MMSE AND ML ESTIMATION OF CHAOTIC SEQUENCES WITH CODED SIGNS WITH APPLICATIONS IN CODING DISCRETE-TIME ANALOG SIGNALS

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ABSTRACT

We consider estimation of a set of tent-map chaotic sequences. These sequences are obtained by iterating a one dimensional chaotic function in the (-1,1) domain. We demonstrate how to construct a set of *n* sequences, in which information in the form of *n* arbitrary samples in the (-1,1) domain is embedded. Additionally the signs of all sequences jointly form a valid digital codeword. Each sequence in the set consists of *m* elements, hence the set of sequences is equivalent to a rate 1/*m* analog error correcting-code. We present performance in terms of coding gain when maximum likelihood (ML) and minimum-mean-square-error (MMSE) estimation approaches are employed.

Index Terms— Chaos, Error correction coding, Estimation, Communication system signaling

1. INTRODUCTION

Chaotic signals have received much attention in the past years. Several estimation techniques for discrete-time chaotic signals immersed in additive white Gaussian noise have been proposed. These include maximum likelihood (ML) [1-4], Bayesian [5] and iterative techniques such as variants of expectation maximization [6].
All of these methods exhibit a threshold effect, which has been thoroughly investigated from an information theoretic point of view in [7] and a bound on the threshold signal to noise ratio (SNR) has been derived.
We shall focus on symmetric tent-map chaotic sequences, generated by iterating the function:

$$\phi(z) = 1 - 2|z| . \tag{1}$$

An ensemble of samples, uniformly distributed in (-1,1), preserves its distribution when this tent function is applied to it. We shall therefore assume this distribution.
ML estimation for tent-map sequences was presented in [1] and MMSE estimation was presented in [5]. The ML estimation approach was shown to have a linear complexity with respect to the sequence length, and since it possesses an invariance property, it was sufficient to estimate the initial value, from which estimates of other elements in the sequence could be obtained.
This is not true for MMSE estimation. Thus, MMSE estimation of each element in the sequence individually, does not form a valid sequence when concatenated. Additionally, the complexity is exponential. We extend previously published work by deriving a closed form expression for MMSE estimation of all the elements in the sequence.
Even though MMSE estimation provides better estimation with respect to the mean squared error (MSE), both ML and MMSE approaches posses the same threshold. In ML estimation the threshold is a consequence of errors in estimating the signs of the elements, implying that if we limited ourselves to estimating a set of sequences whose signs jointly form a valid binary codeword the threshold could be extended to lower SNR values by decoding them during the estimation process. Once the signs are determined, ML and MMSE estimations can be carried out yielding better performance.
Coding of analog signals with chaotic sequences was initially proposed in [8], and signs coding in this context was proposed later in [9]. However, the combination with MMSE estimation is presented here for the first time.
The outline of this contribution is as follows. Section 2 presents the relevant background on symmetric tent sequences. Section 3 briefly reviews ML estimation and section 4 presents MMSE estimation in a closed form. Section 5 describes how signs coding may be applied. Section 6 discusses log-likelihood-ratio (LLR) calculation. Simulation results are presented in section 7, and finally, we conclude our work in section 8.

2. SYMMETRIC TENT SEQUENCES

Consider a length *m* symmetric tent sequence, $\{z_k\}$, where $k=0,1..m-1$. The sign of element k is denoted by s_k. Given an initial value z_0 and the signs vector $\underline{s}$, we can express element k as a linear function of z_0:

$$z_k = \phi^k(z_0) = a_k z_0 + b_k , \tag{2}$$

where the coefficients are obtained in a recursive manner:

$$\begin{aligned} &a_0 = 1;\; b_0 = 0; \\ &a_k = -2 \cdot s_{k-1} \cdot a_{k-1};\; b_k = 1 - 2 \cdot s_k \cdot b_{k-1}. \end{aligned} \tag{3}$$

For length-m symmetric tent-map sequences there are exactly 2^m possible signs-vectors. We shall therefore enumerate them using variable p. We shall use the notation $\underline{s}^{(p)}$ for denoting signs-vector p. Let $\Delta_p = (\alpha_p, \beta_p)$ denote a section of (-1,1) which is associated with $\underline{s}^{(p)}$. The association is with respect to the signs-vectors of tent-map sequences with initial values in this section as expressed in the following formula:

$$z_0 \in \Delta_p : sign\left\{z_0, \phi(z_0), \ldots \phi^{n-1}(z_0)\right\} = \underline{s}^{(p)}. \tag{4}$$

The inverse tent function may be easily expressed using the signs vector as well:

$$z_k = \phi_{s_k}^{-1}(z_{k+1}) = \tfrac{1}{2} s_k \cdot (1 - z_{k+1}). \tag{5}$$

3. MAXIMUM LIKELIHOOD ESTIMATION

We consider estimation of a sequence $\{z_k\}$, from its samples immersed in white Gaussian noise:

$$r_k = z_k + v_k. \tag{6}$$

The variance of the noise samples is denoted by σ_v^2. In [1] it was shown that ML estimation consists of two steps: filtering and smoothing. During the filtering process, the signs are estimated. The smoothing process is performed by back-propagation (5) using the estimated signs.

