IMAGE RECONSTRUCTION FROM THE PHASE OR MAGNITUDE OF ITS COMPLEX WAVELET TRANSFORM

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ABSTRACT

This paper investigates the reconstruction of an image from the phase or magnitude of its complex wavelet transform (CWT). We view the CWT as an approximation to the analytic representation of some real wavelet coefficients and develop the conditions under which a 1D signal is uniquely specified by its analytic phase or magnitude. Then, we extend the uniqueness conditions to multi-resolution and higher dimensions in order to match the situation of the CWT. In the development of the uniqueness conditions, we also gain some insights about the quality of reconstructed images and the geometrical structure of the CWT phase and magnitude representation. Our results for the CWT may also be applied to other localized phase and magnitude representations.

***Index Terms*—** Wavelet transforms, Discrete Fourier transforms, Image representations

1. INTRODUCTION

The recent Complex Wavelet Transform (CWT) (see [1] and the references therein) provides multi-resolution and localized phase and magnitude to represent signals. In [2], we proposed a new image inpainting algorithm which estimates the localized CWT phase and magnitude with simple geometrical models and then reconstructs an inpainted image from the estimated phase and magnitude. If both the phase and magnitude can be estimated with our geometrical models accurately, a good estimate of the image can be reconstructed immediately by the inverse CWT. However, for many images, only one of the phase and magnitude can be described by our simple models with reasonable accuracy due to the complex nature of images. Therefore, we have to face the problem of image reconstruction from only its CWT phase or magnitude. Similar reconstruction problem may be encountered in many other image processing applications involving the modeling and estimation of localized phase and/or magnitude, or in image representation with partial information. For example, an algorithm for image reconstruction from Gabor-like localized phase has been proposed and analyzed in [3, 4].

We observed that for most images, if either the CWT phase or magnitude can be estimated correctly, the image can be reconstructed perfectly or with very high visual and objective quality by POCS-like iterative algorithms [2]. Typically, the reconstruction from phase has better fidelity than the reconstruction from magnitude. In this paper, we address a fundamental problem: under what conditions an image can be reconstructed from its CWT phase or magnitude. We propose to view the CWT phase and magnitude as an approximation to the analytic representation of some real wavelet coefficients and develop the conditions under which a 1D signal is uniquely specified by its analytic phase or magnitude. Then, we discuss the extension of the uniqueness conditions to multi-resolution and higher dimensions. We also discuss some insights about the difference between phase and magnitude, the quality of the reconstructed image, and geometrical structure of the phase and magnitude representation. Our results on the CWT may also be applicable to other localized phase and magnitude representations.

This work is related to the signal reconstruction from its Fourier phase or magnitude. It is well known that under certain conditions, a signal is uniquely specified by its Fourier phase or magnitude and may be reconstructed with iterative algorithms [5, 6, 7, 8]. The reconstruction from the CWT phase or magnitude shows significantly different characteristics from the reconstruction from Fourier phase or magnitude. First, symmetric signals (e.g., edges) are not unique given their Fourier phase or magnitude, but they can be reconstructed from only their CWT phase or magnitude. Second, the reconstruction from Fourier magnitude requires significant amount of information of the phase (in [8], the sign of every phase variable is required), while the reconstruction from the CWT magnitude without any phase information is very common.

This paper is organized as follows. Section 2 discusses the analytic representation of real signals and briefly review the CWT. Section 3 develops the uniqueness conditions of the 1D analytic phase or magnitude representation and discuss its extension to multi-resolution and higher dimensions. Section 4 shows some simulation examples and section 5 concludes the paper.

2. THE COMPLEX WAVELET TRANSFORM

2.1. The analytic representation

A real-valued signal $s(t)$ can be viewed as the real part of a complex-valued analytic signal $c(t) = s(t) + \boldsymbol{j}\,\widehat{s}(t)$, where $\widehat{s}(t)$ denotes the Hilbert transform of $s(t)$. In the frequency domain,

$$C(\omega) = S(\omega)\ F_a(\omega) = \begin{cases} 2S(\omega), & \omega > 0 \\ S(0), & \omega = 0 \\ 0, & \omega < 0 \end{cases}$$

where $F_a(\omega)$ is the frequency response of an analytic filter $f_a(t)$ which suppresses all the negative frequencies. The phase and magnitude of the analytic signal reveal fundamental signal characteristics [9]: the magnitude defines the amplitude of local sinusoid and the phase relates to small local shifts.

