# TWO-DIMENSIONAL FREQUENCIES ESTIMATION USING JOINT DIAGONALIZATION

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## ABSTRACT

A new algorithm for estimating the frequencies of the two-dimensional (2-D) exponentials is presented. Via joint diagonalization of a set of matrices, the estimation of the automatically paired 2-D frequency parameters are obtained. Hence the proposed algorithm can avoid the computationally expensive 2-D parameters pairing procedure. Simulation results validate the high performance of the new algorithm.

***Index Terms*—** Frequency estimation, two-dimensional sinusoid, joint diagonalization, spatial smoothing, harmonic retrieval.

## 1. INTRODUCTION

Two-dimensional (2-D) frequencies estimation has been one of the central problems in many fields such as radar imaging, medical imaging, feature extraction, array processing, and wireless communications. The 2-D FFT method for 2-D frequencies estimation is simple but it suffers from the Rayleigh resolution limit. To improve the resolution, a variety of superresolution techniques have been developed. The maximum likelihood (ML) method [1] requires multidimensional searching in the parameter space and its computational load is heavy. The 2-D MUSIC algorithm [2] searches for the peaks of the 2-D orthogonality-measure spectrum to obtain the estimates of the 2-D frequencies, so its computational complexity is still high due to 2-D search. The matrix enhancement and matrix pencil (MEMP) method in [3] can avoid any search procedure and can solve the problem of multiple 2-D sinusoids sharing a same frequency component by utilizing an enhanced matrix instead of the original data matrix. However it requires computationally expensive 2-D parameters pairing procedure and the incorrect pairs are always obtained when there are repeated frequencies.

In this paper, we develop a new 2-D frequency estimation algorithm based on joint diagonalization. The joint diagonalization method was firstly used for the 1-D frequency estimation in [4]. Here we consider the more complicated case, i.e. we extend it to the 2-D case. By jointly diagonalizing a set of matrices, the estimation of frequency parameter matrix is obtained and the 2-D frequency pairs can be directly derived from it. The main advantage of the proposed algorithm is that additional 2-D parameters pairing can be avoided.

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## 2. PROBLEM FORMULATION

Consider $K$ superimposed 2-D complex-valued sinusoidal signals:

$$x_{m,n} = \sum_{k=1}^{K} s_k \exp(j2\pi f_{1k} m + j2\pi f_{2k} n) + v_{m,n} \qquad (1)$$

where $x_{m,n}$ represents the 2-D observed signal and $m = 0, \cdots, M-1$, $n = 0, \cdots, N-1$ with $(M, N)$ being the size of the observed data matrix. $s_k$ and $\{f_{1k}, f_{2k}\}$ are the amplitude and frequency pair of the $k$th signal respectively. $v_{m,n}$ is assumed to be 2-D additive complex white noise with variance $\sigma^2$. The 2-D harmonic retrieval issue is to estimate the frequency pairs $\{f_{1k}, f_{2k}\}$ from the $M \times N$ observed data.

The matrix form of (1) can be written as

$$\tilde{\mathbf{x}} = \tilde{\mathbf{A}}\mathbf{s} + \tilde{\mathbf{v}} \qquad (2)$$

with the column-ordered vectors

$$\tilde{\mathbf{x}} = [x_{0,0}, x_{1,0}, \cdots, x_{M-1,0}, x_{0,1}, \cdots, x_{M-1,1}, \cdots, x_{0,N-1}, \cdots, x_{M-1,N-1}]^T \qquad (3)$$

$$\mathbf{s} = [s_1, \cdots, s_K]^T \qquad (4)$$

$$\tilde{\mathbf{A}} = [\tilde{\mathbf{a}}_1, \cdots, \tilde{\mathbf{a}}_K] \qquad (5)$$

where the superscript $T$ denotes transpose and $\tilde{\mathbf{v}}$ is the noise term. The $k$th column of $MN \times K$ matrix $\tilde{\mathbf{A}}$ is

$$\tilde{\mathbf{a}}_k = [\, 1, e^{j2\pi f_{1k}}, \cdots, e^{j2\pi(M-1)f_{1k}}, e^{j2\pi f_{2k}}, \cdots, e^{j2\pi(M-1)f_{1k}+j2\pi f_{2k}}, \cdots, e^{j2\pi(N-1)f_{2k}}, \cdots, e^{j2\pi(M-1)f_{1k}+j2\pi(N-1)f_{2k}} \,]^T. \qquad (6)$$

One can find that the subspace method can not be directly applied based on equation (2) because the full rank correlation matrix of $\tilde{\mathbf{x}}$ can not be constructed. In order to restore the full rank of the correlation matrix we adopt the 2-D spatial smoothing method which will be described in detail in the following section.

