BAYESIAN ESTIMATION OF TRANSITION PROBABILITIES IN HYBRID SYSTEMS VIA CONVEX OPTIMIZATION

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ABSTRACT

In practice, the transition probability matrix (TPM) in the approach to track a maneuvering target is often unknown. We propose a new method to estimate the optimal TPM according to the *maximum a posteriori* (MAP) or maximum likelihood (ML) criterion via convex optimization. We apply the proposed method to the nonlinear/non-Gaussian cases, where the interacting multiple model (IMM) particle filter (IMMPF) is employed to estimate the corresponding base state. Simulation results of tracking a maneuvering target show the efficacy of the proposed method with improved performance.

Index Terms— MAP/ML Estimation, IMMPF, Hybrid systems, Convex optimization, Adaptive estimation

1. INTRODUCTION

Hybrid systems are defined as the combination of continuous and discrete dynamic systems, where a bank of state space models are used to describe a nonstationary environment with multiple modes that are switched from one to another according to Markov chain. A popular approach to track a maneuvering target is to model it as a hybrid system with known transition probability matrix (TPM). However, the TPM is often unknown in practice, and to estimate the TPM with high accuracy is difficult. Moreover, the inadequate TPM will result in significant loss of performance. Thus, the accurate estimation of the TPM is necessary for the hybrid systems.

In order to estimate TPM, a variety of criteria such as minimum mean square error (MMSE), maximum *a posteriori* (MAP), and maximum likelihood (ML), can be employed. In [1], Doucet and Ristic assumed that the rows of the TPM obey Dirichlet distribution and derived a result with simple form. In [2], Jilkov and Li gave an approximate *a posteriori* density of the TPM, and the TPM is estimated recursively using the MMSE criterion.

In this paper, we propose a new method to estimate the TPM under the MAP or ML criterion via convex optimization, where the TPM is assumed to be random or non-random, and time-invariant. For lack of *a priori* information or non-random cases, the MAP estimation is extended to the ML estimation. We apply the method to the nonlinear/non-Gaussian cases where the IMM particle filter (IMMPF) [3] is employed. To our best knowledge, the results of the TPM estimation in the nonlinear/non-Gaussian cases are not reported in the open literature.

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2. SYSTEM MODEL

Consider a multiple model stochastic system

$$x(k) = f[k, x(k-1), v(k-1), M(k)] \tag{1}$$

$$z(k) = h[k, x(k), w(k), M(k)] \tag{2}$$

where $x(k)$ is the base state, and $v(k-1)$ and $w(k)$ is noise, respectively. $M(k)$ is the modal state of a Markov chain with r states in the interval $(t_{k-1}, t_k]$ with the transition probabilities defined as

$$P_{ij} \triangleq P\{M(k) = j\,|M(k-1) = i\}, i, j = 1, \ldots, r. \tag{3}$$

The task here is to estimate the augmented hybrid states,

$$y(k) = \{x(k), M(k)\} \tag{4}$$

based on the joint pdf defined by

$$p[y(k)\,|Z^k] = p[x(k), M(k) = i\,|Z^k], \tag{5}$$

where $Z^k \triangleq \{z(l), l = 0, \ldots, k\}$. The mode probability at time k is defined by

$$\mu_{k,i} \triangleq P\{M(k) = i\,|Z^k\}. \tag{6}$$

As shown in [4] that the MMSE-optimal estimation the base state x obtained by the Baysian *full-hypothesis-tree* (FHT) is infeasible due to its exponentially growing computation and memory. This means that the suboptimal approximations with limited complexity are needed. The suboptimal approximations are referred to as *multiple model* (MM) algorithms, where the *interacting multiple model* (IMM) algorithm [4] is attractive. In this paper, the IMM particle filter (IMMPF) [3] is employed.

