NASH BARGAINING AND PROPORTIONAL FAIRNESS FOR LOG-CONVEX UTILITY SETS

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ABSTRACT

For comprehensive convex compact positive utility sets, the Nash bargaining solution (NBS) is obtained by maximizing a product of utilities, a strategy which is also known as "proportional fairness". However, the standard assumption of convexity may not be fulfilled. This is especially true for wireless communication systems, where interference and adaptive techniques can lead to complicated non-convex utility sets (e.g. the 2-user SIR region with linear receivers). In this paper, we show that the Nash bargaining framework can be extended to certain non-convex utility sets, whose logarithmic transformation is strictly convex comprehensive. As application examples, we consider feasible sets of signal-to-interference ratios (SIR), based on axiomatic log-convex interference functions. The resulting SIR region is known to be log-convex. However, strict log-convexity and compactness is required here. We derive conditions under which this is fulfilled. In this case, there is a single-valued Nash bargaining solution, which is equivalent to the proportionally fair operating point. The results are shown for a total power constraint, as well as for individual power constraints.

Index Terms— **resource allocation, SIR feasible set, cooperative game theory, Nash bargaining**

1 INTRODUCTION

Wireless communication systems use cooperative resource allocation strategies in order to efficiently exploit the available power and bandwidth. Cooperation is often facilitated by centralized architectures, like cellular systems (e.g. UMTS-HSDPA). But cooperation can also be useful between decentralized system components. For example, partial cooperation between multiple base stations is currently being discussed in the 3GPP-LTE standardization.

By letting users cooperate, they can efficiently manage or even combat interference. Consider a wireless system, with K users from an index set $\mathcal{K} = \{1, 2, \dots, K\}$, where $K \geq 2$. If the users are coupled by interference, then there is a general trade-off between the users' utilities $\boldsymbol{u} = [u_1, \dots, u_K]^T$ chosen from the *utility set* $\mathcal{U}$. In this paper, we focus on utility sets with the following properties:

- *$\mathcal{U}$ is a non-empty compact subset of* $\mathbb{R}^K_{++}$, where $\mathbb{R}_{++}$ is the set of positive reals. That is, we consider a communication scenario where all users participate (have non-zero utility).
- *$\mathcal{U}$ is comprehensive*. That is, for all $\boldsymbol{u} \in \mathcal{U}$ and $\boldsymbol{u}' \in \mathbb{R}^K_{++}$, the component-wise inequality $\boldsymbol{u}' \leq \boldsymbol{u}$ implies $\boldsymbol{u}' \in \mathcal{U}$. This may be interpreted as free disposability of utility [1].

Throughout the paper, all vector inequalities (e.g. $\boldsymbol{u}' \leq \boldsymbol{u}$) are component-wise. By "compact" we mean *relatively compact* in $\mathbb{R}^K_{++}$. The class of all sets with the above properties is denoted by $\mathcal{D}^K$.

A fundamental problem is to find a suitable *operation point* or *solution outcome* from the boundary of the utility set $\mathcal{U}$. Various strategies exist and were investigated within the framework of *bargaining game theory* [1]. The users can *cooperate* to find an unanimous agreement on some solution outcome $\boldsymbol{\varphi}(\mathcal{U})$. This outcome is generally better than the *Nash equilibrium*, which results from a non-cooperative approach. The gain from cooperation can be substantial (see e.g. [2]).

1.1 The Conventional Nash Bargaining Solution (NBS)

We begin by briefly reviewing the *Nash Bargaining Solution* (NBS), which was introduced by Nash [3] and extended later (see e.g. [1] and the references therein). The NBS in its standard form requires that the utility set $\mathcal{U}$ is *convex*. An extension to certain non-convex sets will be derived later in Section 2.

Definition 1. Consider a convex set $\mathcal{U} \in \mathcal{D}^K$. The NBS is the unique (single-valued) solution that fulfills the following axioms.

