# OPTIMAL POWER ALLOCATION FOR SUM CAPACITY OF A SLOTTED THREE-CELL SYSTEM

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## ABSTRACT

Substantial gains are realizable through basestation cooperation during assignment of transmission powers to users. While binary power assignment is optimal for sum capacity in a two-cell system, it does not hold true in general for multi-cell systems. In this paper, we study an optimal power allocation for sum capacity in a three-cell system. The proposed branching algorithm is of linear computational complexity, and provides a power assignment which achieves a sum capacity arbitrarily close to the theoretical value. This result provides the impetus for next generation wireless systems to enable basestations to perform joint power assignment across multiple cells.

***Index Terms*—** Information rates, Power control, Basestation

## 1. INTRODUCTION

The paradigm of basestation (BS) cooperation during resource allocation is an attractive proposition for improving the overall system capacity. Cooperative schemes can be deployed either in a centralized manner at a Radio Network Controller (RNC) or in a distributed manner with peer-to-peer communication among BSs to optimize metrics such as the overall sum throughput, per user outage probability, per user utility based functions, or a combination thereof.

For a slotted two-cell system with transmit power constraints, [1, 2] shows that the sum capacity achieving optimal power allocation corresponds to binary power control, wherein each BS transmits to its user with either maximum or zero power. Additionally, the optimality loss incurred by extending binary power control scheme to a multi-cell scenario is shown to be negligible in certain propagation environments. The sum capacity of a three-cell system, however, is generally not achievable by the binary power control scheme. In this paper, we study the optimal power assignment to achieve sum capacity in a slotted three-cell system, assuming cooperative BSs. It is of theoretical interest, as well as practical importance to design joint BS power assignment schemes, which approach the theoretical sum capacity, irrespective of the propagation scenarios.

Our key contribution is a *linear complexity* branching algorithm that produces a power assignment to scheduled users in a slotted three-cell system, such that the resulting sum capacity can be arbitrarily close to the optimum sum capacity. The term “slot” refers to separating users within each cell, through one or more dimensions in either temporal, frequency, or orthogonal codes. Due to inter-cell interference, the sum capacity of a three-cell system, in general, is non-convex in the set of feasible power allocation vectors. The non-convex master optimization problem is divided into multiple subprograms with enumerable local optima. We specify an exit condition, which depends on a relative error target on the maximum difference between the computed and the theoretical sum capacity. Thus, the proposed branching algorithm guarantees that the computed sum capacity is arbitrarily close to the theoretical limit. Furthermore, the number of subprograms required to achieve the error threshold is shown to be a function of the channel gains.

## 2. THREE CELL SYSTEM MODEL

Consider a slotted three-cell system, in which each cell schedules a mobile for transmission during a slot. The BSs exchange information regarding the channel gains from each mobile through a central RNC (Figure 1). Denote $P_{max}$ as the maximum transmit power per BS, and $G_{i,j}$ as the channel gain from mobile $j$ to BS $i$, which is the composite channel gain including the effects of path loss, lognormal shadowing, and fast fading. Moreover, denote $\sigma^2$ as the variance of AWGN. For national simplicity, we use a normalized channel gain $g_{ij} = \frac{G_{ij}P_{max}}{\sigma^2}, 1 \leq i, j \leq 3$ throughout this paper.

For a given power assignment $P_1$,$P_2$ and $P_3$ to the scheduled mobile per cell, the achievable sum rate in the reverse link can be expressed as

$$R(P_1, P_2, P_3) \triangleq \log_2 \prod_{i=1}^{3} (1 + \mathrm{SINR}_i)$$

$$\text{where, } \mathrm{SINR}_i \triangleq \frac{P_i g_{ii}}{1 + \sum_{j \neq i} P_j g_{ij}} \forall \quad 1 \leq i \leq 3$$

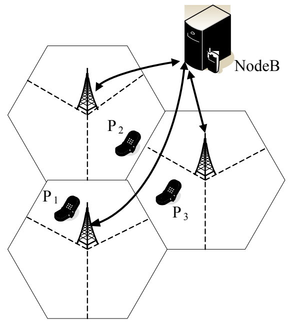


**Fig. 1**. Cooperative power assignment from BS to mobiles

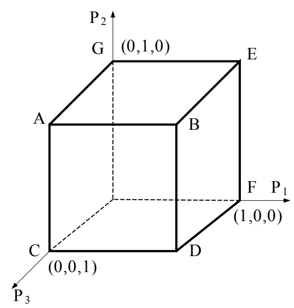


**Fig. 2**. Feasible region of power assignments for $M(\mathbf{P})$.

The BSs perform joint power allocation by solving an optimization problem, which involves maximizing the achievable sum rate, subject to a maximum transmit power constraint per mobile. The master optimization problem $M(\mathbf{P})$ can be formulated as,

