ASYMPTOTIC GENERALIZED EIGENVALUE DISTRIBUTION OF TOEPLITZ BLOCK TOEPLITZ MATRICES

M. Oudin, J.P. Delmas

GET/INT
9 rue Charles Fourier
91011 Evry Cedex, France

ABSTRACT

In many detection and estimation problems associated with processing of second order stationary 2-D discrete random processes, the observation data are the sum of two zero-mean second order stationary processes: the process of interest and the noise process. In particular, the main performance criterion is the Signal to Noise Ratio (SNR). After linear filtering, the optimal SNR corresponds to the maximal value of a Rayleigh quotient which can be interpreted as the largest generalized eigenvalue of the covariance matrices associated with the signal and noise processes, which are Toeplitz block Toeplitz structured. In this paper, an extension of Szegö's theorem to the generalized eigenvalues of Hermitian Toeplitz block Toeplitz matrices is given, under the hypothesis of absolutely summable elements, providing information about the asymptotic distribution of those generalized eigenvalues and in particular of the optimal SNR after linear filtering.

Index Terms— Toeplitz block Toeplitz matrix, Szegö's theorem, generalized eigenvalues.

1. INTRODUCTION

2-D discrete random processes appear in many signal processing applications. A typical example is imagery, where observation data are modelized by such processes. Furthermore, in certain imagery applications, the 2-D discrete random processes are second order stationary (e.g., see [1, chap. 5]). Other examples can be found in different application areas, such as for instance in magnetic recording systems, where the magnetic medium can be modelled by a 2-D second order stationary process (e.g., see [2]), where the two dimensions correspond to the linear and radial coordinates. In that case, the covariance matrix of the random process is structured. More specifically, when a 2-D $n_1 \times n_2$ block of samples from a second order stationary random process is grouped into a vector by stacking its columns, the vector's covariance matrix has a Toeplitz block Toeplitz (TBT) structure (e.g., see [3]). In many detection and estimation problems associated with processing of such 2-D random processes, for which the main performance criterion is the Signal to Noise Ratio (SNR), the data are the sum of two zero-mean second order stationary processes: the process of interest and the noise process. For instance, in [2], the data are the sum of a channel input and a noise refered as a "flux noise". In that case, after linear filtering, the optimal SNR corresponds to the maximal value of a Rayleigh quotient which can be interpreted as the largest generalized eigenvalue of the covariance matrices associated with the signal and noise processes [4].

In this paper, we study the problem of the influence of the size of the samples vector along with the different dimensions, on the generalized eigenvalues of TBT matrices. More specifically, we assume that the number of samples along with both dimensions tends to infinity and analyze the asymptotic generalized eigenvalue distribution of such matrices.

The problem of the asymptotic eigenvalue distribution of Toeplitz matrices has first been analyzed by Grenander and Szegö, whose famous result asserts that the eigenvalues of a sequence of Hermitian Toeplitz matrices asymptotically behave like the samples of the Fourier transform of its entries [5]. However, this analysis has been performed by use of sophisticated mathematics under the general hypothesis that the Toeplitz matrix is generated by measurable and bounded spectrum. Then, Gray has proposed a simpler proof of this result for banded Toeplitz matrices [6] (which are called finite-order Toeplitz matrices) and then for infinite-order Toeplitz matrices, under the assumption of absolutely summable elements, based on the asymptotic equivalence between two matrices [7]. Following this approach and under the same assumption, the Grenander and Szegö result has been later extended to the eigenvalues of block Toeplitz with Toeplitz block matrices where both the size and the number of blocks tend to infinity [8], then to the eigenvalues and generalized eigenvalues of block Toeplitz with non-Toeplitz blocks where only the number of blocks tends to infinity in [9] and [4], respectively. Let us note that extension of the celebrated Grenander and Szegö result has been also considered in the mathematical literature under the general assumption that the involved Toeplitz, block Toeplitz matrices are generated by integrable spectra (e.g., see [10]) for the asymptotic eigenvalue

Thanks to the French MOD (DGA) for funding.

distribution. However, to the best of our knowledge, the extension to generalized eigenvalues of TBT matrices has not been studied until now. In the present work, we propose an extension of Szegö's theorem to the generalized eigenvalues of TBT matrices, under the hypothesis of absolutely summable elements, which relies on an extension of the notion of asymptotic equivalence between matrix sequences established by Gray in [6].