2.2. The complex wavelet transform

The complex wavelet transform (CWT) is a multi-resolution and localized signal representation. In its magnitude and phase form, the

CWT decomposes a discrete signal $s(n)$ into a set of magnitudes $\rho(n;k)$ and phases $\theta(n;k)$, where k is in an index set Φ of scales. The CWT magnitude represents a smoothed measurement of the local signal energy for the designated frequency band, and the CWT phase indicates the location of that energy relative to the position of each coefficient.

$$\left\{\rho(n;k)e^{j\,\theta(n;k)} = c(n;k) : k \in \Phi\right\} = \mathrm{CWT}(s(n))$$

We argued in [2] that an image may be reconstructed from either its CWT phase or magnitude with POCS-like iterative algorithms (exemplified in Fig. 1 [1]), because the CWT is close to a localized Fourier transform. However, we observed that the reconstruction from the CWT has two important properties which the Fourier transform lacks: (1) important symmetric signals (e.g., edges) are unique given the phase or magnitude; (2) the reconstruction from the magnitude requires no extra phase information.

(a) (b) (c) (d)

Fig. 1. The reconstruction results from (a) the Fourier phase (perfect), (b) the Fourier magnitude (PSNR = 20.2dB), (c) the CWT phase (perfect), and (d) the CWT magnitude (PSNR = 44.1dB)

To investigate the reconstruction from the CWT phase or magnitude, we adopt the view point of the analytic representation. The frequency responses of the CWT filterbank on the first three scales are shown in Fig. 2. The filter for each scale approximates the concatenation of a real wavelet bandpass filter and an analytic filter. We propose to view the CWT phase and magnitude in each scale as the analytic phase and magnitude of the corresponding real bandpass wavelet coefficients. In the next section, we connect the analytic phase and magnitude to the Fourier phase and magnitude, develop the conditions under which a signal is uniquely specified by its analytic phase and magnitude, and discuss its extension to multiresolution and higher dimensions to match the situation of the CWT.

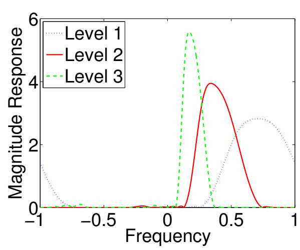

Fig. 2. The frequency response of the CWT

3. THE UNIQUENESS IN TERMS OF ANALYTIC PHASE OR MAGNITUDE

3.1. 1D analytic phase and magnitude

In this subsection, we consider 1D discrete signal $x \in \mathbb{R}^N$. Suppose a discrete analytic filter $F \in \mathbb{C}^{N\times N}$ is used to construct the analytic phase $\theta(x) = \angle(Fx)$ and magnitude $\rho(x) = |Fx|$. If circular boundary extension is used, F is circulant and diagonalizable by the DFT matrix W:

$$Fx = W^{-1}\Lambda_F W x = W^{-1}(\Lambda_F W x)$$

where the diagonal matrix Λ_F contains the frequency response of the analytic filter. Therefore, the analytic phase and magnitude are the Fourier phase and magnitude of sequence $\Lambda_F W x$ (ignore the difference between W and W^{-1}). The advantage of analytic representation over Fourier transform is two-fold. First, the analytic filter can be easily designed to have compact support to achieve time or spatial localization. Second, symmetric x becomes non-symmetric after the mapping to $\Lambda_F W x$ allowing unique representation with phase or magnitude.

After recognizing the connection to the Fourier transform, we know the theory for Fourier phase and magnitude [5, 6, 7, 8] applies to sequence $\Lambda_F W x$ instead of x. However, we found that the global uniqueness conditions for Fourier magnitude differ significantly from our observation: reconstruction from the CWT magnitude is much more common than the reconstruction from Fourier magnitude. Therefore, we use a new method to develop the local uniqueness conditions for analytic phase and magnitude.

