FIXED POINT ERROR ANALYSIS OF LINEAR MULTICHANNEL PRECODING FOR VDSL

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ABSTRACT

Crosstalk interference is the limiting factor in transmission over copper lines. Crosstalk cancellation techniques show great potential for enabling the next leap in DSL transmission rates. An important issue is the effect of finite world length on performance. In this paper we provide an analysis of the performance of linear zero-forcing precoders, used for crosstalk compensation, in the presence of quantization noise. We quantify analytically the trade off between quantization level and transmission rate degradation. We demonstrate, through simulations on real lines, the accuracy of our estimates. Finally, we show how to use these estimates as a design tool for DSL linear crosstalk precoders. ***Index Terms***— **Multichannel DSL, vectoring, linear precoding, capacity estimates, quantization.**

I. INTRODUCTION

DSL systems are capable of delivering high date rates over copper lines. A major problem of DSL technologies is the electromagnetic coupling between the twisted pairs within a binder group. Reference [1] and the recent experimental studies in [2], [3] have demonstrated that *vectoring* and *crosstalk cancellation* allow a significant increase of the data rates of DSL systems. In particular, linear precoding has recently drawn considerable attention [4], [5] as a natural method for crosstalk precompensation. In [2], [3] it is shown that linear precoding affords a per-loop capacity boost ranging from $2\times$ to $4\times$, and also substantially reduces per-loop capacity spread and outage, which are very important metrics from an operator's perspective.

References [4], [5] advocate the use of a diagonalizing pre-compensator, and demonstrate that, without modification of the Customer Premise Equipment (CPE), one can obtain near optimal performance. Recent work in [6], [7], [8] has shown that a low-order truncated series approximation of the inverse channel matrix affords significant complexity reduction in the computation of the precoding matrix. Implementation complexity (i.e., the actual multiplication of the transmitted symbol vector by the precoding matrix) remains high, however, especially for multicarrier transmission which requires one matrix-vector multiplication for each tone. Current advanced DSL systems use thousands of tones. In these conditions, using minimal word length in representing the precoder matrix is important. However, using coarse quantization will result in substantial rate loss. The precoder matrix elements will be quantized in any practical implementation. The number of quantization bits per matrix coefficient will have a direct impact on system complexity and performance. In this paper we study exactly this trade-off. We show that both absolute and relative transmission loss decay exponentially as a function of the number of quantizer bits and provide very explicit bounds for the loss in each tone. Under analytic channel models as in [9] we provide refined and explicit bounds for the transmission loss across the band and compare these to simulation results. This explicit relationship between the number of quantizer bits and the transmission rate loss due to quantization is a very useful tool in the design of practical systems.

The structure of the paper is as follows. In section II, we present the signal model for a precoded discrete multichannel system and provide a model for the precoder errors we study. In section III a general formula for the transmission loss of a single user is derived in the case of full channel state information where the rate loss of a single user results from quantization errors. In section IV we provide simulation results on measured lines, which support our analysis. Proofs of the various mathematical claims made here is deferred to the Journal version.

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II. PROBLEM FORMULATION

II-A. Signal model

In this section we describe the signal model for a precoded discrete multitone (DMT) system. We assume that the transmission scheme is Frequency Division Duplexing (FDD), where the upstream and the downstream transmissions are performed at separate frequency bands. Moreover, we assume that all modems are synchronized. Hence, the echo signal is eliminated, as in [10], [1], and the received signal model at frequency f is given by

$$\mathbf{x}(f) = \mathbf{H}(f)\mathbf{s}(f) + \mathbf{n}(f), \tag{1}$$

where $\mathbf{s}(f)$ is the vectored signal sent by the optical network unit (ONU), $\mathbf{H}(f)$ is a $p \times p$ matrix representing the channels, $\mathbf{n}(f)$ is additive Gaussian noise, and $\mathbf{x}(f)$ (conceptually) collects the signals received by the individual users. The users estimate rows of the channel matrix $\mathbf{H}(f)$, and the ONU uses this information to send $\mathbf{P}(f)\mathbf{s}(f)$ instead of $\mathbf{s}(f)$. This process is called **crosstalk pre-compensation**. In general such a mechanism yields

$$\mathbf{x}(f) = \mathbf{H}(f)\mathbf{P}(f)\mathbf{s}(f) + \mathbf{n}(f). \tag{2}$$