3. BAYESIAN ESTIMATION OF TRANSITION PROBABILITIES

3.1. The TPM Likelihood [2]

Let $\mathbf{P} = [P_{ij}]$ be the TPM to be estimated in hybrid systems. Under the framework of IMM and according to the total probability theorem, the likelihood of the TPM at $k+1$ can be denoted by [2]

$$\begin{aligned} &p[z(k+1)|\mathbf{P}, Z^k] \\ &= \sum_{j=1}^{r} \big\{ p[z(k+1)|M(k+1) = j, \mathbf{P}, Z^k] \\ &\quad \times \sum_{i=1}^{r} P\{M(k+1) = j|M(k) = i, \mathbf{P}, Z^k\} \\ &\quad \times P\{M(k) = i|\mathbf{P}, Z^k\} \big\}. \end{aligned} \tag{7}$$

Denoting the estimate of $\mathbf{P}$ at k as $\bar{\mathbf{P}}(k)$, it is reasonable to replace $\mathbf{P}$ with $\bar{\mathbf{P}}(k)$ in (7), the likelihood of $\mathbf{P}$ can be approximated by

$$p[z(k+1)|\mathbf{P}, Z^k] \approx \sum_{j=1}^{r} \tilde{\Lambda}_{k+1,j} \sum_{i=1}^{r} P_{ij} \tilde{\mu}_{k,i} \tag{8}$$

where

$$\tilde{\Lambda}_{k+1,j} \triangleq p[z(k+1)|M(k+1) = j, \bar{\mathbf{P}}(k), Z^k] \tag{9}$$

$$\tilde{\mu}_{k,i} \triangleq P\{M(k) = i|\bar{\mathbf{P}}(k-1), Z^k\}. \tag{10}$$

Define

$$\tilde{\boldsymbol{\mu}}_k \triangleq [\tilde{\mu}_{k,1}, \tilde{\mu}_{k,2}, \dots, \tilde{\mu}_{k,r}]^T$$
$$\tilde{\mathbf{\Lambda}}_{k+1} \triangleq [\tilde{\Lambda}_{k+1,1}, \tilde{\Lambda}_{k+1,2}, \dots, \tilde{\Lambda}_{k+1,r}]^T,$$

which are approximated using IMMPF particles, so that (8) can be written as

$$p[z(k+1)|\mathbf{P}, Z^k] \approx \tilde{\boldsymbol{\mu}}_k^T \mathbf{P} \tilde{\mathbf{\Lambda}}_{k+1}. \tag{11}$$

If $\mathbf{P}$ is random, then using Bayes' rule the posterior density of $\mathbf{P}$ can be approximated by

$$\begin{aligned} p[\mathbf{P}|Z^{k+1}] &= \frac{p[z(k+1)|\mathbf{P}, Z^k]}{p[z(k+1)|Z^k]} p[\mathbf{P}|Z^k] \\ &\approx \frac{\tilde{\boldsymbol{\mu}}_k^T \mathbf{P} \tilde{\mathbf{\Lambda}}_{k+1}}{\tilde{\boldsymbol{\mu}}_k^T \bar{\mathbf{P}}(k) \tilde{\mathbf{\Lambda}}_{k+1}} p[\mathbf{P}|Z^k] \end{aligned} \tag{12}$$

where

$$\begin{aligned} p[z(k+1)|Z^k] &= \int p[z(k+1)|\mathbf{P}, Z^k] p[\mathbf{P}|Z^k] d\mathbf{P} \\ \approx \int \tilde{\boldsymbol{\mu}}_k^T \mathbf{P} \tilde{\mathbf{\Lambda}}_{k+1} p[\mathbf{P}|Z^k] d\mathbf{P} &= \tilde{\boldsymbol{\mu}}_k^T \bar{\mathbf{P}}(k) \tilde{\mathbf{\Lambda}}_{k+1}. \end{aligned} \tag{13}$$

Here, $\bar{\mathbf{P}}(k)$ is actually the MMSE estimate of $\mathbf{P}$ at k [2]. For a given K, we have

$$p[\mathbf{P}|Z^K] \approx c \prod_{k=1}^{K} [\tilde{\boldsymbol{\mu}}_{k-1}^T \mathbf{P} \tilde{\mathbf{\Lambda}}_k] p[\mathbf{P}] \tag{14}$$

where $p[\mathbf{P}] = p[\mathbf{P}|Z^0]$ and c is a constant. If $\mathbf{P}$ is non-random or random with unknown initial density, (14) can be extended as the likelihood function up to K

$$p[Z^K|\mathbf{P}] \approx \prod_{k=1}^{K} [\tilde{\boldsymbol{\mu}}_{k-1}^T \mathbf{P} \tilde{\mathbf{\Lambda}}_k]. \tag{15}$$

We note here that the approximation we made in (11) is the *local linearization* of the likelihood function of $\mathbf{P}$.

