A COMPRESSIVE BEAMFORMING METHOD

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ABSTRACT

Compressive Sensing (CS) is an emerging area which uses a relatively small number of non-traditional samples in the form of randomized projections to reconstruct sparse or compressible signals. This paper considers the direction-of-arrival (DOA) estimation problem with an array of sensors using CS. We show that by using random projections of the sensor data, along with a full waveform recording on one reference sensor, a sparse angle space scenario can be reconstructed, giving the number of sources and their DOA's. The number of projections can be very small, proportional to the number sources. We provide simulations to demonstrate the performance and the advantages of our *compressive* beamformer algorithm.

Index Terms— Compressive Sensing, DOA Estimation, Acoustic, Basis pursuit, Convex optimization.

1. INTRODUCTION

The problem of direction-of-arrival (DOA) estimation is extensively studied in array signal processing, sensor networks, remote sensing, etc. To determine a DOA using multiple sensors, generalized cross correlation (GCC), minimum variance distortionless response (MVDR), and multiple signal classification (MUSIC) algorithms are commonly used [1]. By construction, all of these methods require Nyquist-rate sampling of received signals to estimate a small number of DOA's in angle space, which is very expensive in some applications such as radar or radio astronomy. As an example, the Allen Telescope Array northeast of San Fransisco has a frequency coverage from 0.5 to 11.2 GHz for scientific studies. In this paper, we propose a method that takes a very small set of informative measurements that still allow us to estimate DOA's.

Recent results in Compressive Sensing (CS) state that it is possible to reconstruct a K-sparse signal $\boldsymbol{x} = \boldsymbol{\Psi s}$ of length N from $O(K \log N)$ measurements [2]. CS takes non-traditional linear measurements, $\boldsymbol{y} = \boldsymbol{\Phi x}$, in the form of randomized projections. A signal $\boldsymbol{x}$, which has a sparse representation in a transform domain $\boldsymbol{\Psi}$, can be reconstructed from $M = C\left(\mu^2(\boldsymbol{\Phi}, \boldsymbol{\Psi}) \log N\right) K$ compressive measurement exactly with high probability by solving a convex optimization problem of the following form

$$\min \|\boldsymbol{x}\|_1, \qquad \text{subject to} \quad \boldsymbol{y} = \boldsymbol{\Phi \Psi x}. \tag{1}$$

which can be solved efficiently with linear programming.

We use a *basis-pursuit* strategy to formulate the DOA estimation problem as a dictionary selection problem where the dictionary entries are produced by discretizing the angle space and then synthesizing the sensor signals for each discrete angle. Sparseness in angle space implies that only a few of the dictionary entries will be needed to match the measurements. According to the results of CS, it should be possible to reconstruct the sparse dictionary-selector vector from M compressive measurements. Note that we do not take compressive measurements (random projections) of the angle space vector directly. Instead, we can only take random projections of the received signals at the sensors, but we have a model for these as delayed and weighted combinations of multiple source signals coming from different angles.

When the source signals are known, e.g., in active radar, it is possible to directly create the dictionary entries by delaying the known reference signals [3]. When the source signals are unknown and incoherent, we show that we can eliminate the high-rate samplers from all but one of the array elements by using CS to perform the beamforming calculation. We must devote one sensor to acquiring a reference signal, and this operation must be done at a high rate, i.e., Nyquist-rate sampling; the other sensors only need to do compressive sensing. By using the data from the reference sensor, we show that one can relate the compressive measurements at all other sensors to the angle space vector θ linearly, because we assume that the locations of the sensors with respect to the reference sensor are known. This enables us to find the sparse dictionary selector vector by solving an ℓ_1 minimization problem, which is detailed in Section 2.

Our compressive beamforming approach based on ℓ_1 minimization is substantially different from approaches in the literature, such as GCC, MVDR, and MUSIC which require Nyquist sampling at the sensors. In addition, we do not have Gaussian source assumptions, such as GCC, nor have any assumptions about the source signals being narrow or wideband, such as MVDR and MUSIC. In the literature, there are other convex optimization approaches to determine multiple source DOA's, based on regularization [4, 5]. However,

This work is supported under the MURI by the U.S. Army Research Office under contract number DAAD19-02-1-0252.

the common theme of these methods is that they still require Nyquist-rate sampling, followed by conventional beamforming at a small number of angles. Regularized construction on the angle space is then done to constrain the calculation of the conventional beamformer output.

2. THEORY: CS FOR DOA ESTIMATION

We consider cases where the source signal is known or unknown, as well as cases with one source, multiple sources, and additive noise.