This paper is organized as follows. In Section 2, we give an interpretation of the generalized eigenvalues of Hermitian matrices from the point of view of SNR after linear filtering. Then, in Section 3, the notation of TBT matrices and Circulant Block Circulant (CBC) matrices are introduced and preliminary results about asymptotic equivalence between block matrix sequences are given. Finally, a generalized eigenvalue distribution theorem of TBT matrices is proved in Section 4.

2. SNR AFTER LINEAR FILTERING OF BIDIMENSIONAL PROCESSES

In this Section, we give an interpretation of the generalized eigenvalues of two Hermitian matrices. Thus, let $\mathbf{x}$ be a data vector where the 2-D data are arranged in alphabetical order of dimension $n_1 \times n_2$ composed by the sum of $\mathbf{s}$, the signal of interest and $\mathbf{n}$, the noise process. Both processes are modelled by second-order stationary processes with $n_1 n_2 \times n_1 n_2$ Hermitian covariance matrices $\mathbf{R}_s = E\{\mathbf{s}\mathbf{s}^H\}$ and $\mathbf{R}_n = E\{\mathbf{n}\mathbf{n}^H\}$, respectively. Since the processes are assumed to be second-order stationary, the covariance matrices are TBT structured. Moreover, due to the presence of background noise in $\mathbf{n}$, $\mathbf{R}_n$ is almost always positive definite. After linear filtering with $n_1 \times n_2$-dimensional filter $\mathbf{w}$ of the data $\mathbf{x}$, the SNR is given by the Rayleigh quotient

$$\mathrm{SNR} = \frac{\mathbf{w}^H \mathbf{R}_s \mathbf{w}}{\mathbf{w}^H \mathbf{R}_n \mathbf{w}}$$

whose stationary points are given by the generalized eigenvalues of $(\mathbf{R}_s, \mathbf{R}_n)$ (e.g., see [4]), denoted $\lambda_k(\mathbf{R}_s, \mathbf{R}_n)$ which are real-valued and strictly positive. Moreover, since $\mathbf{R}_n$ is positive definite and Hermitian, these generalized eigenvalues are given by the eigenvalues[1] of $\mathbf{R}_n^{-1}\mathbf{R}_s$, $\lambda_k(\mathbf{R}_s, \mathbf{R}_n) = \lambda_k(\mathbf{R}_n^{-1}\mathbf{R}_s)$. In particular, the maximum of this SNR is given by the maximum generalized eigenvalue of $(\mathbf{R}_s, \mathbf{R}_n)$. To prove our main result (theorem 1), we need some notations and background.

3. NOTATIONS AND PRELIMINARY RESULTS

We first recall the definition of the structured TBT and CBC matrices and extend the asymptotic equivalence of matrix sequences to block matrix sequences. Then, we give some preliminary lemmas necessary to prove our main result in Section 4.

Definition 1 (TBT matrix):

Given an index (n_1, n_2), a TBT matrix $\mathbf{A}^{n_1,n_2}$ is defined as:

$$\mathbf{A}^{n_1,n_2} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_{-1} & \cdots & \mathbf{A}_{-(n_1-1)} \\ \mathbf{A}_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{A}_{-1} \\ \mathbf{A}_{n_1-1} & \cdots & \mathbf{A}_1 & \mathbf{A}_0 \end{bmatrix}$$

and each block $\mathbf{A}_{m_1}$, $m_1 = -(n_1-1), ..., n_1-2, n_1-1$ is a $(n_2 \times n_2)$ Toeplitz matrix,

$$\mathbf{A}_{m_1} \stackrel{\text{def}}{=} \begin{bmatrix} a_{m_1,0} & a_{m_1,-1} & \cdots & a_{m_1,-(n_2-1)} \\ a_{m_1,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{m_1,-1} \\ a_{m_1,(n_2-1)} & \cdots & a_{m_1,1} & a_{m_1,0} \end{bmatrix}$$

The approach of this paper is to relate the generalized eigenvalues of TBT matrices to those of simpler structured associated CBC matrices that we now formalize.

Definition 2 (CBC matrix):

Given an index (n_1, n_2), a CBC matrix $\mathbf{C}^{n_1,n_2}$ is defined as a TBT matrix previously defined, where Toeplitz blocks are replaced by circulant blocks.