3.2. MAP/ML Estimation of the TPM via Convex Optimization

The MAP estimation of $\mathbf{P}$ can be cast into the following optimization problem,

$$\begin{aligned} \text{maximize} \quad & p[\mathbf{P}|Z^{k+1}] \\ \text{subject to} \quad & \sum_{j=1}^{r} P_{ij} = 1, \ i = 1, \dots, r \\ & P_{ij} \geqslant 0, \ i = 1, \dots, r, \ j = 1, \dots, r \end{aligned} \tag{16}$$

where $\mathbf{P} = [P_{ij}]$. Substituting (14) into (16), (16) is equivalent to

$$\begin{aligned} \text{minimize} \quad & -\ln p[\mathbf{P}|Z^{k+1}] \\ & = -\ln c - \sum_{n=1}^{k+1} \ln[\tilde{\boldsymbol{\mu}}_{n-1}^T \mathbf{P} \tilde{\mathbf{\Lambda}}_n] - \ln p[\mathbf{P}] \\ \text{subject to} \quad & \sum_{j=1}^{r} P_{ij} = 1, \ i = 1, \dots, r \\ & P_{ij} \geqslant 0, \ i = 1, \dots, r, \ j = 1, \dots, r \end{aligned} \tag{17}$$

We see that (17) is a convex optimization problem [6] if $p[\mathbf{P}]$ is log-concave, which can be solved by the nonlinear programming solver *solnp.m* downloaded from *http://www.stanford.edu/ yyye /matlab.html* by Y. Ye at each step. We note that by substituting (15) into (16), the ML estimation of $\mathbf{P}$ can be thus formulated.

3.3. Suboptimal Approach

For updating the likelihood, we replace the MMSE estimate $\bar{\mathbf{P}}(k)$ in (9) and (10) with the corresponding MAP estimate, such that

$$\bar{\mathbf{P}}(k) = \int \mathbf{P} p[\mathbf{P}|Z^k] d\mathbf{P} \approx \text{aug} \max_{\mathbf{P}} p[\mathbf{P}|Z^k] \tag{18}$$

since they both converge to the true TPM.

However, the above approximation may introduce errors to the posterior pdf (12) at the start of few time steps especially when the *a priori* knowledge is lacking. This comes from the fact that the transition probabilities are estimated asymptotically from the *empirical* average of the previous transitions. At the start of few time steps, the mode is possibly unchanged and transition information is lacking, which means that there will be no enough information to estimate all the elements of the TPM. For example, assuming the dominant mode is i and kept unchanged, the optimal estimate of the TPM can be obtained by solving (17), whose elements of i-th column will be 1, and the rest will be 0. It is obvious that the results of the estimation are unreasonable.

The unreasonable results are due to the fact that the coefficients are only determined by $\tilde{\mathbf{\Lambda}}_n$, and the mode probabilities $\tilde{\boldsymbol{\mu}}_{n-1}$ will not take effect, which results in the wrong estimation of the TPM, and so are $\tilde{\boldsymbol{\mu}}_{n-1}$ and $\tilde{\mathbf{\Lambda}}_n$, and the errors introduced into (12).

In order to reduce the effect of the errors, the following suboptimal approach is required to solve (17), which is referred to as greedy strategy. The greedy strategy is what tries to make the best possible local decision to approach the global optimum.

Let $\mathbf{P}_i^T$ be the i-th row of $\mathbf{P}$ and $\mathbf{P} = [\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_r]^T$. Assuming

$$\begin{aligned} \int \mathbf{P}_l p[\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_r|Z^k] d\mathbf{P}_1 \dots \mathbf{P}_{i-1} d\mathbf{P}_{i+1} \dots d\mathbf{P}_r \\ = \bar{\mathbf{P}}_l(k) p[\mathbf{P}_i|Z^k] \end{aligned} \tag{19}$$

for $l, i = 1, 2, \dots, r, l \neq i$, $p[\mathbf{P}_i|Z^{k+1}]$ can be written as (for proof, see Appendix A in [2])

$$p[\mathbf{P}_i|Z^{k+1}] = \{1 + \eta_i(k+1)[\mathbf{P}_i - \bar{\mathbf{P}}_i(k)]^T \tilde{\mathbf{\Lambda}}_{k+1}\} p[\mathbf{P}_i|Z^k] \tag{20}$$

where

$$\eta_i(k+1) = \frac{\tilde{\mu}_{k,i}}{\tilde{\boldsymbol{\mu}}_k^T \bar{\mathbf{P}}(k) \tilde{\mathbf{\Lambda}}_{k+1}}. \tag{21}$$

