LIMITING RATE BEHAVIOR AND RATE ALLOCATION STRATEGIES FOR AVERAGE CONSENSUS PROBLEMS WITH BOUNDED CONVERGENCE

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ABSTRACT

Average consensus algorithms are gossiping protocols for averaging original sensor measurements via near neighbor communications. In this paper, we consider the average consensus algorithm under communication rate constraints. Without any communication rate restrictions, the algorithm ideally allows every node state to converge to the initial average in the limit. Noting that brute force quantization does not guarantee convergence due to error propagation effects, in our recent work we proposed two source coding methods which use side information (predictive coding and Wyner-Ziv coding) to achieve convergence with vanishing quantization rates in the case of block coding. In this work, we focus on a simplified predictive coding scheme with variable quantization rates over the iterations and on a communication network with regular topology. We characterize the asymptotic rate which allows to achieve a bounded convergence in terms of the initial conditions (*i.e*, the rate at the first iteration, and the initial state correlation), and the connectivity of the network. Moreover, we study the optimal rate allocation among the average consensus iterations subject to the constraints that the total number of quantization bits is fixed.

Index Terms— Sensor networks, distributed algorithms, communication systems.

1. INTRODUCTION

Assume that each sensor of a network observes a real scalar value $x_i(0)$ where i denotes the sensor index. Distributed average consensus algorithms are decentralized methods allowing all nodes to compute the average of the initial observations ($1/n\sum_{i=1}^{n} x_i(0)$) in an iterative fashion via only near neighbor communications. Different types of consensus protocols and their performance characteristics have been studied extensively in the literature [1, 2, 3]. We are specifically interested in the *synchronous* linear consensus algorithm where each sensor updates its own state by a weighted sum of differences between its neighbors' values and its local value:

$$x_i(k+1) = x_i(k) + \sum_{j \in N_i} [W]_{ij} \left(x_j(k) - x_i(k)\right) \quad (1)$$

W is the weight matrix with non-negative entries where $[W]_{ii} = 1 - \sum_{j=1}^{n} [W]_{ij}$ and N_i is the neighbor set of node i, *i.e.*, $N_i = \{j : [W]_{ij} > 0,\ j \neq i\}$. In vector form (1) is:

$$x(k+1) = Wx(k) \quad (2)$$

where $x(k) = [x_1(k) \dots x_n(k)]^T$. We have recently studied the average consensus algorithm under finite rate quantization constraint [4]. In particular, we have modeled the quantization noise as an uncorrelated additive noise which is a commonly used model when the quantization rate is sufficiently high [5, 6]. Other treatments of the quantized consensus can be found in [7, 8]. Our key observation was that both temporal and spatial correlation among the node states increases as the system progress through the time epochs. Hence, we proposed a communication scheme based on side information which utilizes the increasing correlation among the states. We showed that under these schemes a consensus in the mean of order two can be achieved and the asymptotic average is bounded from the initial average in the mean squared error (MSE) sense. We also derived necessary and sufficient conditions on the quantization noise variances for bounded MSE convergence and proved that there exists rate regions with vanishing behavior such that bounded convergence is satisfied. Furthermore, in [9], we have characterized the MSE performance of the coding scheme in terms of the connectivity, *fixed* quantization rate, and the initial conditions as the node density increases or in homogeneously distributed networks.

In this paper, we extend the analysis given in [9] such that quantization rate is not fixed through the iterations and we fully characterize the asymptotic rate behavior in terms of the initial conditions (*i.e*, the rate at the first iteration, and the statistics of $x(0)$) and the connectivity of the network. Furthermore, we study the rate allocation problem among the iterations subject to the constraints that the total number of quantization bits is fixed and the bounded asymptotic convergence is guaranteed.