These CBC matrices have an eigenvalue decomposition (EVD) that extends those of circulant matrices (e.g., see [3, Th. 5.8.1]). More specifically, a $n_1 n_2 \times n_1 n_2$ CBC matrix $\mathbf{C}^{n_1,n_2}$ is diagonalizable by the unitary matrix

$$\mathbf{U}_{n_1,n_2} = \mathbf{U}_{n_1} \otimes \mathbf{U}_{n_2} \qquad (1)$$

where $(\mathbf{U}_{n_p})_{p=1,2}$ are the unitary discrete Fourier transform (DFT) matrices of terms $(\mathbf{U}_{n_p})_{k,l} = \frac{1}{\sqrt{n_p}} e^{-j2\pi \frac{(k-1)(l-1)}{n_p}}$, of size $n_p \times n_p$, where the associated eigenvalues are the 2-D DFT of its first row

$$\lambda_{k_1,k_2} = \sum_{m_1=0}^{n_1-1} \sum_{m_2=0}^{n_2-1} c_{m_1,m_2} e^{-j2\pi\left(\frac{m_1 k_1}{n_1} + \frac{m_2 k_2}{n_2}\right)}.$$

Now, we define 2-D asymptotic equivalence.

Definition 3 (2-D Asymptotic Equivalence):

Let $\{\mathbf{A}^{n_1,n_2}\}$ and $\{\mathbf{B}^{n_1,n_2}\}$ be sequences of $n_1 n_2 \times n_1 n_2$ matrices. These matrices are said to be 2-D asymptotically equivalent and noted $\mathbf{A}^{n_1,n_2} \sim \mathbf{B}^{n_1,n_2}$ if the following conditions hold:

[1] Note that these generalized eigenvalues are also given by the eigenvalues of the Hermitian matrix $\mathbf{R}_n^{-1/2}\mathbf{R}_s\mathbf{R}_n^{-1/2}$, but theorem 1 is more directly proved by using the eigenvalues of $\mathbf{R}_n^{-1}\mathbf{R}_s$.

- $\|\mathbf{A}^{n_1,n_2}\| \leq M < \infty$
- $\|\mathbf{B}^{n_1,n_2}\| \leq M < \infty$
- $\lim_{n_1,n_2\to\infty} |\mathbf{A}^{n_1,n_2} - \mathbf{B}^{n_1,n_2}| = 0$

where $\|.\|$ is the spectral norm and $|.|$ is a normalized Frobenius norm defined by

$$|\mathbf{A}^{n_1,n_2}|^2 = \frac{1}{n_1 n_2} \sum_{m_1=1}^{n_1} \sum_{m_2=1}^{n_2} |a_{m_1,m_2}|^2 .$$

Then, we give the following lemma about the asymptotic eigenvalue distribution of 2-D asymptotically equivalent matrices.

Lemma 1

Let $\{\mathbf{A}^{n_1,n_2}\}$ and $\{\mathbf{B}^{n_1,n_2}\}$ be sequences of 2-D asymptotically equivalent matrices of dimension $n_1n_2 \times n_1n_2$ with eigenvalues $\lambda_k(\mathbf{A}^{n_1,n_2})$ and $\lambda_k(\mathbf{B}^{n_1,n_2})$ for $k = 0...(n_1n_2 - 1)$, respectively, then for any positive integer s

$$\lim_{n_1,n_2\to\infty} \frac{1}{n_1n_2} \sum_{k=0}^{n_1n_2-1} (\lambda_k^s(\mathbf{A}^{n_1,n_2}) - \lambda_k^s(\mathbf{B}^{n_1,n_2})) = 0$$

and hence if either limit exists individually,

$$\lim_{n_1,n_2\to\infty} \frac{1}{n_1n_2} \sum_{k=0}^{n_1n_2-1} \lambda_k^s(\mathbf{A}^{n_1,n_2}) = \lim_{n_1,n_2\to\infty} \frac{1}{n_1n_2} \sum_{k=0}^{n_1n_2-1} \lambda_k^s(\mathbf{B}^{n_1,n_2}).$$

Proof: The proof is a mere extension of the proof of [7, Th. 2] to the 2-D case. ■

From now on, we consider Hermitian TBT matrices only, for which $a_{-i_1,-i_2} = a^*_{i_1,i_2}$ is equivalent to a real-valued Fourier transform $a(\omega_1,\omega_2) = \sum_{i_1,i_2} a_{i_1,i_2} e^{-j\sum_{p=1}^{2}\omega_p i_p}$.