$$\mathbf{P}_{opt} = \underset{(P_1,P_2,P_3)\in\mathcal{P}}{\text{argmax}} \; R(P_1, P_2, P_3) \qquad (1)$$

$$\mathcal{P} = \{(P_1, P_2, P_3) : 0 \le P_1, P_2, P_3 \le 1\}$$

Since the objective function is continuous over the compact feasible region $\mathcal{P}$, $M(\mathbf{P})$ has a solution, by the *Maximum-Minimum* theorem [3]. A few assumptions are made with respect to the optimization problem in (1): (a) the channel gains are constant over one slot; (b) each BS has global knowledge of the channel gains; (c) each user employs Gaussian distributed signaling; and (d) each user is capable of choosing a continuum of rates and powers.

Note that due to inter-cell interference, $M(\mathbf{P})$ is non-convex, which makes it difficult to solve. In this paper, we propose to branch $M(\mathbf{P})$ into a finite number of subprograms, each of which is easier to solve. In the following, we first state a proposition, according to the results in [1].

**Proposition 1** The solution to $M(\mathbf{P})$ lies on the boundary $\partial\mathcal{P}$ of the constraint set $\mathcal{P}$.

**Proof** For any $\alpha > 1$, $R(\alpha P_1, \alpha P_2, \alpha P_3) > R(P_1, P_2, P_3)$. This implies that given any feasible power vector in the interior of the cubic feasible region shown in Figure 2, it can be linearly scaled to lie on the boundary of $\mathcal{P}$ for a strictly higher sum rate. Consequently, the power allocation achieving the highest sum rate lies on either the $BACD$ or $BDFE$ or $AGEB$ face of $\mathcal{P}$. $\square$

The consequence of Proposition 1 is that the optimal power allocation consists of *at least one mobile, which is transmitting with maximum power*. Binary power control [1] corresponds to assigning the power allocation from the vertex of the cube that achieves the highest sum rate; however, this may not be the optimal power assignment, as evidenced by the following example:

$$G = 10^5 \cdot \begin{pmatrix} 0.034478 & 0.000011 & 0.000079 \\ 0.006218 & 0.044339 & 0.023319 \\ 0.000073 & 0.000002 & 2.443243 \end{pmatrix} \qquad (2)$$

$$\mathbf{P}_{opt} = \begin{bmatrix} 0.39 & 1 & 0.2 \end{bmatrix} \quad \mathbf{P}_{vert} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$R(\mathbf{P}_{opt}) = 24.9342 \text{ b/s/hz} > R(\mathbf{P}_{vert}) = 24.5670 \text{ b/s/hz}$$

The following section presents a linear complexity algorithm, which solves a series of subprograms per face of $\mathcal{P}$ resulting in a solution arbitrarily close to $R(\mathbf{P}_{opt})$. This algorithm is inspired by the branch-and-bound algorithm in [4] for maximizing products of fractional functions.

## 3. OPTIMAL POWER ALLOCATION: A BRANCHING ALGORITHM

Since at least one mobile in the optimal vector $\mathbf{P}_{opt}$ transmits with maximum power, we propose to branch $M(\mathbf{P})$ into 3 subproblems. Each subproblem consists of allocating maximum power to the selected user in one cell, and solving for the optimal power allocation for the transmitting users in the remaining cells. Stated otherwise, this is equivalent to finding the maximum rate over the faces $BACD$, $BDFE$ and $AGEB$ of $\mathcal{P}$. In addition, each subproblem is solved through a series of subprograms, each of which solves an optimization problem to maximize the sum capacity of a two-cell system for a fixed SINR (rate) in the third cell.

To solve for the highest sum rate over face $BACD$ of $\mathcal{P}$, define $S_{12}(\eta)$ as the subprogram with the following conditions: (a) user in cell 3 transmits with maximum power, and (b) $\text{SINR}_3$ satisfies $1 + \text{SINR}_3 = \eta$. Note that $\eta \in [\eta_{min}, \eta_{max}]$, where $\eta_{max} = 1 + g_{33}$ corresponds to the highest achievable $\text{SINR}_3$ by setting $P_1 = P_2 = 0$, while $\eta = \eta_{min} = 1 + \frac{g_{33}}{1+g_{31}+g_{32}}$ corresponds to the lowest achievable $\text{SINR}_3$, obtained by setting $P_1 = P_2 = 1$. It is shown later in this paper that the highest sum throughput on face $BACD$ of $\mathcal{P}$ can be achieved by solving $S_{12}(\eta_i)$ for "sufficiently" many $\eta_i \in [\eta_{min}, \eta_{max}]$.

