(m)}}(j, k)\}$, yielding B resamples $\{(L_{X_{(m)}}(j, k))^{*(b)}, b = 1, \dots, B\}$. Then Eqs. (1) and (2) are used on each of these resamples to obtain the bootstrap subset estimates $\hat{\theta}_{(m)}^{*(b)}$, from which the sample variance is finally estimated: $\hat{\sigma}_{(m)}^{2*} = \widehat{\text{Var}}^* \hat{\theta}_{(m)}^{*(\cdot)}$. Table 2 (left) sketches this procedure.

Bootstrap null distribution estimation. A bootstrap estimate of the distribution of the test statistic T_θ under H_0 is obtained from the empirical distribution of:

$$T_\theta^* = \sum_{m=1}^M \frac{1}{\hat{\sigma}_{(m)}^{2**}} \left(\hat{\theta}_{(m)}^* - \frac{\sum_{n=1}^M \frac{\hat{\theta}_{(n)}^*}{\hat{\sigma}_{(n)}^{2**}}}{\sum_{n=1}^M \frac{1}{\hat{\sigma}_{(n)}^{2**}}} \right)^2. \quad (6)$$

First, the bootstrap resampling of Table 1 is applied to the *complete* set $\{L_X(j, k)\}$ of Leaders, yielding the B resamples $\{L_X^{*(b)}(j, k)\}$, $b = 1, \dots, B$. Each of these resamples is then cut into M subsets $\{(L_X^{*(b)}(j, k))_{(m)}\}$, and the subset estimations $\hat{\theta}_{(m)}^{*(b)}$ are computed from Eqs. (1) and (2). Let us emphasize that resampling from the complete set of Leaders, rather than from subsets, is a crucial issue, as it ensures that the $\hat{\theta}_{(m)}^*$ all have the same conditional distributions and thus that T_θ^* reproduces the statistics of T_θ under H_0 , shall X satisfy H_0 or H_1 (cf. Fig. 2). The variance estimates $\hat{\sigma}_{(m)}^{2**}$ of $\hat{\theta}_{(m)}^{*(b)}$ are obtained by first applying bootstrap to each $\{L_X^{*(b)}(j, k)\}$, giving the B_2 double bootstrap resamples

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for  $b = 1, \dots, B$ 
  for  $j = 1, \dots, j_2$ 
    From  $\{L_X(j, 1), \dots, L_X(j, n_j)\}$ 
    random draw, with replacement, circular and
    overlapping blocks to form an unsorted collection
     $\{L_X^{*(b)}(j, 1), \dots, L_X^{*(b)}(j, n_j)\}$ 
  end
end

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Table 1: Moving blocks bootstrap for obtaining B bootstrap resamples $\{L_X^{*(b)}(j, k)\}$, $b = 1, \dots, B$, from a set of Leaders $\{L_X(j, k)\}$.

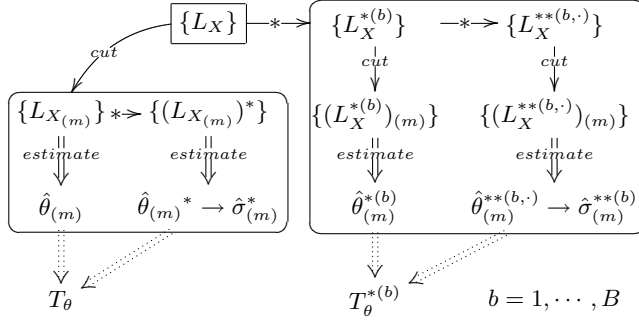


Table 2: Procedure for obtaining T_θ (left) and T_θ^* (right) from the wavelet Leaders $\{L_X(j, k)\}$ of X . "cut", "estimate" and "*" stand for cutting a set into M subsets, computing estimates $\hat{\theta}$ from Eqs. (1) and (2), and bootstrap resampling as in Table 1, respectively.

