APPROXIMATE BER EXPRESSION OF ML EQUALIZER FOR OFDM OVER DOUBLY SELECTIVE CHANNELS

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ABSTRACT

Maximum Likelihood (ML) equalization for Orthogonal Frequency Division Multiplexing (OFDM) over time- and frequency- selective channels is analyzed in this paper. An approximate expression for bit error rate (BER) performance of the ML equalizer with a limited number of taps is developed, which subsumes the matched filter bound (MFB) equalization performance as a special case, and which saves on intensive time-consuming empirical simulations. Numerical simulations validate our approximate expression.

Index Terms— Time-varying channels, Multipath channels, Maximum likelihood detection

1. INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) is a high-rate transmission technique, which mitigates inter- symbol interference (ISI) by the insertion of cyclic prefix (CP) at the transmitter and its removal at the receiver. One major drawback of OFDM is its sensitivity to channel temporal variations, which arise from the relative motion between the transmitter and the receiver as well as from the presence of the carrier frequency offset (CFO) due to the oscillators' mismatch. The orthogonality between the subcarriers is thus destroyed and inter-carrier interference (ICI) that adversely degrades the system performance is generated. While the ICI attributed to CFO may be compensated for at the receiver, the ICI from the mobility-induced Doppler spread is not easily suppressed, since signals from different paths have different Doppler frequencies.

For slowly time-varying channel, assuming that the channel varies temporally in a linear fashion, a frequency-domain equalizer was presented in [1]. Block-wise linear equalizers based on zero-forcing (ZF) and minimum mean squared error (MMSE) criteria were considered in [2]. On the other hand, to mitigate the effects of ICI and to reduce the equalization complexity of block-wise equalizers, an iterative MMSE equalizer with linear preprocessing was also proposed in [3]. Special channel structures are exploited to develop low-complexity linear equalizer [4]. In [5], a Viterbi-type maximum likelihood (ML) equalization in the frequency domain was proposed to suppress the ICI, by utilizing the structure of the ICI and the null (or virtual) subcarriers, which are originally set in each OFDM symbol to mitigate interferences from/to adjacent OFDM channels.

In theory, ML equalization that takes into account all the ICI terms achieves the minimum block error rate. But in practice, only a limited number of ICI terms could be compensated due to hardware limitation. To see the effects of the number of taps of the ML equalizer on the performance, an approximate expression for bit error rate (BER) performance of the ML equalizer is developed, which subsumes the matched filter bound (MFB) equalization performance [6] as a special case. The benefit conferred by the use of this BER expression to measure the ML performance is many fold. It helps to enhance our understanding of the effect of channel statistics on BER. At the same time, since it is simple to generate, it renders time-consuming simulations obsolete and make performance comparisons easy. Numerical simulations are provided to validate our approximate expression.

2. OFDM OVER DOUBLY SELECTIVE CHANNELS

We consider point-to-point Orthogonal Frequency Division Multiplexing (OFDM) transmissions over doubly (time- and frequency-) selective fading channels. For notational simplicity, we only deal with one OFDM symbol duration.

Let the number of subcarriers be N. At the transmitter, a serial information data sequence $\{s_0, s_1, \ldots, s_{N-1}\}$ undergoes serial-to-parallel (S/P) conversion to be stacked into one OFDM symbol. Subsequently, an N-points inverse discrete Fourier transform (IDFT) follows to produce the N dimensional data, which is parallel-to-serial (P/S) converted. A cyclic prefix (CP) of length N_{cp} is appended to the tail of the data sequence in order to mitigate the multipath effects. The transmitted symbols $\{u(n)\}$ can be expressed as

$$u(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} s_k e^{j\frac{2\pi kn}{N}}, \quad n \in [-N_{cp}, N-1]. \qquad (1)$$

The channel has maximum order L. We assume that $N_{cp} \geq L$ so that there is no inter-symbol interference (ISI) between

OFDM symbols. Without loss of generality, the discrete-time baseband equivalent description of the nth received sample is expressed as [5]

$$y(n) = \sum_{l=0}^{L} h(n;l)u(n-l) + w(n), \quad (2)$$

where $h(n;l)$ denotes the lth channel tap at time n and $w(n)$ the additive white Gaussian noise (AWGN) with zero mean and variance σ_w^2. The channel frequency response at frequency $2\pi k/N$ and at time n is $H_k(n)=\sum_{l=0}^{L} h(n;l)e^{-j\frac{2\pi kl}{N}}$.