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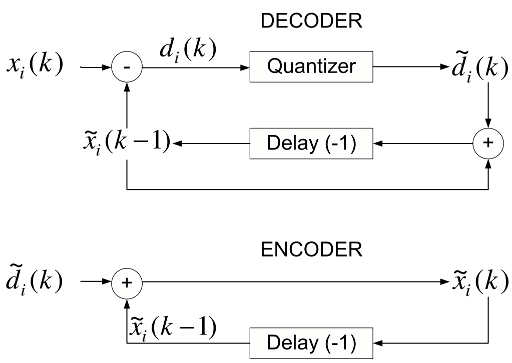

Fig. 1. Differential encoder/decoder diagram.

2. QUANTIZED CONSENSUS MODEL

In this section, we define our model for the rate constrained average consensus (RCAC). To exploit the local temporal correlation, we utilize a simple, suboptimum first order predictive coding model: Node i quantizes the difference between its *unquantized* current state value and quantized previous state value, *i.e.* $x_i(k) - \tilde{x}_i(k-1)$ as in Fig. 1. We define $d_i(k)$ as the prediction or innovation to be transmitted, $Q[\cdot]$ as uniform quantizer, $\tilde{d}_i(k)$ as the quantized prediction error, $\tilde{x}_i(k)$ as the noisy state reconstruction and $n_i(k)$ as the quantization error. Then, we derive the noisy state reconstruction as:

$$\begin{aligned} d_i(k) &= x_i(k) - \tilde{x}_i(k-1) && (3)\\ \tilde{d}_i(k) &= Q[d_i(k)] = Q[x_i(k) - \tilde{x}_i(k-1)] \\ &= x_i(k) - \tilde{x}_i(k-1) + n_i(k) && (4)\\ \tilde{x}_i(k) &= \tilde{d}_i(k) + \tilde{x}_i(k-1) \\ &= x_i(k) + n_i(k) && (5) \end{aligned}$$

where (3) is due to the fact that at each iteration the difference between the current state and the previous quantized state is transmitted, finally, (4) and (5) follow from modeling the quantization error as additive noise and the fact that the previous quantized state value is also known at the neighbors' decoders, respectively.

Once the prediction error ($d_i(k)$) is transmitted and the noisy states ($\tilde{x}_i(k)$) are reconstructed at the nodes, the state values are updated by the recursion:

$$x(k+1) = W\tilde{x}(k) = W\left(x(k) + n(k)\right) \tag{6}$$

where $\upsilon(k) = [n_1(k)\, n_2(k) \ldots n_n(k)]^T$, and $n(k)$ is assumed to be uncorrelated with the messages and is also spatially and temporally uncorrelated, zero mean random vector[1].

In the rest of the paper, we assume that each node encodes and transmits a long block of state variables where the block entries are i.i.d. random variables. Such an assumption is required for utilizing vector quantization properties. We note that each entry of the block follows the update equation given in (6) independently. Moreover, the statistical properties of the states and quantization rates mentioned in the paper are per dimension quantities. We focus on the quantization scheme where quantization rate is uniform among the sensors but variable through the iterations; *i.e.*, can be changed from iteration to iteration.

[1]See Remark 3 in [9] regarding the discussion about the validity of these assumptions.

3. ASYMPTOTIC RATE BEHAVIOR

In this section, we first model the quantization rate for a given iteration in terms of quantization rates in previous iterations, initial conditions, and the connectivity of the network. Then, we characterize the behavior of the quantization rate as the number of iterations grows. By (6), the covariance of the sensor states evolves according to the following recursion:

$$\Sigma(k+1) = W\left(\Sigma(k) + \Upsilon(k)\right)W^T \tag{7}$$

where $\Sigma(k) = \mathbb{E}\{x(k)x^T(k)\}$ and $\Upsilon(k) = \mathbb{E}\{n(k)n^T(k)\}$, with $\mathbb{E}\{\cdot\}$ denoting the statistical expectation. As in [9], we focus on a 2-D regular network.[2] Boyd *et. al.* show that for a random network where n nodes are distributed uniformly on a 2-D unit torus with connectivity radius r, the degree of each node is $O(nr^2)$ with high probability[10]. Therefore, our analysis is also valid for sufficiently large random networks. Under regularity conditions, weight matrix of the network (W) and initial correlation matrix ($\Sigma(0)$) are block circulant with circulant blocks and they are simultaneously diagonalizable by the Kronecker product of two ($\sqrt{n} \times \sqrt{n}$) FFT matrices. Therefore, the recursion in (7) can be written in terms of eigenvalues as:

$$\sigma_i(k) = |\omega_i|^2\left(\sigma_i(k-1) + \upsilon_i(k-1)\right) \tag{8}$$

where $\sigma_i(k)$, ω_i, $\upsilon_i(k)$ are the eigenvalues of $\Sigma(k)$, W and $\Upsilon(k)$, and $i \in \{1, 2, \ldots, n\}$. The eigenvalues of W are ordered, *i.e.* $1 = w_1 \geq w_2 \geq \ldots \geq w_n$. Moreover, the noise covariance matrix is diagonal with equal entries and eigenvalues of the matrix are given as:

$$\upsilon(k) = \frac{1}{n2^{2R(k)}}\left(\sum_{i=1}^{n}(\omega_i - 1)^2\left(\sigma_i(k-1) + \upsilon(k-1)\right)\right) \tag{9}$$

where $R(k)$ is the quantization rate at iteration k. The derivation of (9) can be found in [9].

We will use the system of difference equations in (8), (9) and the constraint that $R(k) > 0$, $\forall k \in \{1, 2, \ldots\}$, to characterize the rate behavior. The constraint on $R(k) > 0$ guarantees that the above system represents a valid quantization

[2]A regular graph is a graph where each vertex has the same number of neighbors.

scheme (non-positive rate is not possible). Since we are interested in the asymptotic behavior, we exclude the case where $R(k)$=0 for finite k. In [4], we have shown that the MSE of the average consensus protocol can be characterized as the summation of the noise variances over iterations (under regularity conditions, summation of the eigenvalue of noise covariance matrix over iterations):

$$\mathrm{MSE}_\infty = \lim_{s\to\infty} \sum_{k=0}^{s} \upsilon(k). \tag{10}$$

By definition, *bounded MSE convergence* constraint implies $\mathrm{MSE}_\infty = \sum_{k=0}^{\infty} \upsilon(k) < \infty$. We can enforce bounded convergence by choosing:

$$\frac{\upsilon(k+1)}{\upsilon(k)} = \beta \tag{11}$$

$$\text{subject to} \sum_{k=0}^{\infty} \upsilon(k) = \upsilon(0) \sum_{k=0}^{\infty} \beta^k < \infty. \tag{12}$$

Such a constrained is satisfied if $0 < \beta < 1$. We are well aware that geometric convergence of the noise variances is neither the unique way to obtain bounded convergence nor results in optimum (minimum) rate regions. More general cases can be captured by allowing (β) to be a function of the iteration index,(k). For mathematical brevity, we focus on the special case of geometric convergence.

Remark 1. *Given* $(0 < \beta < 1)$*, network connectivity* $(w_1, \ldots, w_n)$*, and initial conditions* $(\sigma_1(0), \ldots, \sigma_n(0), R(0))$*, quantization rates follow the non-linear recursion forall* $k \geq 1$*:*

$$\begin{aligned} 2^{2R(k+1)} &= \frac{2^{2R(k)}}{\beta} \frac{\sum_{j=2, w_j\neq 0}^{n} (w_j - 1)^2 \sigma_j(k)}{\sum_{j=2, w_j\neq 0}^{n} \frac{(w_j-1)^2}{w_j^2} \sigma_j(k)} \\ &+ \frac{1}{n\beta} \sum_{j=2}^{n} (w_j - 1)^2). \end{aligned} \tag{13}$$

Since universal solution for the non-linear recurrence systems neither exists nor the behavior of quantization rates (*i.e* decreasing, increasing, vanishing, *etc.*) is obvious from (13), we focus on the limiting behavior of the above system. The behavior is characterized by the following lemma:

Lemma 1. *Quantization rates for the average consensus protocol under predictive coding scheme defined through equations* (8)-(9) *converges to:*

$$\lim_{k\to\infty} R(k) = R^\star = \frac{1}{2} log_2 \left(\frac{1}{n(\beta - K^\star)} \sum_{i=2}^{n} (w_i - 1)^2 \right) \tag{14}$$

where:

$$K^\star = \frac{\sum_{j=2, w_j\neq 0}^{n} \frac{(w_j-1)^2 w_j^2}{\beta - w_j^2}}{\sum_{j=2, w_j\neq 0}^{n} \frac{(w_j-1)^2}{\beta - w_j^2}} \tag{15}$$

and $w_j^2 < \beta < 1$ *for all* $j \in \{1, \ldots, n\}$.

The proof is omitted due to the space constraints and given in [11]. We note that limiting value of the quantization rate is neither a function of initial node correlation nor the initial quantization rate. It is only a function of the connectivity and the behavior of the noise variance (β). On the other hand, node correlation and the initial quantization rate affect the MSE through (10). We also note that achievable noise variance behavior depends on the connectivity; *i.e.*, as the connectivity increases, the range of achievable β also increases.

Next, we characterize the rate regions ($\{R(0), R(1), \ldots\}$) such that bounded convergence is achieved with vanishing quantization rate ($\lim_{k\to\infty} R(k) \to 0$). Given the network connectivity (w_i) and the network size (n); β_0 defining the rate region which achieves a bounded convergence with vanishing quantization rate is found by solving $R^\star = 0$ in (14):

$$\beta_0 = \frac{1}{n} \sum_{i=2}^{n} (w_i - 1)^2 + \frac{\displaystyle\sum_{i=2, w_i\neq 0}^{n} \frac{(w_i - 1)^2 w_i^2}{\beta_0 - w_i^2}}{\displaystyle\sum_{i=2, w_i\neq 0}^{n} \frac{(w_i - 1)^2}{\beta_0 - w_i^2}} \tag{16}$$

We are particularly interested in the solution $w_2^2 < \beta_0 < 1$. If such a solution exist, corresponding rate region can be found for a desired MSE_∞ follows:

1. Choose initial quantization rate $R(0)$ such that $\mathrm{MSE}_\infty = \upsilon(0)/(1 - \beta^\star)$.
2. Choose the quantization rates $R(k)$ iteratively by (13) for all $k > 1$.

4. RATE ALLOCATION PROBLEM

In this section, we study an achievable rate distortion region for average consensus. We constraint $\mathrm{MSE}_\infty \leq D$ and choose β to minimize the total number of bits spent $R_D = \sum_{k=0}^{\infty} R(k)$.

Since a closed form solution for (13) does not exists, we focus on numerical solutions of the optimal distributions of the quantization rates among the iterations by imposing a specific trend for β. For the regular graph with 64 nodes, with connectivity radius 0.25 and initial observations uncorrelated (*i.e.*, $\sigma_i(0) = \sigma_j(0), \forall i, j$), the total number of bits spent over one thousand iterations versus the MSE is given in Fig. 2. For each MSE-rate pair. the algorithm is initialized with a quantization rate $5 \leq R(0) \leq 20$ and the $R(k)$ is calculated by (13) for a given $0.70 \leq \beta \leq 0.95$. We note that such β ensure that the limit in (14) is non-negative. We notice the fact that given the total number of bits, one can achieve a better MSE performance by choosing a larger β.

Fig. 3 shows the behavior of the quantization rates over iterations for, $\beta = 0.95$, $\beta = 0.90$ and $\beta = 0.85$ constraining the total number of bits to be fixed to $\sum_{k=0}^{200} R(k) \simeq$