- *Weak Pareto Optimality (WPO)*. The users should not be able to collectively improve upon the solution outcome, i.e.,

$$\boldsymbol{\varphi}(\mathcal{U}) \in \{\boldsymbol{u} \in \mathcal{U} : \text{there is no } \boldsymbol{u}' \in \mathcal{U} \text{ with } \boldsymbol{u}' > \boldsymbol{u}\} .$$

- *Symmetry (SYM)*. If $\mathcal{U}$ is symmetric, then the outcome does only depend on the employed strategies and not on the identities of the users, i.e., $\varphi_1(\mathcal{U}) = \cdots = \varphi_K(\mathcal{U})$. This does not mean that the game is necessarily symmetric, but rather that all users have the same priorities.
- *Independence of Irrelevant Alternatives (IIA)*. If the feasible set shrinks but the solution outcome remains feasible, then the solution outcome of the smaller set should be the same, i.e.,

$$\varphi(\mathcal{U}) \in \mathcal{U}', \text{ with } \mathcal{U}' \subseteq \mathcal{U} \implies \varphi(\mathcal{U}') = \varphi(\mathcal{U}) .$$

- *Scale Transformation Covariance (STC)*. The optimization strategy is invariant with respect to a component-wise scaling of the region. That is, for every $\mathcal{U} \in \mathcal{D}^K$, and all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^K$ with $\boldsymbol{a} > 0$ and $(\boldsymbol{a} \circ \mathcal{U} + \boldsymbol{b}) \in \mathcal{D}^K$, where '$\circ$' means component-wise multiplication, we have

$$\boldsymbol{\varphi}(\boldsymbol{a} \circ \mathcal{U} + \boldsymbol{b}) = \boldsymbol{a} \circ \boldsymbol{\varphi}(\mathcal{U}) + \boldsymbol{b} .$$

In the game-theoretical literature (e.g. [1]), the Nash bargaining framework usually contains a *disagreement point*, which ensures a solution in case that the players are unable to reach a unanimous agreement. However, this is not required for the problem under consideration. The existence of a solution is always guaranteed by the assumed properties of the utility set.

For convex sets $\mathcal{U} \in \mathcal{D}^K$, the single-valued NBS fulfilling these four axioms is obtained by maximizing the product of utilities

$$\max_{\boldsymbol{u}\in\mathcal{U}} \prod_{k\in\mathcal{K}} u_k \ . \tag{1}$$

It was observed in [4] that this strategy is equivalent to *proportional fairness* (PF). Since $\log \max \prod_k u_k = \max \log \prod_k u_k = \max \sum_k \log u_k$, the optimum (1) can be found by solving

$$\max_{\boldsymbol{u}\in\mathcal{U}} \sum_{k\in\mathcal{K}} \log u_k \ . \tag{2}$$

Proportionally fair resource allocation (2) was originally introduced in the context of stability and fairness of rate control algorithms for communication networks [4] . The proportional fair operating point $\hat{\boldsymbol{u}}$ is the one, at which the difference to any other utility vector $\boldsymbol{u} \in \mathcal{U}$ measured in the aggregated proportional change $\sum_k (u_k - \hat{u}_k)/\hat{u}_k$ is non-positive. This relates the NBS to a known fairness criterion .

1.2 Wireless Systems and SIR Regions

The standard assumption of convexity need not be fulfilled in practice. This is particularly true for wireless systems, where interference and adaptive communication techniques can lead to utility sets with a complicated structure.

An example is the signal-to-interference(-plus-noise) ratio (SIR). The SIR is an important measure for the user performance in a wireless system. Many other measures can be directly related to the SIR (e.g. bit error rate, capacity, ...).

Consider K users, with transmit powers $\boldsymbol{p} = [p_1, \dots, p_K]^T$. The noise power at each receiver is σ^2. Hence, the SIR at each receiver depends on the *extended power vector*

$$\underline{\boldsymbol{p}} = \left[\begin{smallmatrix} \boldsymbol{p} \\ \sigma^2 \end{smallmatrix} \right] = [p_1, \dots, p_K, \sigma^2]^T \ . \tag{3}$$

The resulting SIR of user k is $\mathrm{SIR}_k(\underline{\boldsymbol{p}}) = p_k/\mathcal{I}_k(\underline{\boldsymbol{p}})$, where $\mathcal{I}_k$ is the interference (plus noise) as a function of $\underline{\boldsymbol{p}}$. The set of all possible transmit power vectors is denoted by $\mathcal{P}$ (specified later in Section 3).