In order to be general for both the redundant transforms and critically down-sampled transforms, we assume that x is a bandpass signal living in a known DFT frequency band of $[K_1, K_2]$ ($0 \le K_1 \le K_2 < \frac{N}{2}$). Again for generality, we only assume that the analytic filter $F = A + \boldsymbol{j}\,B$ ($A, B \in \mathbb{R}^{N\times N}$) is circulant and suppresses all the negative frequency components.

The local uniqueness of analytic phase $\theta(x) = \angle(Fx)$ and magnitude $\rho(x) = |Fx|$ can be determined by the Jacobians of $\theta(x)$ and $\rho(x)$ with respect to x. Suppose $\rho(x)$ has no zero element (*i.e.*, $\rho(x) > 0$), then, the Jacobians exist:

$$J_{\theta(x)} = D^{-2}_{\rho(x)}\,(D_{Ax}B - D_{Bx}A)$$
$$J_{\rho(x)} = D^{-1}_{\rho(x)}\,(D_{Ax}A + D_{Bx}B)$$

where D_u denotes the square matrix with vector u on its diagonal. The Jacobians specify the tangent plane of $\theta(x)$ and $\rho(x)$ at x. Given the band $[K_1, K_2]$ and $\theta(x)$ or $\rho(x)$, if for all $v \ne 0 \in \mathbb{R}^N$ in band $[K_1, K_2]$, $J_{\rho(x)}v \ne 0$ or $J_{\theta(x)}v \ne 0$, then x is locally unique within a small neighborhood of x. For phase, local uniqueness implies global uniqueness, because if $\theta(x) = \theta(y)$, then $\theta(ax+by) = \theta(x)$ for all $a, b \ge 0$.

Let $S_x(z)$ be the z transform of WFx. Then, $S_x(z)$ is non-zero only in the range between z^{-K_1} and z^{-K_2}.

$$S_x(z) = \sum_{k=K_1}^{K_2} a_k z^{-k}$$

Since the analytic filter F removes all the negative frequencies, the number of non-zero terms in $S_x(z)$ is at most $\frac{N}{2}$.

Theorem 1 (The uniqueness given phase or magnitude). *With $S_x(z)$ defined above, if and only if at least one of a_{K_1} and a_{K_2} is non-zero and $S_x(z)$ has no zeros on the unit circle or in complex conjugate reciprocal pairs, within frequency band $[K_1, K_2]$, (1) given the analytic phase $\theta(x)$, x is globally unique up to a scale factor ; (2) given the analytic magnitude $\rho(x)$, x is locally unique when $a_0 \ne 0$ or locally unique up to a phase shift when $a_0 = 0$.*

[1] All the figures and results in this paper are generated with an implementation of the CWT following [2].

Proof: The outline of the proof is as follows. First, by the definition of the Jacobians, $\Delta\rho = J_{\rho(x)}\Delta x$, $\Delta\theta = J_{\theta(x)}\Delta x$, and $\Delta\rho + \boldsymbol{j}\, D_{\rho(x)}\Delta\theta = D_{e^{-j\,\theta(x)}}F\Delta x$ for all $\Delta x \in \mathbb{R}^N$. Let $v = \Delta x$, we have

$$\begin{aligned}(Fx)^* \odot (Fv) &= D_{\rho(x)} D_{e^{-j\,\theta(x)}}(Fv) \\ &= D_{\rho(x)}(\Delta\rho + \boldsymbol{j}\, D_{\rho(x)}\Delta\theta)v \\ &= (D_{\rho(x)}J_{\rho(x)} + \boldsymbol{j}\, D^2_{\rho(x)}J_{\theta(x)})v \end{aligned} \quad (1)$$

where $\odot$ and $*$ denote element-wise multiplication and complex conjugation respectively [2]. Therefore, $J_{\theta(x)}v = 0$ or $J_{\rho(x)}v = 0$ is equivalent to $(Fx)^* \odot (Fv)$ being pure real or pure imaginary respectively.