2.1. DOA Estimation of a Known Source Signal

Assume that we know the source signal $s(t)$ and we want to determine the DOA of this source, using an array of L sensors with an arbitrary geometry. The sensor positions are assumed known and are given by $\eta_i = [x_i,\ y_i,\ z_i]^T$. When the source is in the far-field of the array, sensor i simply receives a time-delayed and attenuated version of this source

$$\zeta_i(t) = ws\left(t + \Delta_i(\boldsymbol{\pi}_S) - \frac{R}{c}\right), \tag{2}$$

where w is the attenuation, $\boldsymbol{\pi}_S = (\theta_S, \phi_S)$ is the angle pair consisting of the unknown azimuth and elevation angles of the source, R is the range to the source, and $\Delta_i(\boldsymbol{\pi}_S)$ is the relative time delay (or advance) at the i-th sensor for a source with bearing $\boldsymbol{\pi}_S$ with respect to the origin of the array.

Finding the DOA is equivalent to finding the relative time delay, so we ignore the attenuation and assume that the R/c term is known, or constant across the array. The time delay Δ_i in (2) can be determined from geometry:

$$\Delta_i(\boldsymbol{\pi}_S) = \frac{1}{c}\eta_i^T \begin{bmatrix} \cos\theta_S \sin\phi_S \\ \sin\theta_S \sin\phi_S \\ \cos\phi_S \end{bmatrix}, \tag{3}$$

where c is the speed of the propagating wave in the medium.

The source angle pair $\boldsymbol{\pi}_S$ lies in the product of space $[0, 2\pi)_\theta \times [0, \pi)_\phi$, which must be discretized to form the angle dictionary, i.e., we enumerate a finite set of angles for both azimuth and elevation to generate the set of angle pairs $\mathcal{B} = \{\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots, \boldsymbol{\pi}_N\}$, where N determines our resolution. Let $\boldsymbol{b}$ denote the sparsity pattern which selects members of the discretized angle-pair set $\mathcal{B}$, i.e., a non-zero positive value at index j of $\boldsymbol{b}$ selects a target at the az-el pair for $\boldsymbol{\pi}_j$. When we have only one source, we expect the sparsity pattern vector $\boldsymbol{b}$ to have only one non-zero entry, i.e., maximal sparseness.

We can relate the bearing sparsity pattern vector $\boldsymbol{b}$ linearly to the received signal vector at the i-th sensor as follows:

$$\boldsymbol{\zeta}_i = \boldsymbol{\Psi}_i \boldsymbol{b}, \tag{4}$$

$$\boldsymbol{\zeta}_i = \left[\zeta_i\left(t_0\right),\ \zeta_i\left(t_0 + \frac{1}{F_s}\right), \ldots,\ \zeta_i\left(t_0 + \frac{N_t - 1}{F_s}\right)\right]^T, \tag{5}$$

where F_s is the sampling frequency, t_0 is the appropriate initial time, and N_t is the number of data samples. In (4), the j-th column of $\boldsymbol{\Psi}_i$ corresponds to the time shift of the source signal $s(t)$ corresponding to the j-th index of the sparsity pattern vector $\boldsymbol{b}$, which indicates the proper time shift corresponding to the angle pair $\boldsymbol{\pi}_j$:

$$[\boldsymbol{\Psi}_i]_j = \left[s\left(t'_0 + \Delta_i(\boldsymbol{\pi}_j)\right), \ldots, s\left(t'_{K-1} + \Delta_i(\boldsymbol{\pi}_j)\right)\right]^T, \tag{6}$$

where $t' = t - R/c$. The matrix $\boldsymbol{\Psi}_i$ is the dictionary (or, sparsity basis) corresponding to all discretized angle pairs $\mathcal{B}$ at the i-th sensor.

In CS, rather than sampling ζ_i at its Nyquist rate, which would enable recovery of $s(t)$, we measure linear projections with M random vectors which can be written in matrix form for the i-th sensor:

$$\boldsymbol{\beta}_i = \boldsymbol{\phi}_i \boldsymbol{\zeta}_i = \boldsymbol{\phi}_i \boldsymbol{\Psi}_i \boldsymbol{b}, \tag{7}$$

where $\boldsymbol{\phi}_i$ is an $M \times N_t$ matrix, whose rows are random vectors selected to have minimum mutual correlation with $\boldsymbol{\Psi}_i$. Then the sparsity pattern vector $\boldsymbol{b}$ can be found from the set of compressive samples from all the sensors $\boldsymbol{\beta}_{i=1:L}$ by the solving the following ℓ_1 minimization problem