In order to model interference, we follow the axiomatic approach proposed in [5], [6].

Definition 2. We say that $\mathcal{I} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is an *interference function* if it fulfills the axioms:

- **A1** (non-negativity) $\mathcal{I}(\underline{\boldsymbol{p}}) \geq 0$ for all $\underline{\boldsymbol{p}} > 0$
- **A2** (scale invariance) $\mathcal{I}(\alpha\underline{\boldsymbol{p}}) = \alpha\mathcal{I}(\underline{\boldsymbol{p}}) \quad \forall \alpha \in \mathbb{R}_+$
- **A3** (monotonicity) $\mathcal{I}(\underline{\boldsymbol{p}}) \geq \mathcal{I}(\underline{\boldsymbol{p}}')$ if $\underline{\boldsymbol{p}} \geq \underline{\boldsymbol{p}}'$.

This framework A1–A3 differs slightly from the original notion of *standard interference functions* introduced in [5], where "scalability" was required instead of "scale invariance". The reason is the explicit modeling of the noise in (3). Both models are equivalent if we require additional strict monotonicity with respect to the noise component, i.e.,

$$\mathcal{I}(\underline{\boldsymbol{p}}) > \mathcal{I}(\underline{\boldsymbol{p}}') \text{ if } \underline{p}_{K+1} > \underline{p}'_{K+1} \text{ and } \underline{\boldsymbol{p}} \geq \underline{\boldsymbol{p}}' \ . \tag{4}$$

With (4) and the assumption of constant noise, the function $\mathcal{I}$ is *standard* with respect to the variable vector $\boldsymbol{p}$.

However, in this paper we use the framework A1–A3, which has the advantage of being more general: Its applicability is not restricted to a particular power control problem, but it can also be used for the analysis of utility sets.

As an example, consider the *SIR feasible region*, which is defined as the sub-level set

$$\mathcal{S}(\mathcal{P}) = \{\boldsymbol{\gamma} \in \mathbb{R}^K_{++} : C(\boldsymbol{\gamma}, \mathcal{P}) \leq 1\} \ , \tag{5}$$

where $\boldsymbol{\gamma}$ is a vector of SIR values, whose feasibility is determined by the min-max optimum.

$$C(\boldsymbol{\gamma}, \mathcal{P}) = \inf_{\boldsymbol{p}\in\mathcal{P}} \max_{k\in\mathcal{K}} \frac{\gamma_k \mathcal{I}_k(\underline{\boldsymbol{p}})}{p_k} \ . \tag{6}$$

The structure of the SIR set $\mathcal{S}(\mathcal{P})$ depends on the properties of $C(\boldsymbol{\gamma}, \mathcal{P})$, which in turn depends on the properties of the underlying interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$, as well as on the chosen power set $\mathcal{P}$. It can be observed that $C(\boldsymbol{\gamma}, \mathcal{P})$ itself is an "interference function" fulfilling A1–A3. Moreover, sub-level sets of convex functions are convex, thus $\mathcal{S}(\mathcal{P})$ is a closed convex set from $\mathbb{R}^K_{++}$ if $C(\boldsymbol{\gamma}, \mathcal{P})$ is *convex*. However, convexity of $C(\boldsymbol{\gamma}, \mathcal{P})$ does generally not hold, so SIR regions (5) are typically non-convex.

1.3 Problem Formulation and Contributions

A drawback of the classical Nash bargaining framework is the requirement of convexity. For every compact convex set from $\mathcal{D}^K$, the product maximizer (1) is the single-valued NBS characterized by axioms WPO, SYM, IIA, STC. However, this convexity assumption need not be fulfilled, especially not for interference-coupled wireless systems (see e.g. the example in Section 1.2).

In this paper, we will show that the conventional NBS framework can be extended to certain "log-convex" utility sets. Our approach is based on a change of variable $q_k = \log u_k$, which is a common technique for exploiting "hidden convexity" (see e.g. [7]–[10]). In Section 2, it will be shown that the product maximizer (1) is the single-valued NBS if the set of transformed utilities q_k is a *strictly convex* compact comprehensive set. This result extends the class of bargaining games for which the classical results summarized in Section 1.1 hold, and for which NBS is equivalent to proportional fairness.