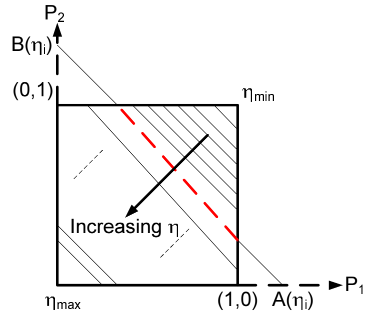


**Fig. 3**. Solving the subproblem $S_{12}(\eta)$ over a face of $\mathcal{P}$

Mathematically, the subprogram $S_{12}(\eta)$ is expressed as:

$$(\hat{P}_1, \hat{P}_2) = \max_{(P_1,P_2)\in\mathcal{P}_{12}(\eta)} Q(P_1, P_2) \tag{3}$$

$$Q(P_1, P_2) \triangleq (1 + \mathrm{SINR}_1) \cdot (1 + \mathrm{SINR}_2)$$

$$\mathcal{P}_{12}(\eta) = \{(P_1, P_2) : 0 \leq P_1, P_2 \leq 1, \frac{P_1}{A(\eta)} + \frac{P_2}{B(\eta)} = 1\}$$

$$\mathrm{SINR}_3 = \eta - 1, A(\eta) = \frac{1 + g_{33} - \eta}{g_{31}(\eta - 1)}, \frac{A(\eta)}{B(\eta)} = \frac{g_{32}}{g_{31}}$$

For a fixed subprogram, Lemma 1 derives the optimizing power allocation to (3).

**Lemma 1** *The optimizing solution $(P_1^*(\eta), P_2^*(\eta))$ of $S_{12}(\eta)$ lies in the set of feasible points in*

$$\begin{aligned}\mathcal{P}_{opt}(\eta) = \{&(A(\eta), 0), (0, B(\eta)), (1, B(\eta) - \frac{g_{31}}{g_{32}}),\\ &(A(\eta) - \frac{g_{32}}{g_{31}}, 1)\} \bigcup \{(P_1, B(\eta) - \frac{g_{31}}{g_{32}}P_1) :\\ &P_1^2(X_2X_4(X_1 + X_3) - X_1X_3(X_2 - X_4))+\\ &2P_1(X_1X_2 + X_3X_4) + (X_1 + X_3 + X_2 - X_4) = 0\}\end{aligned} \tag{4}$$

*where* $X_1 \triangleq \frac{g_{11}-g_{12}g_{31}/g_{32}}{1+g_{13}+g_{12}B(\eta)}$, $X_2 \triangleq \frac{g_{21}-g_{22}g_{31}/g_{32}}{1+g_{23}+g_{22}B(\eta)}$, $X_3 \triangleq \frac{g_{12}g_{31}/g_{32}}{1+g_{13}+g_{12}B(\eta)}$ & $X_4 \triangleq \frac{g_{21}}{1+g_{23}}$.

**Proof** For a given $\eta$, the solution to (3) can be obtained from the following cases:

1. Local Optima in $\mathcal{P}_{12}(\eta)$: Using $P_2 = B(\eta) - \frac{g_{31}}{g_{32}}P_1$, (3) is reexpressed as:

$$\begin{aligned}Q(P_1) = &\frac{(1 + g_{13} + g_{12}B(\eta)) + P_1(g_{11} - g_{12}g_{31}/g_{32})}{(1 + g_{13} + g_{12}B(\eta)) - P_1g_{12}g_{31}/g_{32}} *\\ &\frac{(1 + g_{23} + g_{22}B(\eta)) + P_1(g_{21} - g_{22}g_{31}/g_{32})}{(1 + g_{23} + g_{21}P_1)}\end{aligned} \tag{5}$$

Therefore, $S_{12}(\eta)$ is equivalently represented by the optimization problem,

$$(P_1^*, P_2^*) = \operatorname*{argmax}_{(P_1,P_2)\in\mathcal{P}_{12}(\eta)} \frac{(1 + P_1X_1)(1 + P_1X_2)}{(1 - P_1X_3)(1 + P_1X_4)} \tag{6}$$

Local optima to (6) are obtained by finding the feasible solutions to $Q'(P_1) = 0$, resulting in,

$$\begin{aligned}&P_1^2(X_2X_4(X_1 + X_3) - X_1X_3(X_2 - X_4))+\\ &2P_1(X_1X_2 + X_3X_4) + (X_1 + X_3 + X_2 - X_4) = 0\end{aligned} \tag{7}$$

Note, however, that the solution to (7) is not guaranteed to be feasible in $\mathcal{P}_{12}(\eta)$.