The MSE of the filtered samples was shown to approach $0.75 \cdot \sigma_v^2$ as k, the index of the sample increases. The MSE of the smoothed samples was shown to depend strongly on the sign error probability. We denote the SNR, by:

$$\gamma = \sigma_z^2 / \sigma_v^2 = 1 / \left(3 \cdot \sigma_v^2\right). \tag{7}$$

It was shown that the sign error probability decreases very slowly as SNR increases and is approximately given by [8]:

$$P_s = \left(2 \cdot \sqrt{2\pi\gamma}\right)^{-1}. \tag{8}$$

The resulting MSE of the ML estimation was also derived in [8] and was given by:

$$\sigma_{\hat{z}_k, ML}^2 = \sigma_v^2 \cdot \left[\frac{3}{4^{m-k} - 1} \cdot (1 - P_s)^{m-k-1} + 2P_s \cdot \frac{1 - \left((1 - P_s)/4\right)^{m-k-1}}{1 - (1 - P_s)/4}\right]. \tag{9}$$

The first term in the square parentheses decreases exponentially with the sequence length m. However, the second term sets a "noise floor" for the estimation since it converges to a value proportional to the sign error probability as the sequence length increases. These conclusions motivate generating sequences with coded signs.

4. MMSE ESTIMATION

For calculating the MMSE estimate of element z_k we express it as a function of z_0 according to (2), and obtain its estimate by calculating an expectation of $\phi^k(z_0)$ with respect to the a-posterior distribution of z_0 given the received vector.

$$\hat{z}_{k,MMSE} = E\left\{\phi^k(z_0) \mid \underline{r}\right\} = \int_{-1}^{1} f_{z_0|\underline{r}}(z_0 \mid \underline{r}) \cdot \phi^k(z_0) \cdot dz_0. \tag{10}$$

The conditional distribution $f_{z_0|\underline{r}}(z_0 \mid \underline{r})$ may be expressed as a function of other distributions using Bayes formula as follows:

$$f_{z_0|\underline{r}}(z_0 \mid \underline{r}) = \frac{f_{\underline{r}|z_0}(\underline{r} \mid z_0) \cdot f_{z_0}(z_0)}{f_{\underline{r}}(\underline{r})}. \tag{11}$$

These other distributions are simpler to evaluate:

$$f_{\underline{r}|z_0}(\underline{r} \mid z_0) = \left(\frac{1}{\sqrt{2\pi\sigma_v^2}}\right)^N exp\left\{-\frac{\sum_{k=0}^{m-1}\left(r_k - \phi^k(z_0)\right)^2}{2\sigma_v^2}\right\} \tag{12}$$

$$f_{\underline{r}}(\underline{r}) = \int_{-1}^{1} f_{\underline{r}|z}(\underline{r} \mid z_0) \cdot f_{z_0}(z_0) \cdot dz_0 \tag{13}$$

$$f_{z_0}(z_0) = 0.5 \qquad \forall z \in (-1, 1) \tag{14}$$

Combining these distributions we end up with:

$$\hat{z}_{k,MMSE} = E\left\{\phi^k(z_0) \mid \underline{r}\right\} = \frac{\int_{-1}^{1} f_{\underline{r}|z_0}(\underline{r} \mid z_0) \cdot \phi^k(z_0) dz_0}{\int_{-1}^{1} f_{\underline{r}|z_0}(\underline{r} \mid z_0) dz_0}. \tag{15}$$

Even though (15) appears to be simple, the integrals involve a nonlinear function of the expression $\phi^{m-1}(z_0)$ as evident from (12). This expression is a piece-wise linear function with 2^m sections, hence, the integrals have to be evaluated 2^m times. The complexity for MMSE estimation is thus an exponential function of m, while for ML it was shown to be a linear function of m.