2 Nash Bargaining for Log-Convex Utility Sets

Consider the bijective continuous mapping

$$\mathcal{L}og(\mathcal{U}) = \{\boldsymbol{q} = \log(\boldsymbol{u}) : \boldsymbol{u} \in \mathcal{U}\} \ , \tag{7}$$

where $\log(\boldsymbol{u}) = [\log u_1, \dots, \log u_K]^T$.

Definition 3. By $\mathcal{ST}_c$ we denote the class of all compact comprehensive utility sets $\mathcal{U} \subset \mathbb{R}^K_{++}$, such that $\mathcal{L}og(\mathcal{U})$ is a strictly convex set in $\mathbb{R}^K$. In the following, such sets are sometimes referred to as strictly *log-convex*.

In this section, it will be shown that for any $\mathcal{U} \in \mathcal{ST}_c$, the product maximizer (1) is the single-valued NBS characterized by axioms WPO, SYM, IIA, STC.

By $\mathcal{L}og(\mathcal{ST}_c)$ we denote the class of all sets $\mathcal{L}og(\mathcal{U})$ such that $\mathcal{U} \in \mathcal{ST}_c$. A set $\mathcal{Q}$ belongs to $\mathcal{L}og(\mathcal{ST}_c)$ if and only if it is strictly convex, comprehensive, and compact. Compactness and comprehensiveness of $\mathcal{U}$ are preserved by the log-transformation. That is, $\mathcal{U} \subset \mathbb{R}^K_{++}$ is compact comprehensive if and only if $\mathcal{L}og(\mathcal{U}) \subset \mathbb{R}^K$ is compact comprehensive.

Strict convexity of the transformed set plays an important role for the proof of uniqueness. We also exploit that the axioms

WPO, SYM, IIA, STC have direct equivalences in $\mathcal{L}og(\mathcal{U})$. This is straightforward for axioms WPO, SYM, IIA, which are not affected by the logarithmic transformation. That is, weak Pareto optimality (WPO) in the utility set $\mathcal{U}$ corresponds directly to weak Pareto optimality in the set $\mathcal{L}og(\mathcal{U})$. The same holds for Symmetry (SYM) and Independence of Irrelevant Alternatives (IIA). We will denote the axioms associated with the transformed set by $\text{WPO}_\mathcal{Q}$, $\text{SYM}_\mathcal{Q}$, and $\text{IIA}_\mathcal{Q}$.

Scale transformation covariance (STC) in the utility set $\mathcal{U} \in \mathcal{ST}_c$ also has a direct correspondence for the transformed set $\mathcal{Q} = \mathcal{L}og(\mathcal{U})$. Consider an arbitrary translation $\tilde{\boldsymbol{q}} \in \mathbb{R}^K$, leading to a translated set $\mathcal{Q}(\tilde{\boldsymbol{q}})$, defined as

$$\mathcal{Q}(\tilde{\boldsymbol{q}}) = \{\boldsymbol{q} \in \mathbb{R}^K : \exists \boldsymbol{q}_0 \in \mathcal{Q} \text{ with } \boldsymbol{q} = \boldsymbol{q}_0 + \tilde{\boldsymbol{q}}\} \ .$$

Then, the transformed Nash bargaining solution in the set $\mathcal{Q}$ is

$$\varphi_\mathcal{Q}\big(\mathcal{Q}(\tilde{\boldsymbol{q}})\big) = \varphi_\mathcal{Q}\big(\mathcal{Q}\big) + \tilde{\boldsymbol{q}} \ . \tag{8}$$

We will refer to property (8) as $\text{STC}_\mathcal{Q}$.

It is now shown that the transformed axioms are associated with a unique solution outcome $\varphi_\mathcal{Q}$ in the transformed set.