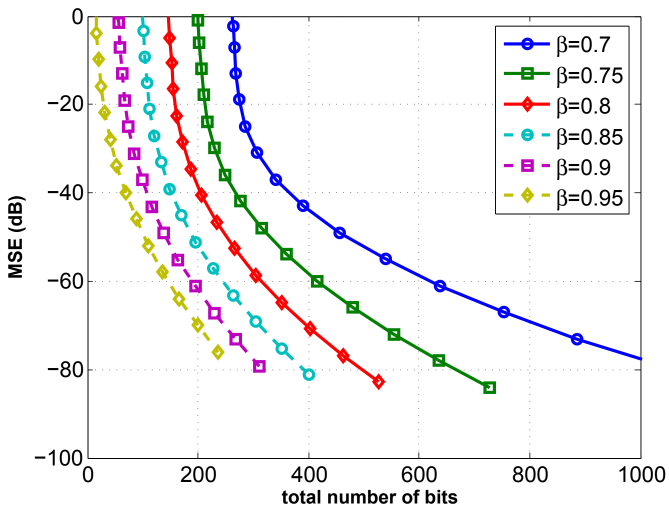

Fig. 2. MSE (dB) vs sum of quantization rates for different β.

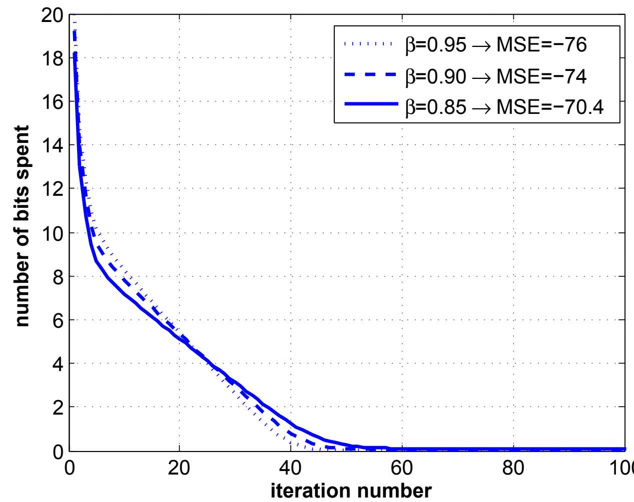

Fig. 3. Number of bits spent vs iteration number for different β.

225. For $\beta = 0.95$, the algorithm starts with 20 bits per node and slowly converges to 0.005 bits per node. On the other hand, for $\beta = 0.90$, the algorithm initializes with 19.2 bits per node, stays under the $\beta = 0.95$ curve initially, then crosses the curve and follows a similar behavior converging to 0.047. The $\beta = 0.85$ curve starts at 18.2 bits per node and converges to 0.091 slower than both 0.90 and 0.95 curves. The MSE performances of these schemes are different, *i.e.*, -76dB, -74dB and -70.4dB respectively, since as we noted in Fig. 2, a larger β results in a better MSE performance for a fixed number of total bits spent. As we mentioned in Section 3, β_0 may have an upper bound which is less than unity and this particular value results in vanishing rate behavior. Therefore, we conclude that if such a solution exists, $\beta^\star$ imposing a vanishing rate behavior achieves the best MSE performance given the total number of bits spent. If such a $\beta^\star$ does not exist, then $\beta^\star \to 1$ performs the best.

5. CONCLUSION

In this paper, we focus on the quantized consensus problem where quantization may change over the iterations. In this scenario, we characterize the limiting behavior of the quantization rate for the average consensus protocol under the first order predictive coding scheme subject to: $1)$ A bounded consensus is achieved, $2)$ the quantization rate has a limit, $3)$the connectivity is as of a regular graph. Under these constraints, we show that the limit is a function of the behavior of the noise variances and the network connectivity. As a consequence, the limit is independent of the initial rate and sensor observation correlation. We also show that the speed at which quantization noise decays is strongly dependent on the connectivity of the network, i.e. the more connected the network is, the faster decay is achievable with $\lim_{k\to\infty} R(k) < \infty$. Furthermore, we analyze the optimal rate allocation problem among the average consensus iterations subject to the constraints that the total number of quantization bits is fixed. By numerically calculating rate-distortion regions, we show that a higher initial rate and a smaller $\beta^\star$ achieves a better MSE than a lower initial rate and a larger $\beta^\star$.

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