$\{L_X^{*(b,b_2)}(j, k)\}$, $b_2 = 1, \dots, B_2$. Each of these double bootstrap resamples is in turn cut into M subsets $\{L_X^{*(b,b_2)}(j, k)_{(m)}\}$, enabling the computation of the double bootstrap subset estimates $\hat{\theta}_{(m)}^{*(b,b_2)}$. Finally, the double bootstrap sample variance estimates are computed: $\hat{\sigma}_{(m)}^{2**} = \widehat{\text{Var}} \hat{\theta}_{(m)}^{*(b,b_2)}$. This procedure is summarized in Table 2 (right).

Bootstrap test. The test is now readily formulated as:

$$d_\theta = 1 \text{ if } T_\theta > T_{\theta,C}^* \text{ and } 0 \text{ otherwise,} \quad (7)$$

where the test critical value $T_{\theta,C}^*$ is the $(1 - \alpha)$ quantile of the empirical distribution of T_θ^* , for a certain preset significance level α . The critical value of α for which the observed test statistic T_θ would be regarded as just decisive against H_0 is called the p-value p_θ of T_θ .

5. PERFORMANCE ASSESSMENT AND RESULTS

Monte-Carlo simulations. To evaluate the performance of the proposed test procedures, we apply them to a large number N_{MC} of realizations of length N of a synthetic multifractal process with a priori known and controlled multifractal properties. We choose a well known and easy to simulate multifractal process called multifractal random walk (MRW) for which $c_1, c_2 \neq 0$; $p \geq 3$: $c_p \equiv 0$. For a definition of this process, see e.g. [9]. The simulation parameters are set to $N_{MC} = 1000$, $N = 2^{15}$, $B = B_2 = 99$ and $\alpha = 0.1$. For multifractal attribute estimation, we use Daubechies wavelets with $N_\psi = 3$. The regression range is set to $j_1 = 3$ and $j_2(M) = \log_2 N - \log_2 M - (2N_\psi - 1)$ (cf. [3]). The multifractal parameters (specified below) are chosen to correspond to realistic situations observed in actual data (for instance, $c_2 \approx -0.025$ is a commonly accepted value in turbulence, cf. [3, 6]).

$\{c_1, c_2\}$	$\{0.75, -0.01\}$		$\{0.8, -0.02\}$	
θ	c_1	c_2	c_1	c_2
$\bar{d}_\theta^{H_0}$	0.113	0.143	0.075	0.139
$\bar{p}_\theta^{H_0}$	0.478	0.469	0.530	0.485

Table 3: Mean rejection rates $\bar{d}_\theta^{H_0}$ and p values $\bar{p}_\theta^{H_0}$ (H_0 , $M = 2$).

Performance assessment. The performance of the test procedures, given that a certain hypothesis $H_{(\cdot)}$ is true, are assessed by their mean rejection rates and p-values,

$$\bar{d}_\theta^{H_{(\cdot)}} = \widehat{\mathbb{E}}_{N_{MC}} \{d_\theta | H_{(\cdot)}\} \quad (8)$$

$$\bar{p}_\theta^{H_{(\cdot)}} = \widehat{\mathbb{E}}_{N_{MC}} \{p_\theta | H_{(\cdot)}\}, \quad (9)$$

where $\widehat{\mathbb{E}}_{N_{MC}}$ stands for the mean over Monte Carlo simulations. For space reasons, we only report results for $\theta \in \{c_1, c_2\}$ and $M = \{2, 4\}$ with equal subset lengths. Similar results can be obtained for c_p , $p \geq 3$ and $\zeta(q)$, other choices of M , and splitting into subsets of non equal length.

5.1. Performance under H_0

The performance under H_0 are studied on processes with constant multifractal attributes $\{c_1, c_2\}$. Table 3 summarizes results for two different sets of parameters: $\{c_1, c_2\} = \{0.75, -0.01\}$ (left) and $\{c_1, c_2\} = \{0.8, -0.02\}$ (right). We observe that the mean rejection rates $\bar{d}_\theta^{H_0}$ are close to the preset significance level α for both c_1 and c_2 , and for both parameter settings. Furthermore, the mean p values $\bar{p}_\theta^{H_0}$ are close to 0.5, indicating a satisfactory null distribution estimation. Indeed, under H_0 , the p-value would be uniformly distributed between 0 and 1 if the test was based on the exact null distribution of the test statistic. The results lead us to the conclusion that the empirical distribution of T_θ^* is a satisfactory approximation of the null distribution of T_θ under H_0 , and that it is robust with respect to the precise values of the multifractal parameters. For sake of completeness, we note that results not reported here show that the slight discrepancies in the observed test sizes are mainly due to small differences between the variance of $\hat{\theta}_{(m)}$ (as measured from Monte-Carlo simulations) and its bootstrap estimation $\hat{\sigma}_{(m)}^{*2}$.