Theorem 1. *For an arbitrary set $\mathcal{U} \in \mathcal{ST}_c$, the solution outcome $\varphi_\mathcal{Q}$ in the transformed set $\mathcal{Q} = \mathcal{L}og(\mathcal{U})$ fulfills WPO_Q, SYM_Q, STC_Q, IIA_Q if and only if*

$$\varphi_\mathcal{Q}\big(\mathcal{L}og(\mathcal{U})\big) = \operatorname*{arg\,max}_{\boldsymbol{q} \in \mathcal{L}og(\mathcal{U})} \sum_{k \in \mathcal{K}} q_k \ . \tag{9}$$

A sketch of the proof is as follows. Given the properties of the region $\mathcal{U} \in \mathcal{ST}_c$ and the image set $\mathcal{L}og(\mathcal{U})$, it is clear that the solution (9) fulfills the axioms WPO_Q, SYM_Q, STC_Q, IIA_Q.

It remains to show the converse. Consider a bargaining strategy on the transformed set $\mathcal{Q} = \mathcal{L}og(\mathcal{U})$, that fulfills the axioms WPO_Q, SYM_Q, STC_Q, IIA_Q. Then, these axioms are fulfilled by a unique solution, which is the optimizer of (9). To show this, consider the set

$$\mathcal{Q}_1 = \{\boldsymbol{q} \in \mathbb{R}^K : \sum_k q_k \le K\} \ .$$

Because of the $\text{STC}_\mathcal{Q}$ property (8), we know that the strategy is invariant with respect to a translation of the region. So, without loss of generality we can assume $\mathcal{Q} \subseteq \mathcal{Q}_1$, and

$$\hat{\boldsymbol{q}} = [1, \ldots, 1]^T = \operatorname*{arg\,max}_{\boldsymbol{q} \in \mathcal{Q}} \sum_{k \in \mathcal{K}} q_k \ .$$

Since $\mathcal{Q}$ is upper-bounded, there is a $\tilde{\boldsymbol{q}} \in \mathbb{R}^K$ such that $\tilde{\boldsymbol{q}} \ge \boldsymbol{q}$, for all $\boldsymbol{q} \in \mathcal{Q}$. Thus, $\mathcal{Q}$ is a sub-set of the set

$$\tilde{\mathcal{Q}}_1 = \{\boldsymbol{q} \in \mathcal{Q}_1 : \boldsymbol{q} \le \tilde{\boldsymbol{q}}\} \ . \tag{10}$$

The set $\tilde{\mathcal{Q}}_1$ is symmetric and strictly convex. Now, let $\tilde{\mathcal{Q}}$ be the smallest symmetric and strictly convex region that fulfills $\tilde{\mathcal{Q}}_1 \supseteq \tilde{\mathcal{Q}} \supseteq \mathcal{Q}$. Since $\tilde{\mathcal{Q}}_1$ is upper-bounded, the set $\tilde{\mathcal{Q}}$ is compact. It is also strictly convex comprehensive, so it is contained in $\mathcal{L}og(\mathcal{ST}_c)$. Because of property SYM_Q, it follows that

$$\varphi_\mathcal{Q}(\tilde{\mathcal{Q}}) = \hat{\boldsymbol{q}} = [1, \ldots, 1]^T \ .$$

Since $\sum_k q_k = K$ is a supporting hyperplane for $\tilde{\mathcal{Q}}$, we have $\varphi_\mathcal{Q}(\tilde{\mathcal{Q}}) = \operatorname{arg\,max}_{\boldsymbol{q} \in \tilde{\mathcal{Q}}} \sum_{k \in \mathcal{K}} q_k$. Now, $\mathcal{Q} \subseteq \tilde{\mathcal{Q}}$ and $\hat{\boldsymbol{q}} \in \mathcal{Q}$. Because of property $\text{IIA}_\mathcal{Q}$ we have

$$\varphi_\mathcal{Q}(\mathcal{Q}) = \varphi_\mathcal{Q}(\tilde{\mathcal{Q}}) = [1, \ldots, 1]^T = \operatorname*{arg\,max}_{\boldsymbol{q} \in \mathcal{Q}} \sum_{k \in \mathcal{K}} q_k \ . \tag{11}$$

Thus, for all $\mathcal{U} \in \mathcal{ST}_c$ the optimization (11) in the transformed domain $\mathcal{Q} = \mathcal{L}og(\mathcal{U})$ leads to the unique optimum (9). Because of the one-to-one logarithmic mapping between the sets $\mathcal{Q}$ and $\mathcal{U}$, we have the following result

Corollary 1. *Let $\mathcal{U} \in \mathcal{ST}_c$. Then WPO, SYM, STC, IIA is fulfilled by a unique solution outcome $\varphi(\mathcal{U})$, which is the maximizer*

$$\varphi(\mathcal{U}) = \operatorname*{arg\,max}_{\boldsymbol{u} \in \mathcal{U}} \prod_{k \in \mathcal{K}} (u_k) \ . \tag{12}$$

3 Nash Bargaining over SIR Feasible Sets

We will now apply the results to the non-convex SIR region $\mathcal{S}(\mathcal{P})$, as defined by (5).