## 3. JOINT DIAGONALIZATION METHOD FOR 2-D FREQUENCIES ESTIMATION

### 3.1. 2-D Spatial Smoothing

The concept of spatial smoothing was firstly used for estimating the direction-of-arrival of the coherent sources and it was extended to two dimensions in [2]. As illustrated in Fig. 1, the $M \times N$ observed data array can be partitioned into $L$ overlapping subarrays with dimensions $p_1 \times p_2$. Obviously the number of the subarrays is

$$L = (M - p_1 + 1)(N - p_2 + 1). \tag{7}$$

Denoting $\mathbf{X}_l$ as the data matrix of the $l$th subarray, the following pseudo code clearly depicts how to construct $\mathbf{X}_l$.

$l = 0$;
for $p = 0 : 1 : M - p_1$
  for $q = 0 : 1 : N - p_2$
    $l = l + 1$;

$$\mathbf{X}_l = \begin{bmatrix} x_{p,q} & \cdots & x_{p,q+p_2-1} \\ x_{p+1,q} & \cdots & x_{p+1,q+p_2-1} \\ \vdots & \ddots & \vdots \\ x_{p+p_1-1,q} & \cdots & x_{p+p_1-1,q+p_2-1} \end{bmatrix} \tag{8}$$

  end
end

Let $\mathbf{x}_l$ denote the column-ordered vector of $\mathbf{X}_l$, i.e.

$$\mathbf{x}_l = \text{vec}(\mathbf{X}_l) \tag{9}$$

where vec denotes concatenation of the columns of a matrix. Then $\mathbf{x}_l$ can be expressed as

$$\mathbf{x}_l = \mathbf{A}\mathbf{s}_l + \mathbf{v}_l \tag{10}$$

where

$$\mathbf{s}_l = [e^{j2\pi p f_{11} + j2\pi q f_{21}} s_1, \cdots, e^{j2\pi p f_{1K} + j2\pi q f_{2K}} s_K]^T \tag{11}$$

$$\mathbf{A} = [\mathbf{a}_1, \cdots, \mathbf{a}_K] \tag{12}$$

and $\mathbf{v}_l$ is the corresponding noise term. Notice that $p_1p_2 \times K$ matrix $\mathbf{A}$ is different from $MN \times K$ matrix $\tilde{\mathbf{A}}$, and the $k$th column of $\mathbf{A}$ is

$$\begin{aligned} \mathbf{a}_k = [\, & 1, e^{j2\pi f_{1k}}, \cdots, e^{j2\pi(p_1-1)f_{1k}}, \\ & e^{j2\pi f_{2k}}, \cdots, e^{j2\pi(p_1-1)f_{1k}+j2\pi f_{2k}}, \cdots, \\ & e^{j2\pi(p_2-1)f_{2k}}, \cdots, e^{j2\pi(p_1-1)f_{1k}+j2\pi(p_2-1)f_{2k}} \,]^T. \end{aligned} \tag{13}$$

Using 2-D spatial smoothing technique, the correlation matrix does not suffer from the rank deficient even in the case that there are multiple 2-D frequencies having a common 1-D frequency component.