To construct a sequence of CBC matrices that are 2-D asymptotically equivalent to $\{\mathbf{A}^{n_1,n_2}\}$ and whose eigenvalues are the samples of $a(\omega_1,\omega_2)$, we define the sequence

$$c^{n_1,n_2}_{m_1,m_2}(a) \stackrel{\text{def}}{=} \frac{1}{n_1n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} a(2\pi\frac{k_1}{n_1}, 2\pi\frac{k_2}{n_2}) e^{-j2\pi\sum_{p=1}^{2}\frac{m_pk_p}{n_p}} \tag{2}$$

and $\{\mathbf{C}^{n_1,n_2}(a)\}$, the sequence of CBC[2] matrices induced by $\{c_{i_1,i_2}(a)\}$. We note that $\mathbf{C}^{n_1,n_2}(a)$ may be written more compactly as

$$\mathbf{C}^{n_1,n_2}(a) = \mathbf{U}^H_{n_1,n_2} \mathbf{\Delta}_{n_1,n_2}(a) \mathbf{U}_{n_1,n_2} \tag{3}$$

where $\mathbf{U}_{n_1,n_2}$ is defined as in (1) and $\mathbf{\Delta}_{n_1,n_2}(a)$ is the $n_1n_2 \times n_1n_2$ diagonal matrix of elements $a(2\pi\frac{k_1}{n_1}, 2\pi\frac{k_2}{n_2})$, arranged in alphabetical order. We are now ready to state the following lemma, proved in [8, Lemma 1].

[2]The CBC matrix structure is easily shown by noticing that the sequence defined by (2) is periodic.

Lemma 2

Let $\{a_{i_1,i_2}\}$ be a Hermitian, absolutely summable sequence with Fourier transform $a(\omega_1,\omega_2)$. Let $\{c_{i_1,i_2}(a)\}$ be defined by (2). Then, the induced sequences of matrices $\{\mathbf{A}^{n_1,n_2}\}$ and $\{\mathbf{C}^{n_1,n_2}(a)\}$ are 2-D asymptotically equivalent.

4. TOEPLITZ BLOCK TOEPLITZ MATRICES GENERALIZED EIGENVALUE DISTRIBUTION THEOREM

The aim of this Section is to extend Szegö's theorem to the case of the generalized eigenvalues of Hermitian TBT matrices, under the assumption that the elements generating the matrices are absolutely summable.

We proceed as in [4] and prove three lemmas used for the proof of Theorem 1. More precisely, we first prove in Lemma 3 that the eigenvalues of Hermitian TBT matrices generated from an absolutely summable sequence are bounded by the minimum and maximum values of the 2-D Fourier transform of the sequence. Then, this lemma is used for the proof of the 2-D asymptotic equivalence between the inverse of a positive definite Hermitian TBT matrix and the inverse of its 2-D asymptotically equivalent CBC matrix, given by Lemma 4. Indeed, Lemma 3 shows that the spectral norm of the inverse of a positive definite Hermitian TBT matrix is bounded [4]. Furthermore, Lemma 5 shows that the product of the inverse of a positive definite Hermitian TBT matrix by a Hermitian BTB matrix is 2-D asymptotically equivalent to the product of the inverse of the Hermitian CBC matrix by a Hermitian CBC, both derived from (2). Finally, using this 2-D asymptotic equivalence and Lemma 1, we straightforwardly obtain Theorem 1.

Lemma 3

Let $\{a_{i_1,i_2}\}$ be a Hermitian, absolutely summable sequence with Fourier transform $a(\omega_1,\omega_2)$. Then, for all eigenvalues $\lambda(\mathbf{A}^{n_1,n_2})$ of the induced sequences of matrices $\{\mathbf{A}^{n_1,n_2}\}$, we have

$$m_a = \min_{\omega_1,\omega_2} a(\omega_1,\omega_2) \leq \lambda(\mathbf{A}^{n_1,n_2}) \leq \max_{\omega_1,\omega_2} a(\omega_1,\omega_2) = M_a.$$