If the interference functions are linear, e.g. $\mathcal{I}(\boldsymbol{p}) = \boldsymbol{v}^T\boldsymbol{p}$, with coupling coefficients $\boldsymbol{v} \ge 0$, and with an unconstrained power set $\mathcal{P} = \mathbb{R}^K_{++}$, then the resulting SIR region $\mathcal{S}(\mathcal{P})$ is known to be log-convex. This was shown in [7], and extended in [8], [9]. Recent work [11] provides conditions under which the transformed set is strictly convex, as required by Nash bargaining. However, all these results are restricted to *linear interference functions*. In this paper we consider the more general interference model A1–A3 (see Definition 2). For this model, strict convexity is generally not fulfilled.

The axiomatic framework A1–A3 was studied in [12], where log-convexity of certain SIR sets was shown for *log-convex interference functions* (as defined in the next section), which includes the linear function as a special case. However, this result neither includes power constraints, nor was strict convexity shown. However, strictness of the logarithmic SIR set is needed for the extended Nash bargaining theory from Section 2 to be applicable.

In the next section we extend the results [12] by deriving conditions under which certain power-constrained SIR regions are *strictly log-convex* compact comprehensive.

3.1 Log-Convex Interference Functions

Having introduced general interference functions in Section 1.2, we will now focus on the important sub-class of *log-convex interference functions*. For explanation, consider $f(\boldsymbol{s}) := \mathcal{I}(\exp\{\boldsymbol{s}\})$. The function $f : \mathbb{R}^K \mapsto \mathbb{R}_+$ is said to be *log-convex* on $\mathbb{R}^K$ if $\log f$ is convex, or equivalently [13]

$$f\big((1-\lambda)\hat{\boldsymbol{s}} + \lambda\check{\boldsymbol{s}}\big) \le f(\hat{\boldsymbol{s}})^{1-\lambda} f(\check{\boldsymbol{s}})^\lambda, \quad \forall \lambda \in (0,1), \ \hat{\boldsymbol{s}}, \check{\boldsymbol{s}} \in \mathbb{R}^K \ .$$

Definition 4. We say that $\mathcal{I} : \mathbb{R}^K_+ \mapsto \mathbb{R}_+$ is a *log-convex interference function* if A1–A3 are fulfilled and in addition $\mathcal{I}(\exp\{\boldsymbol{s}\})$ is log-convex on $\mathbb{R}^K$.

Notice that the log-convexity property in Definition 4 is based on a change of variable $\boldsymbol{p} = \exp\{\boldsymbol{s}\}$ (component-wise exponential). Such a technique was already used by Sung [7] in the context of linear interference functions, and later in [8]–[10].

Some examples of log-convex interference functions are:

Example 1. The linear function

$$\mathcal{I}_k(\boldsymbol{p}) = \boldsymbol{p}^T\boldsymbol{v}_k, \quad k \in \mathcal{K} \ , \tag{13}$$

where $\boldsymbol{v}_k \in \mathbb{R}^K_+$ is a vector of interference coupling coefficients.

Example 2. The coefficients $\boldsymbol{v}$ can adapt to the current interference situation. An example is the "worst-case interference"

$$\mathcal{I}_k(\boldsymbol{p}) = \max_{c_k \in \mathcal{C}_k} \boldsymbol{p}^T\boldsymbol{v}_k(c_k), \quad k \in \mathcal{K} \ . \tag{14}$$

The parameter c_k can stand for some uncertainty, chosen from a compact uncertainty set $\mathcal{C}_k$. Such an interference model is used, e.g., in the context of robust power allocation [14].