Expression (12) may be simplified using the relation

$$\frac{1}{\sigma_v^2} \cdot \sum_{k=0}^{n-1}\left(r_k - \phi^k(z_0)\right)^2 = A \cdot z_0^2 + B \cdot z_0 + C, \tag{16}$$

where the coefficients A,B and C are related to the coefficients a_k and b_k of equation (2) and to the noisy samples r_k in the following way:

$$A = \sum_{k=0}^{n-1} \frac{a_k^2}{\sigma_v^2}; \; B = 2\sum_{k=0}^{n-1} \frac{a_k(b_k - r_k)}{\sigma_v^2}; \; C = \sum_{k=0}^{n-1} \frac{(b_k - r_k)^2}{\sigma_v^2} \tag{17}$$

Notice that these coefficients depend on the signs-vector of the sequence through the coefficients a_k and b_k. Following the definition of Δ_p in (4), we shall use the notation A_p, B_p and C_p for designating these coefficients when the integration in (15) is performed for $z_0 \in \Delta_p$.

Using these notations and some algebraic manipulations we arrive at the following expression for the estimator.

$$\hat{z}_{k,MMSE} = \sum_p b_{k,p} + \frac{\sum_p a_{k,p} \cdot \left[I_{1,p} - \frac{B_p}{2A_p} I_{2,p} \right]}{\sum_p I_{2,p}} \tag{18}$$

$$I_{1,p} = exp\left\{ \frac{B_p^2 - 4A_pC_p}{8A_p} \right\} \cdot \frac{1}{A_p} \cdot \left[exp\left\{ -\frac{(2A_p\alpha_p + B_p)^2}{8A_p} \right\} - exp\left\{ -\frac{(2A_p\beta_p + B_p)^2}{8A_p} \right\} \right] \tag{19}$$

$$I_{2,p} = exp\left\{ \frac{B_p^2 - 4A_pC_p}{8A_p} \right\} \cdot \sqrt{\frac{2\pi}{A_p}} \cdot \left(Q\left(\frac{B_p + 2A_p\alpha_p}{2\sqrt{A_p}} \right) - Q\left(\frac{B_p + 2A_p\beta_p}{2\sqrt{A_p}} \right) \right) \tag{20}$$

where $Q(x)$ is defined as an integral from x to infinity of a unity variance zero mean Gaussian distribution. The last two expressions depend on the received vector $\underline{r}$ through B_p and C_p as evident from (17). Clearly they depend on p. However, they are independent of k, the index of the estimated element. Hence, once $I_{1,p}$ and $I_{2,p}$ are calculated, the MMSE estimation of all the elements in the sequence are obtained through equation (18).

5. CHAOTIC SEQUENCES WITH CODED SIGNS

We wish to generate a set of symmetric tent sequences based on random samples distributed in (-1,1). Additionally the signs of the set of sequences should jointly form a valid codeword. We review here the method proposed in [9].

We begin with *n* samples taken from the domain (-1,1). We perform *q* iterations to each of the samples. The signs of the sequences are coded using some systematic digital code. The obtained parity bits are then taken as signs for (*m*-*q*-1) backwards iterations from each of the initial *n* samples. Thus, each sample is iterated forward and backwards and the set of all signs forms a valid codeword.

6. LOG LIKELIHOOD RATIO OF THE SIGNS

Decoding the signs of the chaotic sequences requires calculation of their LLRs. The LLR of element k of sequence j is formally defined as:

$$LLR_{j,k} = ln\left(\frac{Pr\{z_{j,k} \geq 0 \mid \underline{r}\}}{Pr\{z_{j,k} < 0 \mid \underline{r}\}} \right) \tag{21}$$

where $\underline{r}$ denotes the $n \cdot m$ received samples. While exact calculation is complicated, we can approximate it. One alternative is to consider only received samples which correspond to sequence j. The LLR expression is then simplified to

$$LLR_{j,k} \approx ln\left\{ \frac{Pr\{z_{j,k} \geq 0 \mid \underline{r_j}\}}{Pr\{z_{j,k} < 0 \mid \underline{r_j}\}} \right\} = ln\left(\frac{\sum_{p:\phi^k\left(\frac{1}{2}(\alpha_P+\beta_P)\right)>0} I_{2,p}}{\sum_{p:\phi^k\left(\frac{1}{2}(\alpha_P+\beta_P)\right)<0} I_{2,p}} \right). \tag{22}$$

The last equation implies that the expression of $I_{2,p}$ has to be calculated for all p sections. If element k of a sequence starting in Δ_p, e.g., at $\frac{1}{2}(\alpha_P+\beta_P)$, is positive, $I_{2,p}$ is added to the nominator, otherwise, it is added to the denominator. This approximation is related to MMSE estimation, and therefore, named after it. The calculation has an exponential complexity with respect to the sequences length. The calculation can be further simplified by assuming knowledge of the magnitudes. In practice, the magnitudes can be estimated with the linear-complexity ML approach. In this case we obtain the following expression for the LLR [9]:

$$LLR_{j,k} \approx \frac{2\left|\hat{z}_{j,k}\right| y_{j,k}}{\sigma_y^2} \tag{23}$$

where $y_{j,k}$ denotes a filtered sample corresponding to $z_{j,k}$ with an MSE of σ_y^2, and $\left|\hat{z}_{j,k}\right|$ denotes its ML estimated magnitude. This approach is related to the ML estimation and therefore named after it.