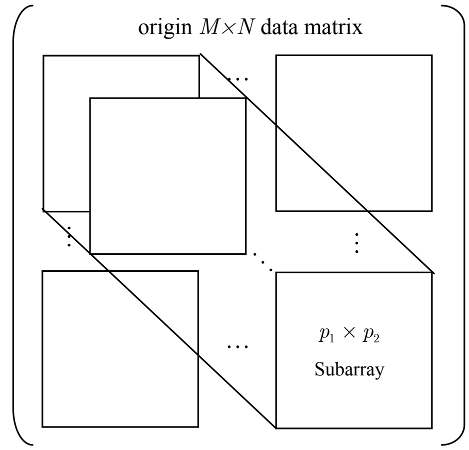


**Fig. 1**. The subarrays for 2-D spatial smoothing.

### 3.2. Eigen-Structure of the Observed Data

According to (10), the correlation matrix can be written as

$$\mathbf{R}_x = E\left\{\mathbf{x}_l\mathbf{x}_l^H\right\} = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma^2\mathbf{I} \tag{14}$$

where the superscript $H$ denotes complex conjugate transpose and $\mathbf{R}_s = E\left\{\mathbf{s}_l\mathbf{s}_l^H\right\}$ with its rank being $K$. The eigenvalue decomposition of $\mathbf{R}_x$ is given by

$$\mathbf{R}_x = \mathbf{U}_s\mathbf{\Lambda}_s\mathbf{U}_s^H + \sigma^2\mathbf{U}_n\mathbf{U}_n^H \tag{15}$$

where $\mathbf{\Lambda}_s = \text{diag}\{\lambda_1, \cdots, \lambda_K\}$ is a diagonal matrix containing the $K$ principal eigenvalues in descending order and $\mathbf{U}_s$ contains the corresponding orthonormal eigenvectors. $\mathbf{U}_n$ contains the $p_1p_2 - K$ orthonormal eigenvectors that correspond to the eigenvalue $\sigma^2$. The range space of $\mathbf{U}_s$ is called signal subspace and its orthogonal complement, named noise subspace, is spanned by $\mathbf{U}_n$.

Based on the $L$ subarrays' data, we can obtain the estimation of correlation matrix without rank deficient. To improve the accuracy of the frequency estimate, both forward smoothing and backward smoothing are utilized. The forward smoothed correlation matrix can be written as

$$\hat{\mathbf{R}}_{\text{F}} = \frac{1}{L}\sum_{l=1}^{L}\mathbf{x}_l\mathbf{x}_l^H \tag{16}$$

and the backward smoothed correlation matrix is

$$\hat{\mathbf{R}}_{\text{B}} = \mathbf{J}\hat{\mathbf{R}}_{\text{F}}^*\mathbf{J} \tag{17}$$

where $\mathbf{J}$ is the exchange matrix (i.e. anti-identity matrix) and the superscript $*$ denotes complex conjugate. The forward-backward smoothed correlation matrix is the average of $\hat{\mathbf{R}}_{\text{F}}$ and $\hat{\mathbf{R}}_{\text{B}}$, i.e.

$$\hat{\mathbf{R}}_{\text{FB}} = \frac{1}{2}\left(\hat{\mathbf{R}}_{\text{F}} + \hat{\mathbf{R}}_{\text{B}}\right). \tag{18}$$

Here we use $\hat{\mathbf{R}}_{\text{FB}}$ as the estimate of the true correlation matrix $\mathbf{R}_x$.

### 3.3. 2-D Frequencies Estimation via Joint Diagonalization

It is easy to see that $\mathbf{A}$ and $\mathbf{U}_s$ span the same range space, therefore there exists a $K \times K$ nonsingular matrix $\mathbf{W}$ satisfying

$$\mathbf{A} = \mathbf{U}_s \mathbf{W}. \tag{19}$$

Since the estimation of signal subspace $\hat{\mathbf{U}}_s$ can be calculated from $\hat{\mathbf{R}}_{\text{FB}}$, the estimation of frequency matrix $\hat{\mathbf{A}}$ can be obtained if we find the matrix $\mathbf{W}$. Based on the analytic constant modulus algorithm (ACMA) [5], we can derive an efficient method for seeking $\mathbf{W}$. Denote by $\mathbf{u}_m^H$ the $m$th row of $\mathbf{U}_s$ and $\mathbf{w}_k$ the $m$th column of $\mathbf{W}$, then

$$\mathbf{a}_k = \mathbf{U}_s \mathbf{w}_k = \begin{bmatrix} \mathbf{u}_1^H \mathbf{w}_k \\ \mathbf{u}_2^H \mathbf{w}_k \\ \vdots \\ \mathbf{u}_{p_1 p_2}^H \mathbf{w}_k \end{bmatrix}, \; k = 1, \cdots, K \tag{20}$$