According to (20), (17) can be simplified and equivalently written as the following r convex optimization problems

$$\begin{aligned} \text{minimize} \ & -\ln p[\mathbf{P}_i|Z^{k+1}] \\ & = -\sum_{n=0}^{k} \ln\{1+\eta_i(n+1)[\mathbf{P}_i - \bar{\mathbf{P}}_i(n)]^T \tilde{\mathbf{\Lambda}}_{n+1}\} \\ & \quad - \ln p[\mathbf{P}_i] \\ \text{subject to} \ & \sum_{j=1}^{r} P_{ij} = 1 \\ & P_{ij} \geq 0, \ j = 1, \ldots, r \end{aligned} \tag{22}$$

where $i = 1, \ldots, r$, $\bar{\mathbf{P}}_i^T(n)$ represents the i-th row of the estimate of $\mathbf{P}$ at time n.

Similarly, we here replace the MMSE estimate $\bar{\mathbf{P}}(k)$ in (22) with the corresponding MAP estimate.

To approach the global solution,the following greedy strategy might be used, where the local optimal estimation of $\mathbf{P}$ is performed using (22) with the observation data of the length of K.

Greedy Strategy:

◇ Step 1: Given the *a priori* pdf (if available) and the *a priori* mode probabilities, and set $k = 1$.
◇ Step 2: Based on the observed (received) data, calculate the mode likelihoods and the mode probabilities using (9) and (10).
◇ Step 3: Find the most likely mode at the previous time step, update the corresponding pdf using (20) and solve the optimization problem of (22).
◇ Step 4: Let $k = k + 1$, then go to step 2 until $k > K$.

Remarks: In maneuvering target tracking, to obtain good base state estimation, the TPM is often diagonally dominant, which means that the mode change will not often happen. The probability of the dominant mode is close to 1 when the mode is unchanged, which often happens when the IMM estimator is used. In such cases, the performance difference of the TPM estimation by (17) and (22) are little, and computation cost of using the latter, however, is much less than that using the former. It should be noted that the greedy strategy is needed in any cases at the start of few time steps.

4. SIMULATION RESULTS

In the simulation, the scenario of tracking a maneuvering target is considered, where the target is assumed to obey one of the 3 dynamic models: (1) constant velocity (CV) model, (2) clock-wise coordinated turn (CT) model, (3) anticlockwise CT model. The target state and ownship state are, respectively, denoted as

$$\begin{aligned} \mathbf{x}(k) &= [x(k), y(k), \dot{x}(k), \dot{y}(k)]^T \\ \mathbf{x^o}(k) &= [x^o(k), y^o(k), \dot{x}^o(k), \dot{y}^o(k)]^T \end{aligned}$$

where $(x(k), y(k))$ and $(\dot{x}(k), \dot{y}(k))$ is the target position and velocity, respectively, which are similarly defined in ownship state. The models are denoted as:

$$\begin{aligned} \mathbf{x}(k+1) = & \mathbf{F}^{(M(k+1))}(\mathbf{x}(k))(\mathbf{x}(k) + \mathbf{x}^o(k)) \\ & - \mathbf{x}^o(k+1) + \mathbf{\Gamma}(k)\mathbf{v}(k) \end{aligned} \tag{23}$$

$$z(k) = [x(k), y(k)]^T + \mathbf{w}(k) \tag{24}$$

where $\mathbf{\Gamma}(k) = [\frac{T^2}{2}, T]^T \otimes \mathbf{I}_2$ and $\mathbf{F}^{(M(k+1))}$ is the transition matrix to the mode $M(k+1)$ and T is the sampling interval. Here $\otimes$ is the Kronecker product and $\mathbf{I}_2$ is the identity matrix. The transition matrices corresponding to the three modes are, respectively, given by