Let $\mathbf{D}(f) = diag(\mathbf{H}(f))$ denote the diagonal of $\mathbf{H}(f)$ and take $\mathbf{P}(f) = \mathbf{H}(f)^{-1}\mathbf{D}(f)$ as suggested in [5]. With this we have

$$\mathbf{x}(f) = \mathbf{D}(f)\mathbf{s}(f) + \mathbf{n}(f), \tag{3}$$

showing that the crosstalk is eliminated. Note that with $\mathbf{F}(f) = \mathbf{H}(f) - \mathbf{D}(f)$ we have the following formula for the matrix $\mathbf{P}(f)$

$$\mathbf{P}(f) = (\mathbf{I} + \mathbf{D}^{-1}(f)\mathbf{F}(f))^{-1}. \tag{4}$$

Following [5] we assume that the matrices $\mathbf{H}(\mathbf{f})$ are row-wise diagonally dominant, namely that

$$\|h_{ii}\| >> \|h_{ij}\|, \forall i \neq j. \tag{5}$$

In fact, motivated in part by Gersgorin's theorem [11] we propose the parameter $r(\mathbf{H})$

$$r(\mathbf{H}) = \max_{1 \leq i \leq N} \left(\frac{\sum_{j \neq i} |h_{ij}|}{|h_{ii}|} \right), \tag{6}$$

as a measure for the dominance. In most downstream scenarios the parameter r is indeed much smaller than 1. We emphasize that typical downstream VDSL channels are row-wise diagonally dominant even in mixed length scenarios as demonstrated in [7].

II-B. A model for precoder errors

In practical implementations, the entries of the precoding matrix $\mathbf{P}$ will be quantized. The number of quantizer bits used is dictated by complexity and memory considerations. Indeed, relatively coarse quantization of the entries of the precoder $\mathbf{P}$ allows significant reduction of the time complexity and the amount of memory needed for the precoding process. The key problem is to determine the transmission rate loss of an individual user caused by such quantization. Let

$$\mathbf{P} = (\mathbf{I} + \mathbf{D}^{-1}\mathbf{F})^{-1} + \mathbf{E}, \tag{7}$$

where $\mathbf{E}$ models the errors caused by quantizing the precoder $\mathbf{P}$. The problem is to determine the capacity of the system, and the capacity of each user, in terms of the system parameters and the statistical parameters of the errors. Note that equation (7) allows the study of capacity loss due to three types of errors: errors in the estimation of $\mathbf{H}$, quantization errors in the representation of $\mathbf{H}$, and quantization errors in the representation of the precoder $\mathbf{P}$.

Our focus will be in the study of the effect of quantization errors in the representation of the precoder on the capacity of an individual user. To model this situation, assume that the matrix elements of $\mathbf{E}$ are uniformly bounded by 2^{-d} for some integer d. This is a weak assumption on the type of the quantization process. Informally it is equivalent to an assumption on the number of bits used to quantize an entry in the channel matrix. In particular, our analysis of the capacity loss will be independent of the specific quantization method and our results are valid for any technique that quantizes matrix elements with bounded errors. Note that due to the stationarity of DSL systems, and their long coherence time we may assume that the estimation of the channel matrices is possible at any desired precision.

III. A GENERAL FORMULA FOR TRANSMISSION LOSS

The purpose of this section is to provide a general formula for the transmission rate loss of a single user, resulting from errors in the estimated channel matrix as well as errors in the precoder matrix. The origin of these errors is irrelevant for the formula. As the results of this section are technical, we briefly overview them. First, we develop a useful expression for the equivalent channel in the presence of errors. This is given in formula (9). Next, a formula for the transmission loss is obtained. The formula compares the achievable rate of a communication system using an ideal ZF precoder as in (4) versus that of a communication system whose precoder is given by (7). This is formula (20) which is the key to the whole paper. Note that we use a gap analysis as in [12], [13]. A useful corollary in the form of formula (22) is derived. This will be used in the next section to obtain bounds on capacity loss due to quantization.