2. If the local optima found in the above step are either a minima or infeasible, $Q(P_1)$ is quasiconvex on $[0, 1]$. Thus, the maxima lies at the endpoints given by the set

$$\{(A(\eta_i), 0), (0, B(\eta_i)), (1, B(\eta_i) - \frac{g_{31}}{g_{32}}), (A(\eta_i) - \frac{g_{32}}{g_{32}}, 1)\}.$$

Combining 1 and 2 completes the proof. □

By varying $\eta$, a family of subprograms $S_{12}(\eta_i)$ is obtained, where $\eta_i \in [\eta_{min}, \eta_{max}]$, over linear constraints on parallel lines as shown in Figure 3. The subproblem characterizing the highest sum rate over the interior of the face $BACD$ equals $R(\eta^*)$, where $\eta^* \triangleq \mathrm{argmax}_{\eta\in[\eta_{min},\eta_{max}]} S_{12}(\eta)$. By symmetry, identical subprograms $S_{23}(\eta_i)$ and $S_{31}(\eta_i)$ are solved over faces $BDFE$ and $AGEB$, respectively.

The preceding observations lead to the algorithm presented in this work. Our optimal power allocation algorithm $\hat{M}(\mathbf{P})$ consists of two parts. In the first part, a subprogram $S_{vert}(\mathbf{P})$ computes the highest sum rate over the vertices of $\mathcal{P}$ (excluding the origin). Next, using Lemma 1, the subprograms $S_{12}(\eta_i), S_{23}(\eta_i)$ and $S_{31}(\eta_i)$ are solved over finitely many $\eta_i \in [\eta_{min}, \eta_{max}]$.

To characterize the difference between the estimated sum rate obtained by the proposed branching algorithm and the theoretical optimal sum rate, the following Lemma states the number of required subprograms per face to achieve arbitrarily close to the theoretical sum capacity.

**Lemma 2** *Let $\mathbf{P}_{opt}$ solve $M(\mathbf{P})$, $\mathbf{P}(\eta^*)$ be the terminating solution to $\hat{M}(\mathbf{P})$, and $\mathbf{P}_{vert}$ solve the vertex subprogram $S_{vert}(\mathbf{P})$. Given a target relative error $\epsilon \geq \frac{|R(\mathbf{P}_{opt})-R(\mathbf{P}(\eta^*))|}{R(\mathbf{P}_{vert})}$, $\hat{M}(\mathbf{P})$ requires a minimum number of intervals per face given as:*

$$\begin{aligned}n_i \geq &\frac{\max \|\nabla_{\mathbf{P}}R(\mathbf{P})\|_2}{2\epsilon R(\mathbf{P}_{vert})} *\\ &\max\left(\frac{g_{31} + g_{32}}{\sqrt{g_{31}^2 + g_{32}^2}}, \frac{g_{21} + g_{23}}{\sqrt{g_{21}^2 + g_{23}^2}}, \frac{g_{12} + g_{13}}{\sqrt{g_{12}^2 + g_{13}^2}}\right)\end{aligned} \tag{8}$$

**Proof** If the optimal power allocation $\mathbf{P}_{opt}$ lies on the vertices of $\mathcal{P}$, then $\hat{M}(\mathbf{P})$ computes the optimal power allocation after solving $S_{vert}(\mathbf{P})$. If not, the optimal allocation should lie on one of the faces of $\mathcal{P}$. WLOG, say $\mathbf{P}_{opt}$ lies on the $BACD$ face of $\mathcal{P}$ (see Figure 4), and let $\eta = \eta^*$ correspond to solution of $S_{12}(\mathbf{P})$. The magnitude of error between