5.2. Performance under H_1

To study the power of the proposed tests, we need to define an alternative hypothesis. One could imagine many forms of non stationary processes, a number of them being likely to mimic scaling behaviors when analyzed blindly over the entire time series. Here, we study one of the simplest such alternatives: processes possess piecewise constant multifractal attributes. H_1 is thus analyzed with an alternative consisting of the concatenation of two truly multifractal processes of equal length with different multifractal attributes $\{c_1^{(i)}, c_2^{(i)}\}_{i=1,2}$. Two cases are investigated.

Non constant c_1 , constant c_2 . In the first case, which we denote $H_1(c_1)$, we set $c_1^{(1)} \neq c_1^{(2)}$ and $c_2^{(1)} \equiv c_2^{(2)} = c_2$, i.e., c_2 is constant, while c_1 is not. Thus, \bar{d}_{c_1} assesses the power of the test for time constancy of c_1 . The parameters are set to $c_2 = -0.02$ and $c_1^{(1)} = \{0.70, 0.72, \dots, 0.80\}$, $c_1^{(2)} = 0.8$. Fig. 1 (top) shows test decisions \bar{d}_{c_1} (solid red line) and \bar{d}_{c_2} (dashed blue line) as a function of the step size $c_1^{(1)} - c_1^{(2)}$. The rightmost points $c_1^{(1)} - c_1^{(2)} = 0$ correspond to the mean rejection rates under H_0 of Table 3 (right). We observe that \bar{d}_{c_1} increases fast with $|c_1^{(1)} - c_1^{(2)}|$ and thus that the test

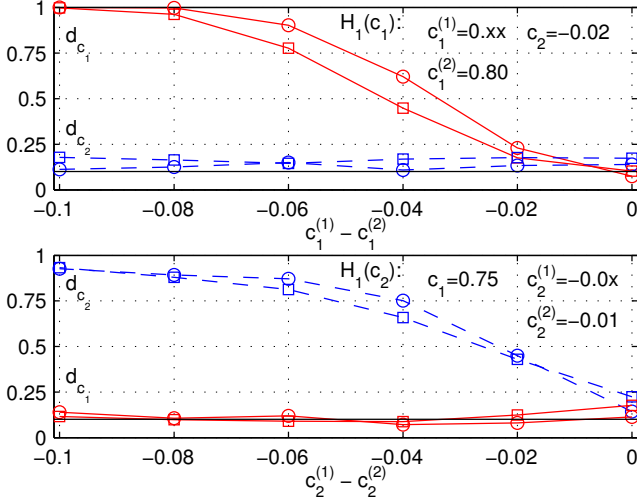


Fig. 1: Test decisions \bar{d}_{c_1} (solid red) and \bar{d}_{c_2} (dashed blue) under $H_1(c_1)$ (top) and $H_1(c_2)$ (bottom) for $M = 2$ (circles) and $M = 4$ (squares) as a function of $c_p^{(1)} - c_p^{(2)}$. The solid black line indicates the preset significance level α .

is powerful: When $c_1^{(1)} - c_1^{(2)} = -0.04$ ($c_1^{(1)} = 0.76$, $c_1^{(2)} = 0.8$), corresponding to values that are in practice considered as being very close, the test rejects the time constancy hypothesis for c_1 with a probability above 0.6 ($M = 2$) and close to 0.5 ($M = 4$). Conversely, the mean test decisions \bar{d}_{c_2} reproduce closely the preset significance level α and remain constant when $c_1^{(1)} - c_1^{(2)}$ varies, indicating that the time constancy test for c_2 is not subject to cross-influence from changes in c_1 . We conclude, first, that the test for time constancy of c_1 is powerful, and second, that the test for constancy of c_2 closely reproduces the level α , independently of $c_1^{(1)} - c_1^{(2)}$.