$$\mathbf{F}^{(1)}(\mathbf{x}(k)) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \otimes \mathbf{I}_2$$

for CV model; and

$$\mathbf{F}^{(j)}(\mathbf{x}(k)) = \begin{bmatrix} 1 & 0 & \frac{\sin(\Omega_k^{(j)}T)}{\Omega_k^{(j)}} & \frac{-(1-\cos(\Omega_k^{(j)}T))}{\Omega_k^{(j)}} \\ 0 & 1 & \frac{(1-\cos(\Omega_k^{(j)}T))}{\Omega_k^{(j)}} & \frac{\sin(\Omega_k^{(j)}T)}{\Omega_k^{(j)}} \\ 0 & 0 & \cos(\Omega_k^{(j)}T) & -\sin(\Omega_k^{(j)}T) \\ 0 & 0 & \sin(\Omega_k^{(j)}T) & \cos(\Omega_k^{(j)}T) \end{bmatrix}$$

$j = 2, 3$

for clockwise CT model and anticlockwise CT model, where the mode-conditioned turning rates are

$$\Omega_k^{(2)} = -\Omega_k^{(3)} = \frac{a_m}{\sqrt{(\dot{x}(k) + \dot{x}^o(k))^2 + (\dot{y}(k) + \dot{y}^o(k))^2}}$$

Here $a_m > 0$ denotes maneuver acceleration. Note that the turning rate is expressed as a function of target velocity, and thus mode 2 and 3 are clearly nonlinear transitions. Assuming $\mathbf{v}(k) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$, where $\mathbf{Q} = \sigma_a^2 \mathbf{I}_2$. $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$, where $\mathbf{R} = \text{diag}\left\{\sigma_x^2, \sigma_y^2\right\}$.

In the simulation, we assume that the ownship is stationary and located at $(0, 0)$, and the target is always observable.

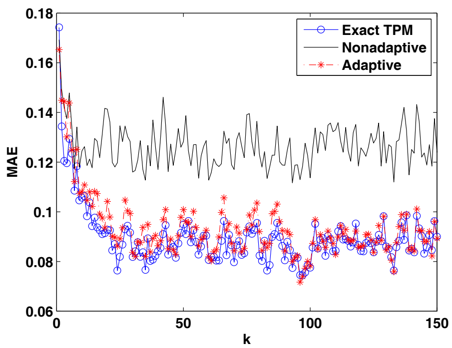
We set the parameters $T = 2$min, $\sigma_a = 1$km/h, $\sigma_x = 0.2$km, $\sigma_y = 0.2$km, $a_m = 2000\text{km/h}^2$. The number of particles is selected as $N = 1800$. The *a priori* mode probabilities are assumed as $\mu_{0,1} = 0.8$, $\mu_{0,2} = \mu_{0,3} = 0.1$. The valid range of $\mathbf{P}$ is assumed as $P_{ii} \in [0.6, 0.99]$ for $i = 1, 2, 3$, and $P_{ij} \in [0.01, 0.3]$ for $i \neq j$. We assume that the TPM is non-random, and the exact $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}. \tag{25}$$

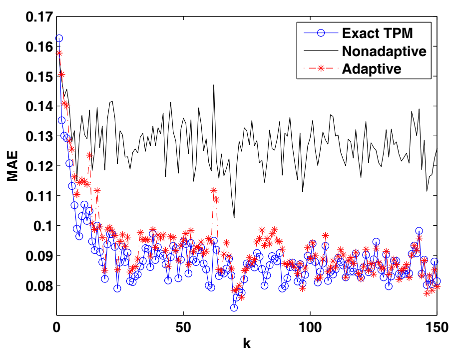
The non-adaptive TPM is assumed as $\frac{1}{3}[\mathbf{1}, \mathbf{1}, \mathbf{1}]^T$, where all the entries of $\mathbf{1}$ is 1. Under the assumption of non-random TPM, the TPM is initialized by $\frac{1}{3}[\mathbf{1}, \mathbf{1}, \mathbf{1}]^T$ and it is adaptively estimated using the ML criterion. For convenience, that the TPM be adaptively estimated, non-adaptively given, and known exactly are referred to as adaptive, non-adaptive, and exact TPM, respectively. The true initial target state is assumed as $\mathbf{x}(0) = [10\text{km}, 10\text{km}, 80\text{km/h}, 80\text{km/h}]$, and the initial particles are drawn from $\mathcal{N}(\mathbf{x}(0), \mathbf{S})$, where $\mathbf{S} = 10\mathbf{I}_4$. 200 Monte Carlo runs are performed to simulate the base state estimate error and the convergence of the TPM estimate. In each run, a true mode sequence is generated according to the true TPM. Because there's no observation at $k = 0$, we don't solve (22) at $k = 1$.