After removing the part of the received signal corresponding to CP, we apply a discrete Fourier transform (DFT) on the received signal to obtain the frequency-domain received signal for $k \in [0, N-1]$ as

$$Y_k = \sum_{n=0}^{N-1} H_{k,n}s_n + W_k, \quad (3)$$

where

$$H_{k,n}=\frac{1}{N}\sum_{m=0}^{N-1} H_k(m)e^{j\frac{2\pi m(n-k)}{N}}, W_k=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} w(n)e^{-j\frac{2\pi kn}{N}}.$$

Clearly, if the channel is time-invariant, then the channel frequency response $H_k(n)$ is constant in time n. It follows that $H_{k,n} = H_k(n)\delta(n-k)$, where $\delta(\cdot)$ stands for Kronecker's delta. However, since channels are in general time-varying due to the relative motion between the transmitter and the receiver, the ICI term $\sum_{n=0,n\neq k}^{N-1} H_{k,n}s_n$ appears, which is evident in the R.H.S. of (3). Hence, the one-tap equalizer cannot compensate for the effects of the ICI, resulting in a performance floor that increases with the speed of the channel time variation.

2.1. ML Equalization for ICI Suppression

From the collection of N received samples in (3), we form a receive vector $\boldsymbol{Y} = [Y_0, Y_1, \ldots, Y_{N-1}]^T$, which is expressed as

$$\boldsymbol{Y} = \boldsymbol{H}\boldsymbol{s} + \boldsymbol{W}, \quad (4)$$

where $H_{k,n}$ is the $(k+1, n+1)$st entry of the channel $\boldsymbol{H}$, $\boldsymbol{s} = [s_0, s_1, \ldots, s_{N-1}]^T$, and $\boldsymbol{W} = [W_0, W_1, \ldots, W_{N-1}]^T$.

The optimal equalization for the block transmission model in (4) is provided by the maximum likelihood criterion, which for a given $N \times N$ channel matrix $\boldsymbol{H}$ realization under the AWGN model detects the transmitted vector $\boldsymbol{s}$ as

$$\boldsymbol{s}_{ML} = \arg\min_{\boldsymbol{s}} \|\boldsymbol{Y} - \boldsymbol{H}\boldsymbol{s}\|^2, \quad (5)$$

where $\|\cdot\|$ denotes Euclidean vector norm.

For simplicity of presentation, we utilize a circular index with respect to N where the index n of a sequence corresponds to n modulo N.

Ignoring the ICI terms which do not significantly affect the kth subcarrier, we may assume that the ICI terms come only from $2K$ $(K > 0)$ neighboring subcarriers, i.e.,

$$H_{k,k+n} = 0 \quad \text{for} K < |n|, \quad n \in [0, N-1].$$

This gives rise to the so-called nearly banded channel matrix structure.

Albeit the channel matrix is sparse, an exhaustive search is nonetheless needed for ML equalization, since ICI terms appear in a circular fashion. To avoid the exhaustive search, successive null subcarriers, which are usually embedded in every OFDM symbol to mitigate interferences from/to adjacent OFDM channels, are utilized to develop a Viterbi-type ML equalization in [5].

Without loss of generality, we put N_{G_1} successive null subcarriers at the top and N_{G_2} successive null subcarriers at the bottom of $\boldsymbol{s}$, i.e.,

$$s_k = 0, \quad k \in [0, N_{G_1}-1], k \in [N - N_{G_2}, N-1]. \quad (6)$$

We assume that $K \leq N_{G_1}$ and $K \leq N_{G_2}$. Then, we can consider the first N_{G_1} and the last N_{G_2} columns of $\boldsymbol{H}$ as zero vectors. Removing the entries of $\boldsymbol{H}$ corresponding to null subcarriers, i.e., removing the first N_{G_1} and the last N_{G_2} columns of $\boldsymbol{H}$, effectively renders the channel matrix exactly banded. Consequently, it follows from (3) that

$$Y_k = \sum_{n=-K}^{K} H_{k,n}s_n + W_k. \quad (7)$$

Since Y_k depends only on $2K+1$ instead of N successive s_n, a dynamic programming, e.g., Viterbi algorithm, can be easily applied to obtain the maximum likelihood sequence [5].