Constant c_1 , non constant c_2 . In the second case, which we denote $H_1(c_2)$, we set $c_2^{(1)} \neq c_2^{(2)}$ and $c_1^{(1)} \equiv c_1^{(2)} = c_1$, i.e., c_1 is constant, while c_2 is not. Therefore, \bar{d}_{c_2} assesses the power of the test for time constancy of c_2 . The parameters are set to $c_1 = 0.75$ and $c_2^{(1)} = \{-0.11, -0.09, \dots, -0.01\}$, $c_2^{(2)} = -0.01$. Fig. 1 (bottom) shows test decisions \bar{d}_{c_1} (solid red line) and \bar{d}_{c_2} (dashed blue line) as a function of the step size $c_2^{(1)} - c_2^{(2)}$. Exchanging the roles of \bar{d}_{c_1} and \bar{d}_{c_2} , conclusions are similar to those obtained under $H_1(c_1)$: satisfactory power of the test for time constancy of c_2 , and insensitivity of the test on c_1 with respect to level change $c_2^{(1)} - c_2^{(2)}$.

Null distribution estimation under H_1 . Fig. 2 shows bootstrap test critical values (as defined in Eq. (7)) $T_{c_1, C}^*$ under $H_1(c_1)$ (left) and $T_{c_2, C}^*$ under $H_1(c_2)$ (right). The circles and the bars correspond, respectively, to $\widehat{\mathbb{E}}_{NMC} T_{\theta, C}^*$ and to $1.64 \cdot \widehat{\text{Std}}_{NMC} T_{\theta, C}^*$. We observe that the $T_{c_p, C}^*$ do not depend on the step size $c_p^{(1)} - c_p^{(2)}$ and thus on the precise hypothesis H_1 . Moreover, the $T_{c_p, C}^*$ equal the critical values under H_0 (given by the rightmost points). This shows that the empirical distribution of T_{θ}^* under H_1 provides us with a robust and accurate null distribution estimation, as test design demands.

6. CONCLUSIONS AND PERSPECTIVES

We have devised a practical procedure for discriminating the existence of true scaling properties against non stationarities. It consists of a bootstrap based test for the constancy along time of wavelet

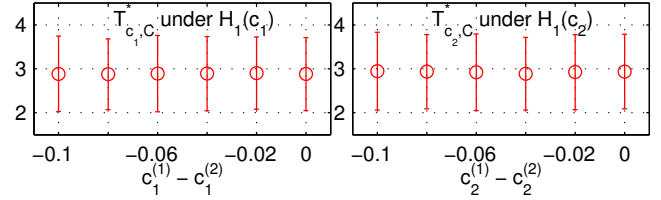


Fig. 2: Bootstrap test critical values $T_{\theta, C}^*$ under H_1 (mean value and $1.64\sigma_{T_{\theta, C}^*}$ bars, obtained through Monte Carlo simulations): Left, $T_{c_1, C}^*$ under $H_1(c_1)$; Right $T_{c_2, C}^*$ under $H_1(c_2)$ ($M = 2$).

Leader based multifractal parameter estimates. We have shown, by means of numerical simulations, that this bootstrap based test procedure is reliable and powerful. Notably, the empirical distribution of T_{θ}^* under H_1 yields an accurate estimation of the null distribution, a central feature for relevant test design. Our procedure successfully addresses this nontrivial issue by combining a "split then bootstrap" for T_{θ} and a "bootstrap then split" for T_{θ}^* . It has heavy computational cost (due to double bootstrap) but remains, to the best of our knowledge, the only procedure practically available. It can be applied to a single observation of real data with unknown statistical characteristics. The impact of choosing M remains to be discussed in terms of trade-off for type-II errors: A test with too small M may miss non stationarities, choosing M too large results in a lack of power due to poor estimations, hence the existence of an optimal M for a given but unknown alternative hypothesis. The procedure can be further extended to testing the constancy along time of the whole structure functions ($S(2^j, q)$, $C(2^j, p)$) or to testing jointly the constancy of a vector of multifractal attributes. This test is being applied to a set of biomedical data.

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