Second, let the z transform of WFv be $S_v(z)$ (in the same way as $S_x(z)$), then the z transform of $W((Fx)^* \odot (Fv))$ is the polynomial $S(z) = S_x^*(1/z^*)S_v(z)$. Therefore, $(Fx)^* \odot (Fv)$ being pure real or pure imaginary is equivalent to $S(z)$ being complex conjugate symmetric or anti-symmetric about z^0 respectively. According to the following lemma, the zeros of $S_x^*(1/z^*)S_v(z)$ are on the unit circle or in complex conjugate reciprocal pairs.

Lemma. *A FIR sequence $S(z)$ has generalized complex conjugate symmetry (*i.e., $S^*(1/z^*) = e^{j\,\alpha}z^M S(z)$ *for some $\alpha \in \mathbb{R}$ and $M \in \mathbb{Z}$), if and only if the zeros of $S(z)$ are on the unit circle or in complex conjugate reciprocal pairs.*

Finally, if and only if $S_x(z)$ have no zeros on the unit circle or in complex conjugate reciprocal pairs, the symmetry of $S(z)$ about z^0 requires that $S_v(z) = re^{j\,\phi}S_x(z)$ $(r, \phi \in \mathbb{R})$. Combined with the fact $x, v \in \mathbb{R}^N$, we conclude that, for $v \neq 0$ in band $[K_1, K_2]$, (1) $J_{\theta(x)}v = 0$ if and only if $v = rx$; (2) $J_{\rho(x)}v = 0$ if and only if $a_0 = 0$ and $v = r\widehat{x}$ (where $\widehat{x}$ is the Hilbert transform of x). It is easy to show that if $a_0 = 0$ and $y = x\cos\alpha + \widehat{x}\sin\alpha$ $(S_y(z) = e^{j\,\alpha}S_x(z))$ for any $\alpha \in \mathbb{R}$, $\rho(x) = \rho(y)$.

Therefore, under the conditions stated in the theorem, x is globally unique up to a scale factor given $\theta(x)$ and x is locally unique up to at most a phase shift given $\rho(x)$.

□

The above uniqueness conditions are about the same for phase and magnitude. Under the uniqueness conditions and given the phase, the signal is living in a 1D linear subspace, therefore, POCS algorithm is guaranteed to converge to the right signal. Given the magnitude, in most cases we have $a_0 \neq 0$ (suppose appropriate down-sampling is performed), therefore, when the initial estimate is close to the right signal, POCS algorithm will still converge to the right signal without any phase information.

3.2. Discussions about the uniqueness theorem

A. Localized signal with Zero elements in the magnitude

In the above proof, we assume that the magnitude $\rho(x)$ contains no zero elements. For real life signals, the magnitude may be zero within smooth areas and be non-zero around edges. The uniqueness theorem may be extended in the following way to incorporate this situation. Suppose we know the locations of zero magnitude and we are only interested in signals with same set of zero magnitude (similar to the known frequency band in the above theorem). The k-th element of the magnitude is zero ($\rho_k(x) = 0$) if and only if $S_x(z)$ has a zero at $e^{-j\,\frac{2k\pi}{N}}$. Then, $S_v(z)$ in the above proof has to have the same zero to maintain $\rho_k(v) = 0$. Therefore, the conclusion above still holds.

[2] The $D_\rho^{-1}(x)$ terms on in $J_{\rho(x)}$ and $J_{\theta(x)}$ are all canceled out, therefore, the zero elements in $\rho(x)$ does not change the conclusion.

B. Geometry specified by phase and magnitude

The uniqueness theorem reveals that the geometry of the manifold specified by the analytic phase or magnitude is very similar to 2D concentric circles. For any $x \in \mathbb{R}^N$ satisfying the uniqueness conditions, any $y \in \{rx : r > 0\}$ have the same phase as x; any $y \in \{x\cos\alpha + \widehat{x}\sin\alpha : \alpha \in \mathbb{R}\}$ has the same magnitude as x, if x is zero mean. Starting from x, by continuously changing the phases α, we can keep the magnitude unchanged and travel all the way through $\widehat{x}$, $-x$ and $-\widehat{x}$, and then back to x again (just like traveling on a circle); or, we may continuously change r to reach bigger or smaller circles.