3. APPROXIMATE BER OF ML EQUALIZER

ML equalization that accounts for all the ICI terms achieves the minimum block error rate, whose performance is expressed as the matched filter bound (MFB) [6]. If the channel matrix is strictly banded, then the Viterbi equalizer described above attains the MFB. However, even though each out-of-band entry of the channel matrix may be small, their overall effect cannot be totally ignored. To analyze the effects of the residual ICI terms of the ML equalization for doubly selective channels on BER, let us develop an approximate expression for BER performance of the ML equalizer with a limited number of taps, which subsumes MFB as a special case.

For our analysis, we assume a wide-sense stationary uncorrelated scattering (WSSUS) channel model, whose correlation function is characterized by

$$E\{h(n_1;l_1)h^*(n_2;l_2)\} = \sigma_{l_1}^2\delta(l_1-l_2)R(n_1-n_2). \quad (8)$$

This implies that channel coefficients having different delays are independent but obey the same time-correlation function $R(n)$.

Let us define

$$\tilde{\boldsymbol{h}}_k = [H_k(0), H_k(1), \ldots, H_k(N-1)]^T. \quad (9)$$

Subsequently, it follows from (8) that

$$E\{\tilde{\boldsymbol{h}}_k \tilde{\boldsymbol{h}}_k^{\mathcal{H}}\} = c\boldsymbol{R}, \quad (10)$$

where $c = \sum_{l=0}^{L} \sigma_l^2$, and $\boldsymbol{R}$ represents the $N \times N$ channel correlation matrix with $[\boldsymbol{R}]_{m,n} = R(m-n)$.

We consolidate the $2K+1$ received signals to be used for the equalization of s_k into a vector as follows

$$\boldsymbol{Y}_k = [Y_{k-K}, \ldots, Y_{k-1}, Y_k, Y_{k+1}, \ldots, Y_{k+K}]^T. \quad (11)$$

From (3), $\boldsymbol{Y}_k$ can be expressed as

$$\boldsymbol{Y}_k = \sum_{n=0}^{N-1} \boldsymbol{F}_{k,n} \tilde{\boldsymbol{h}}_n s_n + \boldsymbol{W}_k, \quad (12)$$

where $\boldsymbol{F}_{k,n}$ is a $(2K+1) \times N$ matrix defined with $\mathcal{W} = e^{-j2\pi/N}$ as

$$\boldsymbol{F}_{k,n} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & \mathcal{W}^{k-n-K} & \cdots & \mathcal{W}^{(k-n-K)(N-1)} \\ 1 & \mathcal{W}^{k-n+1-K} & \cdots & \mathcal{W}^{(k-n+1-K)(N-1)} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \mathcal{W}^{k-n+K} & \cdots & \mathcal{W}^{(k-n+K)(N-1)} \end{bmatrix},$$

and $\boldsymbol{W}_k = [W_{k-K}, \ldots, W_{k-1}, W_k, W_{k+1}, \ldots, W_{k+K}]^T$.

By separating $\boldsymbol{Y}_k$ into one part comprising the signal $\{s_{k-K}, \ldots, s_k, \ldots, s_K\}$ and treating the remaining part as noise, we can reexpress $\boldsymbol{Y}_k$ as

$$\boldsymbol{Y}_k = \sum_{n=-K}^{K} \boldsymbol{F}_{k,n} \tilde{\boldsymbol{h}}_n s_n + \boldsymbol{V}_k, \quad (13)$$

where $\boldsymbol{V}_k$ is the effective noise which is the sum of the residual ICI terms and the additive noise such that

$$\boldsymbol{V}_k = \sum_{n=0, n \notin [k-K, k+K]}^{N-1} \boldsymbol{F}_{k,n} \tilde{\boldsymbol{h}}_n s_n + \boldsymbol{W}_k. \quad (14)$$

For N sufficiently large and a small number of null subcarriers, i.e., $N >> N_{G_1} + N_{G_2}$, the effects of null subcarriers can be ignored. As such, $\boldsymbol{R}$ is approximately diagonalized with DFT matrix $\boldsymbol{F}$ and IDFT matrix $\boldsymbol{F}^{\mathcal{H}}$ as

$$\boldsymbol{R} = \boldsymbol{F} \text{diag}[S(1), S(\mathcal{W}), \ldots, S(\mathcal{W}^{N-1})] \boldsymbol{F}^{\mathcal{H}}. \quad (15)$$