$$\hat{\boldsymbol{b}} = \arg\min ||\boldsymbol{b}||_1 \qquad \text{subject to} \quad \boldsymbol{A}\boldsymbol{b} = \boldsymbol{\beta}, \tag{8}$$

where $\boldsymbol{\beta} = [\boldsymbol{\beta}_1^T, \ldots, \boldsymbol{\beta}_L^T]^T$, and $\boldsymbol{A} = \boldsymbol{\Phi}\boldsymbol{\Psi}$ with $\boldsymbol{\Psi} = [\boldsymbol{\Psi}_1^T, \ldots, \boldsymbol{\Psi}_L^T]^T$, and $\boldsymbol{\Phi}$ the block diagonal matrix of size $LM \times LN_t$ formed with the $\boldsymbol{\phi}_i$'s along its diagonal.

2.2. DOA Estimation of an Unknown Source Signal

In passive sensing problems, the source signal $s(t)$ is not known and is often estimated jointly with the source angle pair $\boldsymbol{\pi}_S$. When $s(t)$ is unknown, we cannot construct $\boldsymbol{\Psi}$ in the ℓ_1 minimization problem (8) to determine the sparsity pattern vector $\boldsymbol{b}$. One alternative is to use the received signal at one sensor (sampled at the Nyquist rate) as the presumed source signal; the rest of the sensors can still collect the compressive samples. We call this sensor the *reference sensor* (RS).

The reference sensor records the signal $\zeta_0(t)$ at a high sampling rate. We can calculate the time shift for sensor i with respect to the RS using (5). Thus, the data at sensor i for an unknown source at bearing $\boldsymbol{\pi}_S$ is $\zeta_i(t) = \zeta_0(t + \Delta_i(\boldsymbol{\pi}_S))$. The sparsity basis matrix $\boldsymbol{\Psi}_i$ for sensor i can be constructed using proper shifts of $\zeta_0(t)$ for each $\boldsymbol{\pi}_j$ in $\mathcal{B}$. Hence, not knowing the source signal incurs a cost of Nyquist rate sampling at one of the sensors, but high data sampling rates from the rest of the array elements are still avoided.

2.3. Effects of Additive Sensor Noises

In general, the i-th sensor receives a noisy version of the RS signal (or the source signal) as $\zeta_i(t) = \zeta_0(t + \Delta_i(\theta_S, \phi_S)) +$

$n_i(t)$. Then the compressive measurements $\boldsymbol{\beta}_i$ at the i-th sensor have the following form:

$$\boldsymbol{\beta}_i = \boldsymbol{\phi}_i \boldsymbol{\zeta}_i = \boldsymbol{\phi}_i \boldsymbol{\Psi}_i \boldsymbol{b} + \boldsymbol{u}_i \qquad (9)$$

where $\boldsymbol{u}_i = \boldsymbol{\phi}_i \boldsymbol{n}_i \sim \mathcal{N}(0, \sigma^2)$ and $\boldsymbol{n}_i$ is the concatenation of the noise samples at the sensor i, which is assumed to be $\mathcal{N}(0, \sigma_n^2)$ Since $\boldsymbol{\phi}_i$ is deterministic, we have $\sigma^2 = (\sum_{n=1}^{N_s} \boldsymbol{\phi}_{il}^2)\sigma_n^2$. Hence, if we constrain the norm of the $\boldsymbol{\phi}_i$ vectors to be one, then $\sigma^2 = \sigma_n^2$.

With the construction of $\boldsymbol{\beta}$ and $\boldsymbol{A}$ in Sect. 2.1, the sparsity pattern vector $\boldsymbol{b}$ can be recovered using the Dantzig selector [6] convex optimization problem:

$$\hat{\boldsymbol{b}} = \arg\min \|\boldsymbol{b}\|_1 \qquad \text{s.t.} \quad \|\boldsymbol{A}^T(\boldsymbol{\beta} - \boldsymbol{A}\boldsymbol{b})\|_\infty < \epsilon_N \sigma. \quad (10)$$

Selecting $\epsilon_N = \sqrt{2 \log N}$ makes the true $\boldsymbol{b}$ feasible with high probability. The optimization problems in (8) and (10) both minimize convex functionals, a global optimum is guaranteed.