The function (14) fulfills A1–A3. Also, $\mathcal{I}_k(\boldsymbol{p})$ is convex on $\mathbb{R}_+^K$, and $\mathcal{I}_k(\mathrm{e}^{\boldsymbol{s}})$ is log-convex on $\mathbb{R}^K$.

In the remainder, we show strict log-convexity of the SIR region under two different kinds of power constraints. The underlying interference functions are assumed to be log-convex, in the sense of Definition 4. Proofs are omitted because of the page restrictions.

3.2 SIR Region under a Total Power Constraint

Assume that the sum of all transmission powers is limited by P_{tot}. The set of all possible power vectors is

$$\mathcal{P}_{tot} = \{\underline{\boldsymbol{p}} > 0 : \sum_{k \in \mathcal{K}} \underline{p}_k \leq P_{tot},\ \underline{p}_{K+1} = \sigma^2\} . \qquad (15)$$

Theorem 2. *Let $\mathcal{I}_1, \ldots, \mathcal{I}_K$ be arbitrary log-convex interference functions, which are strictly monotonic increasing with respect to the noise component. For all $0 < P_{tot} < +\infty$ the SIR region $\mathcal{S}(\mathcal{P}_{tot})$, as defined by (5), is contained in $\mathcal{ST}_c$.*

Hence, the proportionally fair optimizer (2), respectively (1), is the single-valued NBS. The existence of the optimizer is guaranteed by the power constraints (the SIR region is a compact set).

3.3 SIR Region under Individual Power Constraints

Next, we will show the same for individual power limits $\boldsymbol{p}^{max} = [p_1^{max}, \ldots, p_K^{max}]^T$. The set of all possible power vectors is

$$\mathcal{P}_{ind} = \{\underline{\boldsymbol{p}} > 0 : \underline{p}_k \leq p_k^{max},\ k \in \mathcal{K},\ \underline{p}_{K+1} = \sigma^2\} , \qquad (16)$$

and the associated SIR region is $\mathcal{S}(\mathcal{P}_{ind})$, as defined by (5).

Under individual power constraints, strict convexity does not follow as easily as for the sum power constraint. Whether or not the region is strictly convex depends on the interference coupling in the system. In order to characterize interference coupling, we introduce a *dependency matrix*

$$[\boldsymbol{D}]_{kl} = \begin{cases} 1 & \text{if there exists a } \boldsymbol{p} > 0 \text{ such that } \mathcal{I}_k(\boldsymbol{p} + \delta \boldsymbol{e}_l) \\ & \text{is not constant for some values } \delta > 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (17)$$

Here, $\boldsymbol{e}_l$ is the all-zero vector with the lth component set to one. The non-zero entries in $\boldsymbol{D}$ mark the transmitter/receiver pairs which are coupled by interference. A zero entry means that no interference is received, no matter how large the transmission power is. As an example, think of users that are assigned to different orthogonal resources, or separated by adaptive interference rejection techniques. Based on this definition, we define a *dependency set*

$$\mathsf{L}_k = \{l \in \mathcal{K} : [\boldsymbol{D}]_{kl} = 1\} \quad \text{(transmitters on which user } k \text{ depends)}$$

The possible occurrence of decoupled users did not matter under a sum-power constraint. However, in order to analyze the behavior under individual power constraints, a few additional definitions are required.

Definition 5 (strict monotonicity)**.** $\mathcal{I}_k(\boldsymbol{p})$ is said to be strictly monotonic if $\boldsymbol{p}^{(1)} \geq \boldsymbol{p}^{(2)}$, with $p_l^{(1)} > p_l^{(2)}$, for one or more $l \in \mathsf{L}_k$, implies $\mathcal{I}_k(\underline{\boldsymbol{p}}^{(1)}) > \mathcal{I}_k(\underline{\boldsymbol{p}}^{(2)})$.