C. Singular values of the Jacobians

The largest singular values of $D_{\rho(x)}J_{\theta(x)}$ and $J_{\rho(x)}$ are both $\sqrt{2}$ and the associated right singular vectors are $\widehat{x}$ and x respectively ($\|\Delta\theta(x)\| \leq \sqrt{2}\|\Delta x\|$ and $\|\Delta\rho(x)\| \leq \sqrt{2}\|\Delta x\|$). That is, the largest changes in $\theta(x)$ and $\rho(x)$ come from the changes of x in the direction of $\widehat{x}$ and x respectively. Therefore, the phase is most effective in encoding the local shift (α in $\{x\cos\alpha + \widehat{x}\sin\alpha : \alpha \in \mathbb{R}\}$) and magnitude is most effective in encoding the local signal energy (r in $y \in \{rx : r > 0\}$).

The smallest singular value of $D_{\rho(x)}J_{\theta(x)}$ is 0 and the corresponding singular vector is x. In reconstruction from phase, we have to assume that $\|x\|$ is known. The smallest singular value of $J_{\rho(x)}$ is 0 or very close to 0 and the corresponding singular vector is $\widehat{x}$. That is, $x + r\widehat{x}$ has exactly or almost the same magnitude as x for small $r > 0$. Therefore, the reconstruction from magnitude without any information of phase is typically less accurate than the reconstruction from phase with known signal energy for very detailed structures.

3.3. Extension to multi-resolution

The extension of the uniqueness conditions to the multi-resolution case is straightforward. Suppose we first apply a real wavelet filter bank to decompose the signal $s(n)$ into real wavelet coefficients $\{x(n;k) : k \in \Phi\}$ and then apply the analytic filter F to transform $x(n;k)$ to analytic magnitude and phase $\{\rho(n;k), \theta(n;k)\}$. If the wavelet coefficients $x(n;k)$ in each band satisfies the uniqueness condition, $x(n;k)$ can be uniquely specified by $\rho(n;k)$ or $\theta(n;k)$. Since the signal x can be determined by the wavelet coefficients in all the bands $\{x(n;k) : k \in \Phi\}$, x must be uniquely specified by $\{\rho(n;k) : k \in \Phi\}$ or $\{\theta(n;k) : k \in \Phi\}$.

The multi-resolution decomposition greatly helps the reconstruction from magnitude. Lowpass bands are typically recovered first and give a good initial estimate of the highpass bands through the inter-band dependency and redundancy of the CWT.

3.4. Extension to higher dimensions

The extension of the uniqueness condition to higher dimensions follows from the extension of analytic filter to higher dimensions [10]. For example, consider a 2D signal $x \in \mathbb{R}^N$ with $\sqrt{N} \times \sqrt{N}$ pixels. The 2D magnitude $\rho(m,n)$ and phase $\theta(m,n)$ in each band is constructed by filtering $x(m,n)$ with a 2D analytic filter F with single quadrant frequency response. We can construct polynomials $S_x(z_1, z_2)$ and $S_v(z_2, z_2)$ in a similar way as for 1D and let $S(z_1, z_2) = S_x^*(1/z_1^*, 1/z_2^*)S_v(z_1, z_2)$. Since equation (1) in the uniqueness theorem holds for higher dimensions, we conclude that $S(z_1, z_2)$ should have no non-trivial symmetric factors ($f(z_1, z_2) = e^{j\,\alpha}z_1^{M_1}z_2^{M_2}f^*(1/z_1^*, 1/z_2^*)$) for the phase and magnitude to be unique.