Proof: This lemma is proved in the first step of the proof of [8, Lemma 1]. ■

Lemma 4

Let $\mathbf{B}^{n_1,n_2}$ be a positive definite Hermitian, TBT matrix generated from the absolutely summable sequence $\{b_{i_1,i_2}\}$ with Fourier transform $b(\omega_1,\omega_2)$ and the associated 2-D asymptotically equivalent MCBC matrix $\mathbf{C}^{n_1,n_2}(b)$ as defined in Lemma 2. If $\min_{\omega_1,\omega_2} b(\omega_1,\omega_2) = m_b > 0$, then

$$(\mathbf{B}^{n_1,n_2})^{-1} \sim (\mathbf{C}^{n_1,n_2}(b))^{-1}.$$

Proof: Using Lemma 3, the proof is similar to that of Lemma 3 in [4]. ■

Lemma 5

With the assumptions of Lemma 4, if $\mathbf{A}^{n_1,n_2}$ is a Hermitian TBT matrix generated by an absolutely summable sequence $\{a_{i_1,i_2}\}$, the associated CBC matrices $\mathbf{C}^{n_1,n_2}(a)$ and $\mathbf{C}^{n_1,n_2}(b)$ given by (2) satisfy

$$(\mathbf{B}^{n_1,n_2})^{-1}\mathbf{A}^{n_1,n_2} \sim (\mathbf{C}^{n_1,n_2}(b))^{-1}\mathbf{C}^{n_1,n_2}(a).$$

Proof: The proof is the same as for Lemma 4 in [4]. ■

We now introduce the interval

$$I_\omega = [\min_{\omega_1,\omega_2} b^{-1}(\omega_1,\omega_2)a(\omega_1,\omega_2); \max_{\omega_1,\omega_2} b^{-1}(\omega_1,\omega_2)a(\omega_1,\omega_2)]$$

and give a theorem about the asymptotic distribution of the generalized eigenvalues of Hermitian TBT matrices.

Theorem 1

Let $\mathbf{A}^{n_1,n_2}$ and $\mathbf{B}^{n_1,n_2}$ be two Hermitian TBT matrices, such that $\mathbf{B}^{n_1,n_2}$ is positive definite, and generated by absolutely summable sequences $\{a_{i_1,i_2}\}$ and $\{b_{i_1,i_2}\}$, respectively, with $\min_{\omega_1,\omega_2} b(\omega_1,\omega_2) = m_b > 0$. Then for all continuous functions F on I_ω

$$\lim_{n_1,n_2\to\infty} \frac{1}{n_1 n_2}\sum_{k=0}^{n_1n_2-1} F(\lambda_k(\mathbf{A}^{n_1,n_2},\mathbf{B}^{n_1,n_2})) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} F(b^{-1}(\omega_1,\omega_2)a(\omega_1,\omega_2))d\omega_1 d\omega_2.$$

Proof: Using Lemma 5, $(\mathbf{B}^{n_1,n_2})^{-1}\mathbf{A}^{n_1,n_2}$ is 2-D asymptotically equivalent to $(\mathbf{C}^{n_1,n_2}(b))^{-1}\mathbf{C}^{n_1,n_2}(a)$ and thanks to the similarity of $\mathbf{C}^{n_1,n_2}(b)$ and $\mathbf{C}^{n_1,n_2}(a)$ to the diagonal matrices $\mathbf{\Delta}_{n_1,n_2}(b)$ and $\mathbf{\Delta}_{n_1,n_2}(a)$ with the same unitary matrix $\mathbf{U}_{n_1,n_2}$ (3), we have using Lemma 1, for arbitrary integer s,

$$\lim_{n_1,n_2\to\infty} \frac{1}{n_1 n_2}\sum_{k=0}^{n_1n_2-1} [\lambda_k^s(\mathbf{A}^{n_1,n_2},\mathbf{B}^{n_1,n_2}) - \lambda_k^s(\mathbf{\Delta}_{n_1,n_2}^{-1}(b)\mathbf{\Delta}_{n_1,n_2}(a))] = 0$$

with

$$\lim_{n_1,n_2\to\infty} \frac{1}{n_1 n_2}\sum_{k=0}^{n_1n_2-1} \lambda_k^s(\mathbf{\Delta}_{n_1,n_2}^{-1}(b)\mathbf{\Delta}_{n_1,n_2}(a))]$$
$$= \lim_{n_1,n_2\to\infty} \frac{1}{n_1 n_2}\sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} b^{-s}(2\pi\frac{k_1}{n_1},2\pi\frac{k_2}{n_2})a^s(2\pi\frac{k_1}{n_1},2\pi\frac{k_2}{n_2})$$
$$= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} b^{-s}(\omega_1,\omega_2)a^s(\omega_1,\omega_2)d\omega_1 d\omega_2,$$

where the continuity of the Fourier transforms $b(\omega_1,\omega_2)$ and $a(\omega_1,\omega_2)$ guarantees the existence of the integral.