Definition 6 (strict log-convexity)**.** Let $\boldsymbol{p}(\lambda) = \hat{\boldsymbol{p}}^{1-\lambda} \cdot \check{\boldsymbol{p}}^{\lambda}$. A log-convex interference function $\mathcal{I}_k$ is said to be *strictly log-convex* if for all $\hat{\boldsymbol{p}}$, $\check{\boldsymbol{p}}$ which are not constant on the dependency set L_k

$$\mathcal{I}_k\big(\boldsymbol{p}(\lambda)\big) < \big(\mathcal{I}_k(\hat{\boldsymbol{p}})\big)^{1-\lambda} \cdot \big(\mathcal{I}_k(\check{\boldsymbol{p}})\big)^{\lambda} , \quad \lambda \in (0,1) . \qquad (18)$$

Using these properties, the following result can be shown:

Theorem 3. *Assume that $\mathcal{I}_1, \ldots \mathcal{I}_K$ are strictly monotonic and strictly log-convex, and there is no self-interference. If $\boldsymbol{D}\boldsymbol{D}^T$ is irreducible, then the SIR region $\mathcal{S}(\mathcal{P}_{ind})$ contained in $\mathcal{ST}_c$.*

4 Conclusions

For compact convex comprehensive utility sets, the unique solution fulfilling the Nash axioms is obtained as the optimizer of a product maximization problem, also known as proportional fairness.

In this paper we show that the Nash bargaining framework can be extended to certain non-convex compact comprehensive sets, which are strictly convex after a logarithmic transformation.

As an example, we have studied the SIR region of a wireless system, based on axiomatic log-convex interference functions. Under the assumption of power constraints (total and individual), it is shown that the resulting SIR region is comprehensive, compact, and strictly log-convex. Even though the region itself is non-convex, all properties of the classical Nash bargaining solution are preserved.

References

[1] H. J. M. Peters, *Axiomatic Bargaining Game Theory*. Kluwer Academic Publishers, Dordrecht, 1992.

[2] E. G. Larsson and E. A. Jorswieck, "The MISO interference channel: Competition versus collaboration," in *Proc. Allerton Conf. on Commun., Control and Computing, Monticello, USA*, 2007.

[3] J. F. Nash Jr., "The bargaining problem," *Econometrica, Econometric Soc.*, vol. 18, pp. 155–162, 1950.

[4] F. Kelly, A. Maulloo, and D. Tan, "Rate control for communication networks: Shadow prices, proportional fairness and stability," *Journal of Operations Research Society*, vol. 49, no. 3, pp. 237–252, Mar. 1998.

[5] R. D. Yates, "A framework for uplink power control in cellular radio systems," *IEEE J. Select. Areas Commun.*, vol. 13, no. 7, pp. 1341–1348, Sept. 1995.

[6] M. Schubert and H. Boche, "QoS-based resource allocation and transceiver optimization," in *Foundations and Trends in Communications and Information Theory*, 2005, vol. 2, no. 6, pp. 383–529, Online: http://www.nowpublishers.com/cit/.

[7] C. W. Sung, "Log-convexity property of the feasible SIR region in power-controlled cellular systems," *IEEE Communications Letters*, vol. 6, no. 6, pp. 248–249, June 2002.

[8] D. Catrein, L. Imhof, and R. Mathar, "Power control, capacity, and duality of up- and downlink in cellular CDMA systems," *IEEE Trans. Commun.*, vol. 52, no. 10, pp. 1777–1785, 2004.

[9] H. Boche and S. Stańczak, "Convexity of some feasible QoS regions and asymptotic behavior of the minimum total power in CDMA systems," *IEEE Trans. Commun.*, vol. 52, no. 12, pp. 2190 – 2197, Dec. 2004.

[10] M. Chiang, C. W. Tan, D. Palomar, D. O'Neill, and D. Julian, "Power control by geometric programming," *IEEE Trans. Wireless Commun.*, vol. 6, no. 7, pp. 2640–2651, July 2007.

[11] S. Stanczak and H. Boche, "Strict convexity of the log-SIR region," in *IEEE Info. Theory Workshop (ITW), Chengdu, China*, 2006.

[12] H. Boche, M. Schubert, S. Stańczak, and M. Wiczanowski, "An axiomatic approach to resource allocation and interference balancing," in *Proc. IEEE Internat. Conf. on Acoustics, Speech, and Signal Proc. (ICASSP), Philadelphia, USA*, Mar. 2005.

[13] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[14] M. Biguesh, S. Shahbazpanahi, and A. B. Gershman, "Robust downlink power control in wireless cellular systems," *EURASIP Journal on Wireless Communications and Networking*, no. 2, pp. 261–272, 2004.