4. SIMULATION EXAMPLES

In this section, we show some simple reconstruction results to illustrate the uniqueness conditions developed in the previous section. In Fig. 3, an anti-symmetric 1D edge with non-zero mean ($K_1 = 0$ and $a_{K_1} = a_0 \neq 0$) is reconstructed from its analytic phase or magnitude with POCS-like iterative algorithms. The reconstruction from phase is very accurate (relative error is 6.2×10^{-15}). Although the signal is only locally unique given its magnitude, we found that for edges, the reconstruction from magnitude usually converges to the right signal when starting with random phase. We observed that the reconstruction from magnitude has larger relative error than the reconstruction from the phase, which is due to the very small singular value associated with $\widehat{x}$ discussed in the previous section.

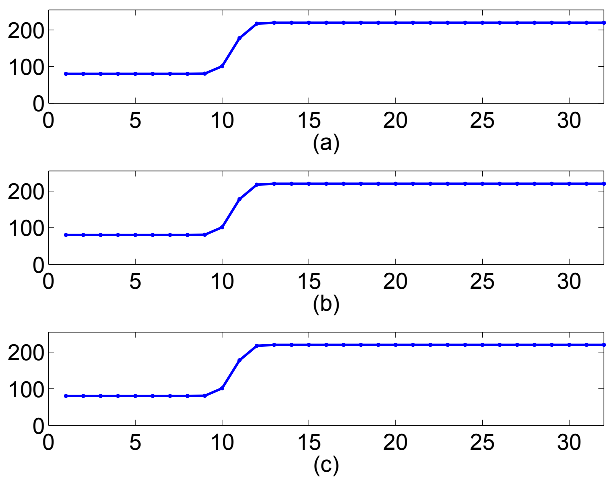

Fig. 3. The reconstruction from the analytic phase or magnitude: (a) x, the original signal with non-zero mean; (b) $\widehat{x}_\theta$, the reconstruction from analytic phase ($\|\widehat{x}_\theta - x\|_2/\|x\|_2 = 6.2 \times 10^{-15}$); (c) $\widehat{x}_\rho$, the reconstruction from analytic magnitude ($\|\widehat{x}_\rho - x\|_2/\|x\|_2 = 4.9 \times 10^{-6}$).

An example of 2D image reconstruction with given CWT phase or magnitude has already been given in Fig. 1. In many real life applications, we may have neither the phase nor the magnitude available and we have to rely on some modeling techniques to estimate either of them. In Fig. 4, we show some image reconstruction results with modeled CWT phase or magnitude. Although the phase and magnitude models used are very simple (see [2] for more details and examples), the reconstructed images have very good objective and visual quality.

5. CONCLUSIONS

This paper investigates the reconstruction of an image from its CWT phase or magnitude. We relate the localized CWT phase and magnitude to the analytic representation and develop the conditions under which a signal is unique with specified CWT phase or magnitude. Our results on the CWT may also be applied to other localized phase and magnitude representations.

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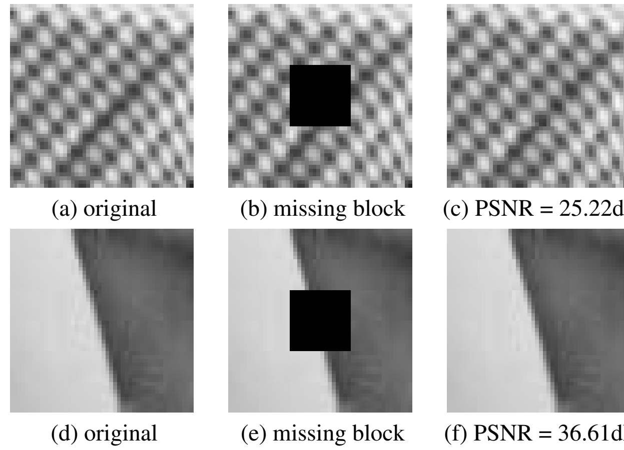

Fig. 4. The reconstruction of missing image blocks (image inpainting [2]): (a-c) a missing block (in Barbara) is reconstructed from modeled CWT phase; (d-f) a missing block (in Lena) is reconstructed from modeled CWT magnitude.

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