All the elements of $\mathbf{a}_k$ have unit modulus, hence

$$\left|\mathbf{u}_m^H \mathbf{w}_k\right|^2 = \mathbf{w}_k^H \mathbf{u}_m \mathbf{u}_m^H \mathbf{w}_k = 1, \; m = 1, \cdots, p_1 p_2. \tag{21}$$

Equation (21) can be also written as

$$\text{vec}^T\left(\mathbf{u}_m \mathbf{u}_m^H\right)\left(\mathbf{w}_k \otimes \mathbf{w}_k^*\right) = 1 \tag{22}$$

where $\otimes$ denotes Kronecker product. Define a $p_1 p_2 \times K^2$ matrix $\mathbf{P}$ whose $m$th row is $\text{vec}^T\left(\mathbf{u}_m \mathbf{u}_m^H\right)$, then we can obtain the matrix form of (22)

$$\mathbf{P}\left(\mathbf{w}_k \otimes \mathbf{w}_k^*\right) = \mathbf{1} \tag{23}$$

where $\mathbf{1} = [1, \cdots, 1]^T$. Define a Householder transformation matrix

$$\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{q}\mathbf{q}^T}{\mathbf{q}^T\mathbf{q}}, \quad \mathbf{q} = \mathbf{1} - \|\mathbf{1}\|\, \mathbf{e}_1 \tag{24}$$

where $\mathbf{e}_1 = [1, 0, \cdots, 0]^T$. Left multiplying both sides of (23) by $\mathbf{H}$ leads to

$$\mathbf{H}\mathbf{P}\left(\mathbf{w}_k \otimes \mathbf{w}_k^*\right) = \|\mathbf{1}\|\, \mathbf{e}_1 \tag{25}$$

The matrix $\mathbf{HP}$ with the first row deleted is defined as a new $(p_1 p_2 - 1) \times K^2$ matrix $\mathbf{Q}$, therefore we obtain

$$\mathbf{Q}\left(\mathbf{w}_k \otimes \mathbf{w}_k^*\right) = \mathbf{0}. \tag{26}$$

Equation (26) means that $\mathbf{w}_k \otimes \mathbf{w}_k^*$ $(k = 1, \cdots, K)$ belongs to the null space of $\mathbf{Q}$. Each base vector of the null space of $\mathbf{Q}$, notated by $\mathbf{y}_i$ $(i = 1, \cdots, K)$, can be expressed as a linear combination of the vectors $\{\mathbf{w}_k \otimes \mathbf{w}_k^*\}_{k=1}^K$

$$\mathbf{y}_i = \sum_{k=1}^K c_{ki}\left(\mathbf{w}_k \otimes \mathbf{w}_k^*\right) = \sum_{k=1}^K c_{ki}\text{vec}\left(\mathbf{w}_k^* \mathbf{w}_k^T\right). \tag{27}$$

Define $K \times K$ matrix $\mathbf{Y}_i = \text{unvec}(\mathbf{y}_i)$ with unvec denoting the inverse vec operation, then we have

$$\mathbf{Y}_i = \sum_{k=1}^K c_{ki}\left(\mathbf{w}_k^* \mathbf{w}_k^T\right) = \mathbf{W}^* \mathbf{C}_i \mathbf{W}^T, \; i = 1, \cdots, K \tag{28}$$

where $\mathbf{C}_i = \text{diag}\{c_{1i}, \cdots, c_{Ki}\}$ is a diagonal matrix. Equation (28) is interesting and important since it indicates that we can jointly diagonalize the set of matrices $\{\mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_K\}$ to find $\mathbf{W}$. In other words, $\mathbf{W}$ is a joint diagonalizer of the set of matrices. Recently, a fast approximate joint diagonalization (FAJD) algorithm [6] has been proposed for avoiding not only the trivial solution but also any degenerate solutions. The FAJD algorithm minimizes the cost function