Recall that $\mathbf{H}(f) = \mathbf{D}(f) + \mathbf{F}(f)$ is a decomposition of the channel matrix at a given frequency to diagonal and non-diagonal terms. Also we let $SNR_i(f)$ be the signal to noise ratio of the i-th receiver at frequency f

$$SNR_i(f) = \frac{P_i(f)|d_{i,i}(f)|^2}{|n_i(f)|^2}. \tag{8}$$

In this formula $P_i(f)$ is the power spectral density (PSD) of the i-th user at frequency f, and $n_i(f)$ is the associated noise term.

III-A. A formula for the channel in presence of errors

We first derive a general formula for the equivalent signal model. The next lemma provides a useful reformulation of the signal model in (2):

Lemma 3.1: The precoded channel (2) with precoder as in (7) is given by

$$\mathbf{x}(f) = \mathbf{D}(f)\mathbf{s}(f) + \mathbf{D}(f)\boldsymbol{\Delta}(f)\mathbf{s}(f) + \mathbf{n}(f), \tag{9}$$

with

$$\boldsymbol{\Delta}(f) = (\mathbf{I} + \mathbf{D}^{-1}(f)\mathbf{F}(f))\mathbf{E}(f) \tag{10}$$

The proof is straightforward.

III-B. Transmission Loss of a Single User

Consider a communication system as defined in (3) and denote by B the frequency band of the system. We let $SNR_i(f)$ be as in (8) and let Γ be the Shannon Gap comprising modulation loss, coding gain and noise margin.

Let R_i be the transmission rate of the i-th user in the system defined in (3). Recall that in such a system the cross talk is completely removed and it is well known that

$$R_i = \int_{f \in B} \log_2(1 + \Gamma^{-1} SNR_i(f)) df. \tag{11}$$

Let

$$R_i(f) = \log_2(1 + \Gamma^{-1} SNR_i(f)) \tag{12}$$

be the transmission rate at frequency f (formally, it is just the density of that rate).

Let $\tilde{R}_i(f)$ be the transmission rate at frequency f of the i-th user, when the precoder in (7) is used. We note that while $R_i(f)$ is a number, the quantity $\tilde{R}_i(f)$ depends on the random variables $\mathbf{E}(f)$ and hence by itself is a random variable. Let $\tilde{R}_i$ be the transmission rate of the i-th user for the equivalent system in (9). Thus,

$$\tilde{R}_i = \int_{f \in B} \tilde{R}_i(f) df. \tag{13}$$

Definition 3.1: The Transmission loss $L_i(f)$ of the i-th user at frequency f is given by

$$L_i(f) = R_i(f) - \tilde{R}_i(f). \tag{14}$$

The **total loss** of the i-th user is

$$L_i = \int_{f \in B} L_i(f) df. \tag{15}$$

We are ready to deduce a formula for the rate loss of the i-th user as a result of the non-ideal precoding system in (9). Our result will be given in terms of the matrix $\mathbf{\Delta}$. Recall that $\mathbf{\Delta}$ depends on the precoder quantization errors $\mathbf{E}(f)$.

Denote by $\mathbf{\Delta}_{i,j}$ the (i,j)-th element of the matrix $\mathbf{\Delta}$ and let

$$\delta_i(f) = \Gamma \sum_{j \neq i} \frac{P_j(f)}{P_i(f)} |\Delta_{i,j}(f)|^2. \quad (16)$$

Let

$$a_i(f) = \delta_i(f)\Gamma^{-1} SNR_i(f) = \sum_{j \neq i} \frac{P_j(f)}{P_i(f)} |\mathbf{\Delta}_{i,j}(f)|^2 SNR_i(f), \quad (17)$$

$$q_i(\mathbf{\Delta}, f) = \frac{|1 + \mathbf{\Delta}_{i,i}(f)|^2}{a_i(f) + 1}, \quad (18)$$

and

$$k_i(f) = \frac{\Gamma^{-1} SNR_i(f)}{\Gamma^{-1} SNR_i(f) + 1}. \quad (19)$$

Note that $a_i(f)$ and hence $q_i(\mathbf{\Delta}, f)$ are independent of the Shannon gap Γ. The next lemma provides a formula for the exact transmission rate loss due to the errors modelled by the matrices $\mathbf{E}(f)$. The result is stated in terms of quantities $q(\mathbf{\Delta}, f)$ and the effective signal to noise ratio, $\Gamma^{-1} SNR_i(f)$.