2.4. DOA Estimation of multiple unknown sources

Now assume we have another source $s_2(t)$ impinging on the array at the bearing $\boldsymbol{\pi}_2$. If $s_2(t)$ is non-coherent with $s_1(t)$ we can show that its effect is similar to additive noise when we are looking in the direction of the first source signal. In order to show that this additive noise behavior is a correct interpretation, we examine the constraint in (10) which yields a sparse solution for $\boldsymbol{b}$ even in the presence of noise.

The recorded RS signal is

$$\zeta_0(t) = s_1(t) + s_2(t) \qquad (11)$$

assuming equal amplitude signals. The shifted RS signal at the i-th sensor is

$$\zeta_0(t + \Delta_i(\boldsymbol{\pi}_n)) = s_1(t + \Delta_i(\boldsymbol{\pi}_n)) + s_2(t + \Delta_i(\boldsymbol{\pi}_n)) \quad (12)$$

when the assumed bearing is $\boldsymbol{\pi}_n$, and this signal is used to populate the n-th column of the $\boldsymbol{A}$ matrix. On the other hand, the true received signal at the i-th sensor is

$$\zeta_i(t) = s_1(t + \Delta_i(\boldsymbol{\pi}_1)) + s_2(t + \Delta_i(\boldsymbol{\pi}_2)) \qquad (13)$$

where we have different time shifts for the two signals.

The terms in the Dantzig Selector (10) constraint, $\boldsymbol{A}^T\boldsymbol{\beta}$ and $\boldsymbol{A}^T\boldsymbol{A}$ are actually auto- and cross-correlations. For $\boldsymbol{A}^T\boldsymbol{\beta}$ we get a column vector whose n-th element is

$$R_{11}(\Delta_i(\boldsymbol{\pi}_n), \Delta(\boldsymbol{\pi}_1)) + R_{12}(\Delta_i(\boldsymbol{\pi}_n), \Delta(\boldsymbol{\pi}_2)) + \qquad (14)$$
$$R_{12}(\Delta_i(\boldsymbol{\pi}_n), \Delta(\boldsymbol{\pi}_1)) + R_{22}(\Delta_i(\boldsymbol{\pi}_n), \Delta(\boldsymbol{\pi}_2)) \qquad (15)$$

where R_{11} is the autocorrelation of signal $s_1(t)$, R_{22} the autocorrelation of $s_2(t)$, and R_{12} the crosscorrelation. For the matrix $\boldsymbol{A}^T\boldsymbol{A}$, the element in the n-th row and r-th column is

$$R_{11}(\Delta_i(\boldsymbol{\pi}_n), \Delta(\boldsymbol{\pi}_r)) + R_{12}(\Delta_i(\boldsymbol{\pi}_n), \Delta(\boldsymbol{\pi}_r)) + \qquad (16)$$
$$R_{12}(\Delta_i(\boldsymbol{\pi}_n), \Delta(\boldsymbol{\pi}_r)) + R_{22}(\Delta_i(\boldsymbol{\pi}_n), \Delta(\boldsymbol{\pi}_r)) \qquad (17)$$

We make two assumptions: first, that the cross correlation is small—this is the incoherence assumption; second, that the signals decorrelate at small lags, i.e., the autocorrelations are peaked at zero lag. Then we can examine the constraint in (10), and observe that in order to make $\boldsymbol{A}^T\boldsymbol{\beta} - \boldsymbol{A}^T\boldsymbol{A}\boldsymbol{b}$ small we should make sure that the large elements in the vector $\boldsymbol{A}^T\boldsymbol{\beta}$ are cancelled by the large terms in $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{b}$. With our assumptions, the two largest elements in $\boldsymbol{A}^T\boldsymbol{\beta}$ occur when $\boldsymbol{\pi}_n = \boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_n = \boldsymbol{\pi}_2$, because these are cases where we have peaks in the autocorrelations, i.e., $R_{11}(\Delta_i(\boldsymbol{\pi}_1), \Delta(\boldsymbol{\pi}_1))$ and $R_{22}(\Delta_i(\boldsymbol{\pi}_2), \Delta(\boldsymbol{\pi}_2))$. When we cancel the element $R_{11}(\Delta_i(\boldsymbol{\pi}_1), \Delta(\boldsymbol{\pi}_1))$, we use the row of $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{b}$ corresponding to $\boldsymbol{\pi}_n = \boldsymbol{\pi}_1$, so the vector $\boldsymbol{b}$ must select the column where $\boldsymbol{\pi}_r = \boldsymbol{\pi}_1$. Likewise, to cancel the element $R_{22}(\Delta_i(\boldsymbol{\pi}_2), \Delta(\boldsymbol{\pi}_2))$, we use the $\boldsymbol{\pi}_n = \boldsymbol{\pi}_2$ row and the $\boldsymbol{\pi}_r = \boldsymbol{\pi}_2$ column. Our assumptions say that all the other elements will be relatively small.