The base state is estimated using the IMMPF under the TPM estimate at previous time step, such that $\hat{x}(k+1) = E[x(k+1)|\bar{\mathbf{P}}(k), Z^{k+1}]$. Fig. 1 shows the curves of the mean absolute error (MAE) of the base state estimate versus time, from which we can see that the MAE under adaptive TPM is less than that under non-adaptive TPM, and approaches that under exact TPM. In Fig. 2, the curves of the convergence of the TPM estimate versus time are plotted. It is seen that the TPM estimate converges in less than 100 time steps.

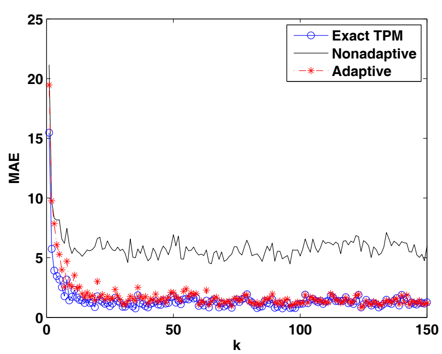
In order to avoid particle impoverishment which may cause the error that is increased as time, we use the technique similar to the



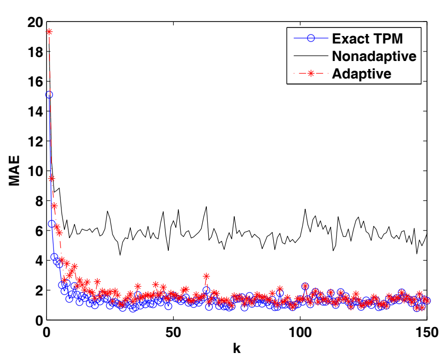
(a) MAE at x-position



(b) MAE at y-position



(c) MAE at x-velocity



(d) MAE at y-velocity

Fig. 1. The MAE of the base state estimate.

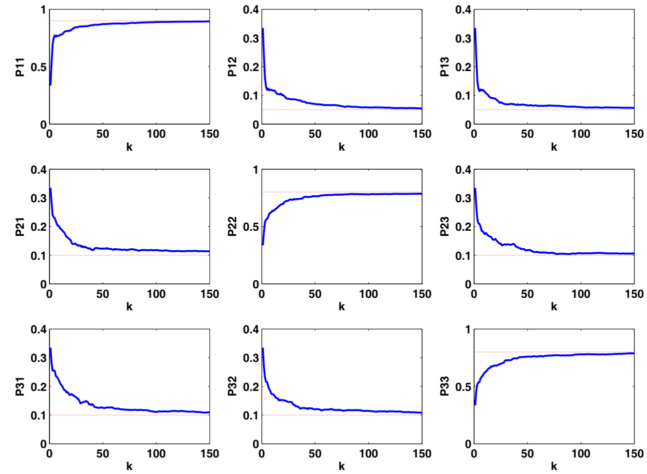


Fig. 2. The convergence of the components of $\mathbf{P}$: dashed lines are true values, solid thick lines are estimates.

regularized particle filter (RPF) when the step of interaction resampling is carried on in IMMPF [3], and the Gaussian kernel [5] is used to construct a continuous pdf in the simulation.

5. CONCLUDING REMARKS

In this paper we have proposed a new method to estimate the TPM in hybrid systems via convex optimization according to the MAP/ML criterion, where the IMMPF is employed to deal with nonlinear/non-Gaussian problems. Simulation results show the efficiency of the proposed method, and the good performance through a maneuvering target tracking example.

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