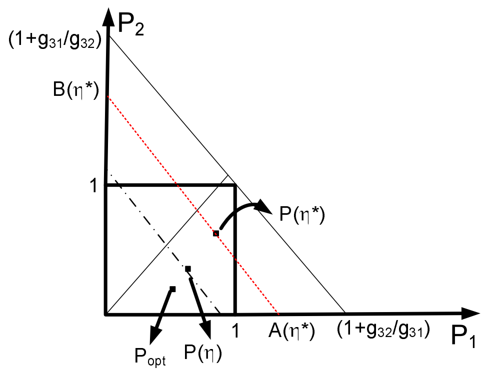


**Fig. 4**. Solving the subproblem $S_{12}(\eta)$ over a face of $\mathcal{P}$

$R(\mathbf{P}(\eta^*))$ and the highest sum rate $R(\mathbf{P}_{opt})$ can be bounded as,

$$|R(\mathbf{P}_{opt}) - R(\mathbf{P}(\eta^*))| \leq |R(\mathbf{P}_{opt}) - R(\mathbf{P}(\eta))| \quad (9)$$
$$= |(\mathbf{P}_{opt} - P(\eta))' \nabla_{\mathbf{P}} R(\mathbf{P})| \quad (10)$$
$$\leq \|(\mathbf{P}_{opt} - P(\eta))\|_2 \|\nabla_{\mathbf{P}} R(\mathbf{P})\|_2 \quad (11)$$
$$\leq \frac{1}{2n_i} \frac{g_{31} + g_{32}}{\sqrt{g_{31}^2 + g_{32}^2}} \max \|\nabla_{\mathbf{P}} R(\mathbf{P})\|_2 \quad (12)$$

Choosing $\mathbf{P}(\eta)$ to be the point closest to $\mathbf{P}_{opt}$, Equation (9) follows, since $R(\mathbf{P}(\eta)) \leq R(\mathbf{P}(\eta^*)) \leq R(\mathbf{P}_{opt})$. Next, choosing $\mathbf{P} = \lambda \mathbf{P}(\eta) + (1-\lambda)\mathbf{P}_{opt}$ where $\lambda \in [0,1]$, the mean value theorem guarantees that (10) holds. (11) follows from the Cauchy-Schwarz inequality. Finally, (12) follows since $\|(\mathbf{P}_{opt} - P(\eta))\|_2$ is less than half the spacing between two consecutive lines. □

Computing the three-cell capacity by solving the vertex subprogram incurs the smallest computational cost, but only provides a suboptimal solution. Lemma 2 implies that the proposed algorithm $\hat{M}(\mathbf{P})$ approaches the sum capacity while incurring a linear complexity $O(\max(n_1, n_2, n_3))$. In contrast, the empirical capacity calculation has a complexity of $O(n^2)$, where $n$ represents the number of points in each dimension.

## 4. RESULTS

Table 2 shows the ergodic sum capacity, corresponding to the system parameters in Table 1. The setup consists of three neighboring sectors in adjacent cells, whose BSs perform the proposed optimal power allocation in each slot. The results were obtained after averaging over 250 different mobile drops in both macro-cell and micro-cell scenarios, with 2000 trials per drop. A Proportional Fair (PF) scheduler was used to select the transmitting mobile in each cell per slot. The proposed algorithm $\hat{M}(\mathbf{P})$ was used to obtain the optimal power allocation after solving 100 subprograms over each face of $\mathcal{P}$. The empirical capacity was derived by finding the highest sum rate over 100 distinct pairs of points on the 3 candidate faces.

**Table 1**. System Parameters

| Parameter | Value |
|---|---|
| Radius (Micro, Macrocell) | 0.5, 1 km |
| $P_{max}$ | 21 dBm |
| Mobile Speed | 60 kmph |
| Path loss (dB) | $127.6 + 37.6 \log_{10}(d_{\text{km}})$ |
| LN Shadowing ($\sigma_{dB}$) | 8 dB |
| Antenna Pattern | $-\min\left((\frac{\theta}{70})^2, 20 \text{ dB}\right)$ |

**Table 2**. Results (m: Microcell, M: Macrocell)

| Users | Vertex | Empirical | Estimated | % Gain |
|---|---|---|---|---|
| 15 (m) | 13.2775 | 13.3776 | 13.3779 | 0.76 |
| 15 (M) | 8.7561 | 8.7651 | 8.7651 | 0.1 |
| 25 (m) | 13.5183 | 13.6104 | 13.6106 | 0.68 |
| 25 (M) | 9.1306 | 9.1423 | 9.1423 | 0.13 |

The closeness between the empirical and estimated capacity shows the near optimality of the proposed algorithm. Moreover, the sum capacity computed over the vertices of $\mathcal{P}$ is close to the sum capacity achieved by the proposed algorithm. Although this suggests that binary power allocation achieves near optimal sum capacity in a three-cell system, our proposed algorithm enables achieving the optimal power assignment, while maintaining linear complexity.


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