If information symbols are independent, have zero mean and the same variance σ_s^2, then the correlation matrix of $\boldsymbol{V}_k$ is found to be

$$E\{\boldsymbol{V}_k \boldsymbol{V}_k^{\mathcal{H}}\} = \left[\frac{c\sigma_s^2 R(0)}{N} + \sigma_w^2\right] \boldsymbol{I}_{2K+1} \quad (16)$$

$$-c\sigma_s^2 \sum_{n=-K}^{K} \text{diag}[S(\mathcal{W}^{n-K}), \ldots, S(\mathcal{W}^{n+K})],$$

where we used the equality $R(0) = \sum_{n=0}^{N-1} S(\mathcal{W}^n)/N$. This shows that the effective noise correlation matrix $E\{\boldsymbol{V}_k \boldsymbol{V}_k^{\mathcal{H}}\}$ is independent of its subcarrier index k. It is a diagonal matrix whose diagonal entries have in general different values. Thus, strictly speaking, the Viterbi-type equalizer with insufficient number of taps is not equivalent to the optimal equalizer, even if $\boldsymbol{V}_k$ can be assumed to be Gaussian. However, since the diagonal entries of $E\{\boldsymbol{V}_k \boldsymbol{V}_k^{\mathcal{H}}\}$ have almost the same value, the ML equalizer is nearly optimal if $\boldsymbol{V}_k$ is Gaussian.

Now, we roughly evaluate the error-rate performance of the Viterbi-type equalizer. We define the estimates of Y_k as $\hat{\boldsymbol{Y}}_k = [\hat{Y}_{k-K}, \ldots, \hat{Y}_k, \ldots, \hat{Y}_K]^T$. Its relation with the symbol estimates $\hat{s}_n$ is given by

$$\hat{\boldsymbol{Y}}_k = \sum_{n=-K}^{K} \boldsymbol{F}_{k,n} \tilde{\boldsymbol{h}}_n \hat{s}_n. \quad (17)$$

Suppose that there are no more than two errors in $2K+1$ successive symbol estimates and consider an error event such that $\hat{s}_k \neq s_k$ and $\hat{s}_{k+m} = s_{k+m}$ for $m \in [-K, K]$ and for $m \neq 0$. Then, the error vector $\boldsymbol{e}_k = \hat{\boldsymbol{Y}}_k - \boldsymbol{Y}_k$ is expressed as

$$\boldsymbol{e}_k = \boldsymbol{F}_{k,k} \tilde{\boldsymbol{h}}_k (\hat{s}_k - s_k) + \boldsymbol{V}_k. \quad (18)$$

From the Central Limit Theorem, for N sufficiently large, $\boldsymbol{V}_k$ can be approximated as a Gaussian vector with zero mean and correlation $\boldsymbol{\Sigma} := E\{\boldsymbol{V}_k \boldsymbol{V}_k^{\mathcal{H}}\}$. Under this approximation, $\tilde{\boldsymbol{h}}_k$ and $\boldsymbol{V}_k$ become statistically independent.

An error event occurs if a real Gaussian variable z of unit variance exceeds $||\boldsymbol{\Sigma}^{-1/2} \boldsymbol{F}_{k,k} \tilde{\boldsymbol{h}}_k (\hat{s}_k - s_k)||/\sqrt{2}$. Let δ_{min} denote the minimum distance between symbols. As a result, the error probability is bounded by

$$\overline{\text{BER}} := Q\left(\frac{\delta_{min}}{\sqrt{2}} ||\boldsymbol{\Sigma}^{-1/2} \boldsymbol{F}_{k,k} \tilde{\boldsymbol{h}}_k||\right), \quad (19)$$

where the Gaussian function $Q(x) = (1/\sqrt{2\pi}) \int_x^\infty e^{-t^2/2} dt$.

The vector $\boldsymbol{\Sigma}^{-1/2} \boldsymbol{F}_{k,k} \tilde{\boldsymbol{h}}_k$ in the norm of (19) is Gaussian with zero mean and its correlation is given by

$$\tilde{\boldsymbol{\Sigma}} := \text{diag}[\tilde{\sigma}_{-K}^2, \ldots, \tilde{\sigma}_K^2], \quad (20)$$

where

$$\tilde{\sigma}_m^2 = \frac{cS(\mathcal{W}^m)}{c\sigma_s^2 \sum_{n=0, n \notin [-K,K]}^{N-1} S(\mathcal{W}^{n+m}) + \sigma_w^2}. \quad (21)$$