Lemma 3.2: Let $\mathbf{H}(f)$ be the channel matrix at frequency f and let $\mathbf{E}(f)$ be the quantization error as in (7). Let $L_i(f)$ be the loss in transmission rate of the i-th user defined in (14). Then:

$$L_i(\mathbf{\Delta}, f) = -\log_2 \left(1 - k_i(f)(1 - q_i(\mathbf{\Delta}, f))\right), \quad (20)$$

where $q_i(\mathbf{\Delta}, f)$ is given in (18) and $k_i(f)$ is given in (19).

In particular, if $\mathbf{\Delta}_{i,i}(f) = -1$ the transmission loss is $\log_2(1 + \Gamma^{-1} SNR_i(f))$, where $SNR_i(f)$ is defined in (8). Finally, if $\Delta_{i,i}(f) \neq -1$ we have

$$L_i(\mathbf{\Delta}, f) \leq Max\left(0, \log_2\left(\frac{1}{q_i(\mathbf{\Delta}, f)}\right)\right) \quad (21)$$

A usefull corollary of the lemma is

Corollary 3.3: Let $\mathbf{H}(f)$ be the $p \times p$ channel matrix at frequency f and let $\mathbf{E}(f)$ be the quantization error as in (7). Let $L_i(f)$ be the transmission rate loss of the i-th user defined in (14). Furthermore assume that $t_i(f) = max_{1 \leq j \leq n} |\mathbf{\Delta}_{i,j}| < 1$ and let $M_i(f) = max_{j \neq i} \frac{P_j(f)}{P_i(f)}$. Then

$$L_i(\mathbf{\Delta}, f) \leq \log_2\left(\frac{1 + (p-1)M_i(f)t_i^2(f)SNR_i(f)}{(1 - t_i(f))^2}\right). \quad (22)$$

The proof of the lemma and the corollary is omitted due to space limitations.

We are now ready to analyze the effect of quantization errors on the transmission rate of an individual user of the DSL system.

We will not make any further assumptions about the particular quantization method employed and we will provide upper bounds for the capacity loss. *We do not assume any specific random model for the values of* $\mathbf{E}(f)$ *because we are interested in obtaining absolute upper bounds on capacity loss*.

We assume that the Power Spectral Density (PSD) of all the users of the binder is the same. Namely, we assume that $P_i(f) = P(f)$ for all $i = 1, ..., p$ for some fixed unspecified function $P(f)$. The justification of this assumption is that in a precoded DSL system each user will use the entire PSD mask allowed by the regulation since in such systems no interference is generated to other systems.

The following theorem (the proof is omitted) describes the transmission rate loss resulting from quantization of the precoder.

Main Theorem *3.4:* Let $\mathbf{H}(f)$ be the channel matrix of p twisted pairs in frequency f. Let $r(f) = r(\mathbf{H}(f))$ as in (6), and assume that $r(f) \leq 1$ for all $f \in B$. Assume that the precoder $\mathbf{P}$ is quantized using $d \geq 4$ bits. The transmission rate loss of the i-th user at frequency f due to quantization with d bits, $L_i(d, f)$, is bounded by

$$L_i(d, f) \leq \log_2(1 + \gamma(d, f)SNR_i(f)) - 2\log_2(1 - v(f)2^{-d}), \quad (23)$$

where

$$\gamma(d, f) = 2(p-1)(1 + r(f))^2 2^{-2d} \quad (24)$$

and

$$v(f) = \sqrt{2}(1 + r(f)). \quad (25)$$

In particular, with $r_{max} = max_{f \in B}(r(\mathbf{H}(f))$, the transmission loss in the band B resulting from quantization with d bits, $L_i(d)$, is at most

$$\begin{aligned} L_i(d) \quad \leq \quad & \textstyle\int_{f \in B} \log_2(1 + \gamma(d)SNR_i(f))df \\ & -2|B| \log_2(1 - (1 + r_{max})2^{-d+0.5}), \end{aligned} \quad (26)$$

where $|B|$ is the total bandwidth, and

$$\gamma(d) = 2(1 + r_{max})^2 (p-1)2^{-2d}. \quad (27)$$

We also introduce the relative loss to be

$$\eta_i(d) = \frac{L_i(d)}{R_i} \quad (28)$$

where R_i is the transmission rate of the i-th user in the Band B as in (3).