Finally, using the approximate LLRs (23) enables decoding the received signs. The resulting sign error probability is generally significantly lower than the one used when magnitudes were initially estimated. Hence, magnitude estimation can be improved using the decoded signs, which in turn, provides better LLRs. Notice that this iterative approach exploits information from all the sequences in the set during the decoding process overcoming the limitation we had so far.

Once signs are decoded, both ML and MMSE sequence estimation are significantly simplified. An ML estimation method was proposed in [4]. The method examines all possible sign vectors, and then, selects the one which provides the lowest MSE. This method is suitable here with a slight twist in the sense that only one signs vector is considered.

For the MMSE estimation (18) the summation over p is replaced with selecting only one section which corresponds to the decoded signs vector. Notice that an estimate of element k in a sequence is now directly related to the estimate of the first element, as shown in the following equations.

$$\hat{z}_{0,MMSE} = \frac{I_{1,p}}{I_{2,p}} - \frac{B_p}{2A_p}; \;\; \hat{z}_{k,MMSE} = a_k \cdot \hat{z}_{0,MMSE} + b_k \tag{24}$$

7. SIMULATION RESULTS

We considered estimation of a set of 250 sequences, each composed of 8 elements. We used a rate 0.5 irregular LDPC code for generating their signs, i.e., 3 forward iterations and 4 backwards iterations were performed on each of the 250 source samples.

Figure 1 presents the obtained sign error probability when different approaches for LLR calculation are employed. The ML LLR exhibits the worst performance due to magnitude inaccuracies. A single iteration in the calculation provides some improvement; however, the MMSE approach provides the lowest error floor.

Figure 2 presents the coding gain, which is the ratio between the additive noise variance and the MSE of the estimated source samples, normalized by the bandwidth expansion, *m*. We present several combinations of LLR estimation techniques and sequence estimation approaches.

As expected, MMSE sequence estimation exhibits best asymptotic performance. However, the iterative LLR calculation approach, when combined with MMSE estimation is worth noting for having the lowest threshold SNR, while maintaining linear complexity with respect to the sequences length.

8. CONCLUSIONS

We have introduced the concept of coding of a set of analog samples with chaotic sequences whose signs jointly form a binary codeword. We have presented a closed form MMSE estimator for the sequences, and discussed several options for LLR calculations.

Further work may include different chaotic sequences and more complex noise models.

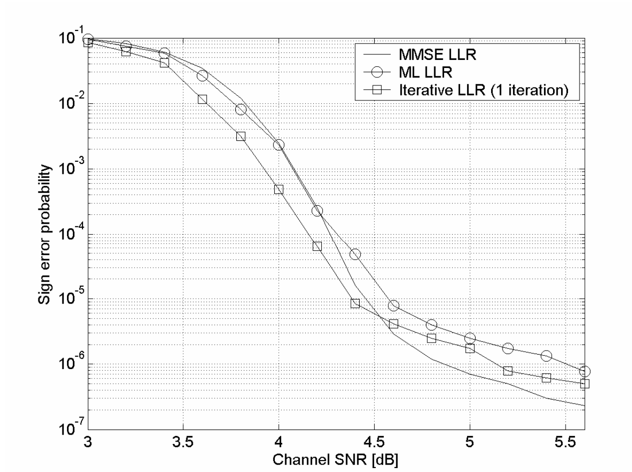

Figure 1. Sign error probability for various approaches for LLR calculations.

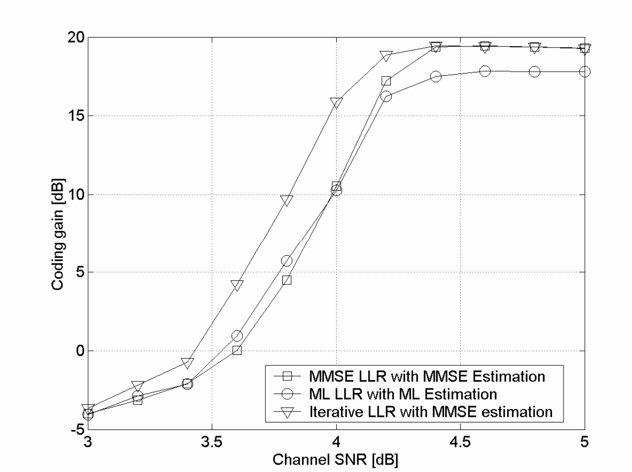

Figure 2. Coding gain for several approaches for LLR calculations and sequence estimation.

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