Finally, extending this result to any polynomial and after invoking the Stone-Weierstrass approximation theorem, Theorem 1 is proved. ■

As shown in [6, 7], and combined with the fact that for all vectors n_1, n_2, the eigenvalues of $(\mathbf{B}^{n_1,n_2})^{-1}\mathbf{A}^{n_1,n_2}$ lie in I_ω, Theorem 1 leads to the following corollary:

Corollary 1

For any positive integer l, the smallest and the largest l generalized eigenvalues of $(\mathbf{A}^{n_1,n_2},\mathbf{B}^{n_1,n_2})$ are convergent in n_1, n_2 and

$$\lim_{n_1,n_2\to\infty} \lambda_{n_1n_2-l+1}(\mathbf{A}^{n_1,n_2},\mathbf{B}^{n_1,n_2}) = \min_{\omega_1,\omega_2} b^{-1}(\omega_1,\omega_2)a(\omega_1,\omega_2)$$

and

$$\lim_{n_1,n_2\to\infty} \lambda_l(\mathbf{A}^{n_1,n_2},\mathbf{B}^{n_1,n_2}) = \max_{\omega_1,\omega_2} b^{-1}(\omega_1,\omega_2)a(\omega_1,\omega_2)$$

where the eigenvalues are ranked in decreasing order.

In particular, the asymptotic optimal SNR after linear filtering is given by

$$\lim_{n_1,n_2\to\infty} \text{SNR} = \max_{\omega_1,\omega_2} b^{-1}(\omega_1,\omega_2)a(\omega_1,\omega_2).$$

5. CONCLUSION

In this paper, we have given an extension of Szegö's theorem to the generalized eigenvalues of Hermitian TBT matrices using a simple proof under the hypothesis of absolutely summable elements, based on the notion of 2-D asymptotic equivalence between 2-D block matrix sequences.

6. REFERENCES

[1] W.K. Pratt, *Digital Image Processing*, Wiley-Interscience, 1978.

[2] P.A. Voois, J.M. Cioffi, "Upper Bounds on Achievable Storage Density: A Two-Dimensional Approach," *IEEE Trans. Magn.*, vol. 33, pp. 844-854, Jan. 1997.

[3] P.J. Davis, *Circulant Matrices*, *Pure and Applied Mathematics*, Wiley, New York, 1979.

[4] M. Oudin, J.P. Delmas, "Asymptotic generalized eigenvalue distribution of block Toeplitz matrices and application to space-time beamforming," in review to *IEEE Trans. Signal Processing*, 2007.

[5] U. Grenander, G. Szegö, *Toeplitz Forms and Their Applications*, University of California Press, Berkeley, 1958.

[6] R.M. Gray, "On the asymptotic eigenvalue distribution of Toeplitz matrices," *IEEE Trans. Inform. Theory*, vol. 18, pp. 725-730, Nov. 1972.

[7] R.M. Gray, "Toeplitz and circulant matrices: a review," *Foundations and Trends in Communications and Information Theory*, vol. 2, pp. 155-239, 2006.

[8] P.A. Voois, "A Theorem on the Asymptotic Eigenvalue Distribution of Toeplitz-Block-Toeplitz Matrices," *IEEE Trans. Signal Process.*, vol. 44, pp. 1837-1841, July 1996.

[9] H. Gazzah, P.A. Regalia, J.P. Delmas, "Asymptotic eigenvalue distribution of block Toeplitz matrices and applications to blind SIMO channel identification," *IEEE Trans. Inform. Theory*, vol. 47, pp. 1243-1251, March 2001.

[10] P. Tilli, "Asymptotic spectral distributions of Toeplitz-related matrices," in *Fast Reliable Algorithms for Matrices with Structure*, T. Kailath and A.H. Sayed, Eds. Philadelphia, PA:SIAM, pp. 153-187, 1999.