The theorem has a number of interesting consequences . For example, if $r_{max} \leq 1$ and we consider large values of d, then the dominant term in the bound is $-2|B| \log_2(1 - 2^{-d+1.5})$ thus for large values of d the loss is about $\frac{\sqrt{32}}{\ln(2)} 2^{-d}$ bps/Hz per user. However, for many practical values of the parameters (e.g. $SNR_i(f) = 80dB$, $d \leq 20$, $p \leq 100$) the first term of (23) is the dominant and can not be ignored. We note that the proof provides the following simple bound on the loss at one tone f:

$$L_i(\mathbf{\Delta}, f) \leq 2^{-d+3.5} + \log_2(1 + 8(p-1)SNR_i(f)2^{-2d}) \quad (29)$$

IV. SIMULATION RESULTS

To check the quality of the bounds stated in our theorem 3.4 and its corollaries, we compared the bounds with simulation results which were based on measured channels. We have used the measurement campaign conducted by France Telecom R&D as described in [9]. We studied a bandwidth of 30 MHz. In each bin we used a measured channel of 28 pairs of length 300m. We quantized the precoder matrix in each frequency and computed the resulting channel capacity of each of the 28 users and computed the relative and absolute capacity loss of each of the users. In each bin we picked the worst case out of 1000 quantization trials and obtained a quantity we called maximal loss. The quantity maximal loss is a random variable depending on the number of bits used to quantize the precoder matrices. Each value of this random variable provides a lower bound for the actual worst case that can occur when the

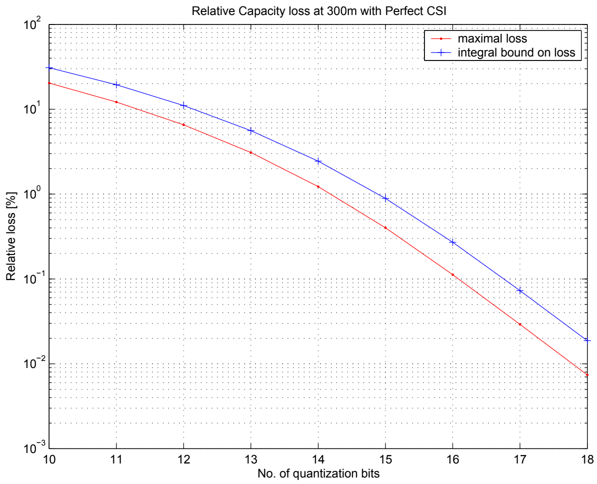

Fig. 1. Relative Capacity loss vs number of quantizer bits with perfect CSI in a system of 28 users

channel matrices are quantized. We compare this lower bound with our upper bounds of theorem 3.4. To obtain relative results we divided both quantities by R_i. We have checked our bounds in the following scenario: each user has flat PSD of -60dBm/Hz, the noise has flat PSD of -140dBm/Hz. The Shannon gap is assumed to be $10.7dB$. As can be seen in Figure 1, the bound given by the integral (26) is sharp.

V. CONCLUSIONS

In this paper we analyzed finite word length effects on the achievable rate of vector DSL systems with zero forcing precoding. The results of this paper provide simple analytic expressions for the loss due to finite word length. These expressions allow simple optimization of linearly precoded DSM level 3 systems.

We validated our results using measured channels. Moreover, we showed that our bounds can be adapted to study the effect of measurement errors on the transmission loss. In practice for loop lengths between 300 and 1200 meters, one needs 14 bits to represent the precoder elements in order to lose no more than one percent of the capacity.

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