If the diagonal entries of $\tilde{\boldsymbol{\Sigma}}$ are non-zero and have different values, averaging (19) with respect to the random channel yields [7, Chap. 5]

$$\overline{\text{BER}} = \sum_{n=-K}^{K} \frac{\alpha_n}{2} \left(1 - \frac{1}{\sqrt{1 + \left[\frac{\delta_{min}^2}{4} \tilde{\sigma}_n^2\right]^{-1}}}\right), \quad (22)$$

where

$$\alpha_n = \prod_{m=-K, m \neq n}^{K} \frac{\tilde{\sigma}_m^2}{\tilde{\sigma}_m^2 - \tilde{\sigma}_n^2}. \tag{23}$$

Eq. (22) is the approximate performance bound when $2K+1$ entries are used for equalization. If we can take into account all the ICI terms, then Eq. (22) becomes equivalent to the matched filter bound (MFB) derived in [6], which is the performance limit.

It follows from the definitions that the received signal-to-noise ratio (SNR) is $c\sigma_s^2/\sigma_w^2$. On careful inspection of (21), as SNR gets large, $\tilde{\sigma}_m^2$ converges to a value as $\tilde{\sigma}_m^2 \rightarrow S(\mathcal{W}^m)/(\sigma_s^2 \sum_{n=0, n\notin[-K,K]}^{N-1} S(\mathcal{W}^{n+m}))$. The implication is that at high SNR, the average error probability eventually becomes saturated due to the residual ICI terms. Increasing K reduces the effect of the residual ICI terms and improves the average error probability at the expense of an increase in computational complexity.

4. NUMERICAL EXAMPLES

Based on perfect channel, we test the performance of Viterbi-type equalizer with $K = 1, 2, 3$. Each OFDM symbol has $N = 128$ subcarriers with 4 successive null subcarriers at the frequency edges, i.e., $N_{G_1} = N_{G_2} = 4$. Let T_c be the chip sampling period, which is set to be equivalent to the transmitted symbol duration. We define the normalized Doppler frequency as $\bar{f}_D = (v/c)f_c T_c$, where v, c and f_c respectively denote mobile velocity, speed of light and carrier frequency. We generate 10^3 Rayleigh channels, having 8 complex zero-mean Gaussian taps with identical power profile. Channel taps are independent of each other and fade according to Jakes model [8]. The length of the cyclic prefix is $N_{cp} = 16$. The simulation results are averaged over the channels.

For BPSK constellation, Fig. 1 compares the approximate BER performance with the empirical ones at $\bar{f}_D = 0.002$. Good agreement between them is evident, which validates our approximate expression in (22). As the number of taps increases, the approximate and empirical BERs show an improvement in tandem. The approximate BER is much easier to generate than the empirical BER. Since time-consuming simulations are not incurred to produce the approximate BER, the effect of channel statistics on BER is also simpler to evaluate. All these make the approximate BER attractive to use.

5. CONCLUSIONS

An approximate expression for BER performance of the ML equalizer with a limited number of taps for OFDM over doubly selective channels has been developed to assess the equalization performance without time-consuming Monte-Carlo si mulations. Numerical simulations demonstrated that the expression approximates the actual BER performance well.

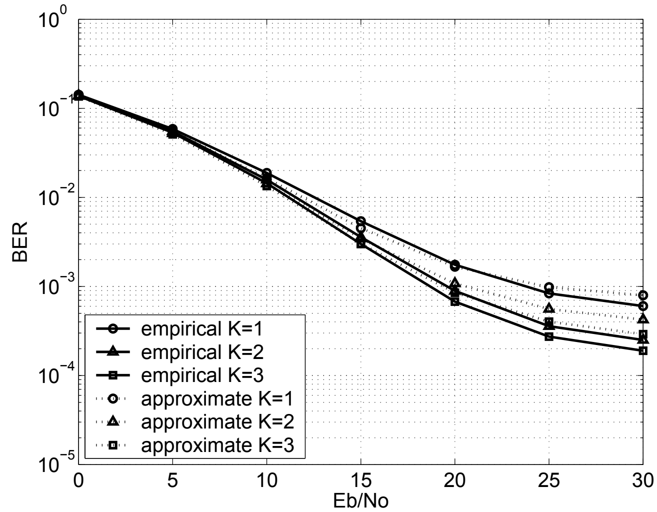

Fig. 1. Empirical and approximate BERs for $\bar{f}_D = 0.002$ with perfect channel.

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