$$J(\mathbf{W}) = \sum_{k=1}^K \left\|\text{off}\left(\mathbf{W}^H \mathbf{Y}_k \mathbf{W}\right)\right\|_{\text{F}}^2 - \beta \log\left|\det(\mathbf{W})\right| \tag{29}$$

where $\det(\cdot)$ denotes the determinant of squared matrix, and $\text{off}(\cdot)$ zeros the diagonal elements of a matrix. The first term of the cost function is the squared off-diagonal error and the minus determinant term can avoid the trivial solution and any degenerate solutions. Note that the performance of equation (29) is independent of the choice of positive weight $\beta$ in the sense that related minimal solutions with different $\beta$ only differ by a scalar [6], therefore a simple choice of $\beta$ is $\beta = 1$. More details about FAJD algorithm can be found in [6].

Once we obtain $\mathbf{W}$, the frequency parameter matrix can be estimated using $\hat{\mathbf{A}} = \hat{\mathbf{U}}_s \mathbf{W}$, i.e. the estimation of each column vector $\hat{\mathbf{a}}_k$ $(k = 1, \cdots, K)$ is obtained.

It is easy to extract the 2-D frequency pair $\{f_{1k}, f_{2k}\}$ from the estimated vector $\hat{\mathbf{a}}_k$. Firstly the $p_1 p_2 \times 1$ vector $\hat{\mathbf{a}}_k$ is converted into the $p_1 \times p_2$ matrix $\hat{\mathbf{A}}_k$ shown in equation (30) which can be found in the top of the next page.

According to equation (30), the estimation of the first component $\hat{f}_{1k}$ can be derived from the phase difference between the consecutive elements of any column of $\hat{\mathbf{A}}_k$, and the second component $\hat{f}_{2k}$ can be derived from the phase difference between the consecutive elements of any row of $\hat{\mathbf{A}}_k$. We can adopt the average of the multiple estimation results as the final estimation for improving the accuracy.

Note that the 2-D frequencies are automatically paired, therefore the proposed algorithm can avoid 2-D parameter pairing.

## 4. SIMULATION RESULTS

We consider a typical simulation example adopted in many literatures such as [3][7]. There are three 2-D harmonics with frequency pairs $(f_{11}, f_{12}) = (0.26, 0.24), (f_{21}, f_{22}) = (0.24, 0.24)$, and $(f_{31}, f_{32}) = (0.24, 0.26)$. The sizes of the observed data and the subarray are set to $(M, N) = (20, 20)$

$$
\hat{\mathbf{A}}_k = \text{unvec}\left(\hat{\mathbf{a}}_k\right) = \begin{bmatrix} 1 & e^{j2\pi \hat{f}_{2k}} & \cdots & e^{j2\pi (p_2-1)\hat{f}_{2k}} \\ e^{j2\pi \hat{f}_{1k}} & e^{j2\pi (\hat{f}_{1k}+\hat{f}_{2k})} & \cdots & e^{j2\pi (\hat{f}_{1k}+(p_2-1)\hat{f}_{2k})} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j2\pi (p_1-1)\hat{f}_{1k}} & e^{j2\pi ((p_1-1)\hat{f}_{1k}+\hat{f}_{2k})} & \cdots & e^{j2\pi ((p_1-1)\hat{f}_{1k}+(p_2-1)\hat{f}_{2k})} \end{bmatrix} \quad (30)
$$

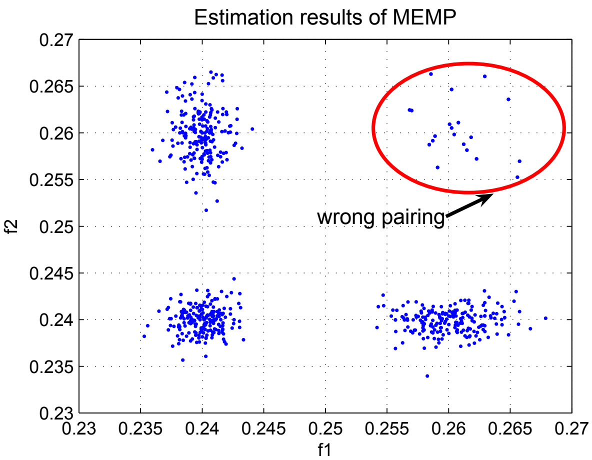


**Fig. 2**. Two hundred independent estimates of three 2-D frequencies using MEMP method.

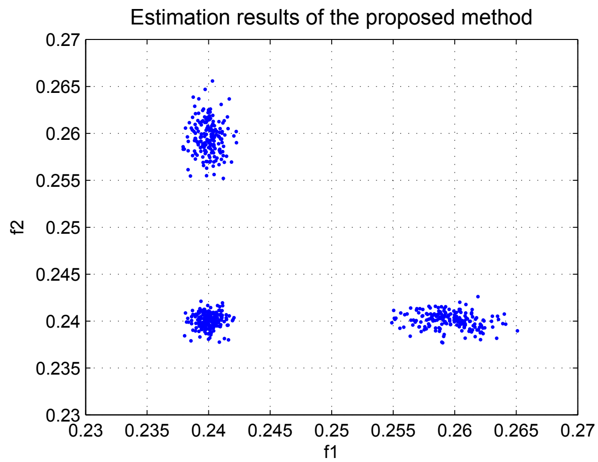


**Fig. 3**. Two hundred independent estimates of three 2-D frequencies using the proposed method.

and $(p_1, p_2) = (10, 10)$, respectively. The signal-to-noise ratio (SNR) is defined by

$$
\text{SNR} = 10 \log \left( \frac{1}{\sigma^2} \right). \quad (31)
$$

Figs. 2 and 3 illustrate the estimated frequencies for 200 Monte Carlo runs at SNR=15 dB using the MEMP method and the proposed algorithm, respectively. One can see that the wrong pair $(0.26, 0.26)$ was obtained using the MEMP method. The proposed algorithm never experienced such a problem since the two components of the frequency pairs were obtained simultaneously without any additional pairing procedure. In addition, the frequency estimations obtained by the new algorithm are more accurate than the MEMP method.

## 5. CONCLUSION

Based on joint diagonalization we develop a new algorithm for 2-D frequency estimation. The estimates of the 2-D frequencies are obtained without 2-D parameter paring by jointly diagonalizing a set of matrices. By avoiding any computationally demanding search in the parameter space, the proposed algorithm reduces greatly the computational complexity. Simulation results demonstrate that the proposed algorithm outperforms the MEMP method.

## 6. REFERENCES


[1] M. P. Clark, L. L. Scharf, "Two-dimensional modal analysis based on maximum likelihood," *IEEE Trans. Signal Process.*, vol. 42, no. 6, pp. 1443-1452, 1994.

[2] J. W. Odendaal, E. Barnard and W. I. Pistorius, "Two-dimensional superresolution radar imaging using the MUSIC algorithm," *IEEE Trans. Antennas Propag.*, vol. 42, no. 10, pp. 1386-1391, 1994.

[3] Y. Hua, "Estimating two-dimensional frequencies by matrix enhancement and matrix pencil," *IEEE Trans. Signal Process.* vol. 40, no. 9, pp. 2267-2280, 1992.

[4] O. J. Micka and A. J Weiss, "Estimating frequencies of exponentials in noise using joint diagonalization," *IEEE Trans. Signal Process.* vol. 47, no. 2, pp. 341-348, 1999.

[5] A.-J van der Veen and A. Paulraj, "An analytical constant modulus algorithm," *IEEE Trans. Signal Process.* vol. 44, no. 5, pp. 1136-1155, 1996.

[6] X.-L. Li and X.-D Zhang, "Nonorthogonal joint diagonalization free of degenerate solution," *IEEE Trans. Signal Process.* vol. 55, no. 5, pp. 1803-1814, 2007.

[7] F.-J. Chen, C. C. Fung, C.-W. Kok and S. Kwong, "Estimation of two-dimensional frequencies using modified matrix pencil method," *IEEE Trans. Signal Process.* vol. 55, no. 2, pp. 718-724, 2007.