The bottom line of this analysis is that the Dantzig Selector constraint, with a well-chosen ϵ, will allow the matching of the two signals at their true bearings. Then the ℓ_1 minimization of the selector vector $\boldsymbol{b}$ will tend to pick the signals whose autocorrelation is large. The preceding analysis can be modified for the case where the signals have different amplitudes, but when the relative amplitudes become too different we expect that the ℓ_1 minimization would pick the larger of the two.

This same reasoning can be extended to the case with P unknown sources at bearings $(\theta_1, \phi_1), (\theta_2, \phi_2), ..., (\theta_P, \phi_P)$, impinging on the array of sources. A possible scenario is shown in Fig. 1. Sensor i receives a delayed combination of

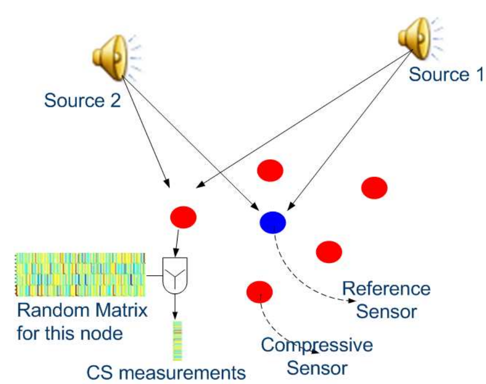

Fig. 1. Sensor setup for compressive beamforming

source signals as

$$\zeta_i(t) = \sum_{p=1}^{P} s(t + \Delta_i(\theta_S, \phi_S)) + n_i(t). \qquad (18)$$

If the non-coherency between sources is satisfied then we can extend the two-source analysis above to the P source case, and claim that the Dantzig Selector constraint will favor the correct source bearings. Thus, the ℓ_1 minimization problem in (10) will reconstruct the appropriate selector vector $\boldsymbol{b}$ from one RS signal and $L-1$ compressed sensor outputs.

3. SIMULATIONS

Finally, a test example is shown to illustrate the ideas presented in the previous section.

Two synthetic speech sources are taken and placed in the far field of a linear array of 11 sensors placed on the x-axis uniformly with 0.25 m spacing. The middle sensor is selected as the reference sensor which is taken to be at the origin. The two sources are placed at angles 33° and 78°. The two sources are WAV files that we assume are unknown. The first source reads "Houston we have a problem," and the second reads "Remember. The force will be with you. Always." The source signals used in our simulation are shown in Fig. 2(a). The RS signal is the sum of the two source signals.

Segments of length $N_t = 8000$ are extracted from the source signals with $t_0 = 5000$ to be used in the processing. Each sensor takes only 15 compressive measurements which makes a total of 165 measurements. Therefore, the total measurement number is much less than the standard time sample numbers of the signals, N_t. This is because we are not trying to reconstruct the signals. We are only reconstructing DOAs in θ space, which has a resolution of 1° and length of 181 for this example. The entries of the random measurement matrices for each sensor is drawn randomly from $\mathcal{N}(0,1)$ independently. WGN is added to the compressive measurements with signal-to-noise ratio (SNR) equals 10 dB. Figure 2(b) shows the compressive measurements, $\boldsymbol{y}$, from all sensors. These measurements are the only information we have about the sources along with the RS data. For the Dantzig Selector constraint, we use $\epsilon = 3\sqrt{2\log N}\sigma = 0.98$ for this example. Solution of the ℓ_1 minimization problem in (10) gives the result in Fig. 2(d).

If all the sensors had samples of their received signals at a high sampling frequency we can apply MVDR and we would obtain the response in Fig. 2(c). The MVDR processing is done at $f = 500$ Hz which is a peak in the FFT of the signals. The number of snapshots was 40, and the length of each snapshot 200 samples. Even though the MVDR shows two significant peaks at the true source bearings we were able to obtain a much sparser result with CS while using many fewer measurements than from standard sampling.

4. SUMMARY

This paper gives a method for using compressive sensing for DOA estimation of multiple targets. The fact that all but one of the array sensors uses compressed measurements will reduce the amount of data that must be communicated between sensors, which has potential in wireless sensor networks where arrays would be formed from distributed sensors [7].

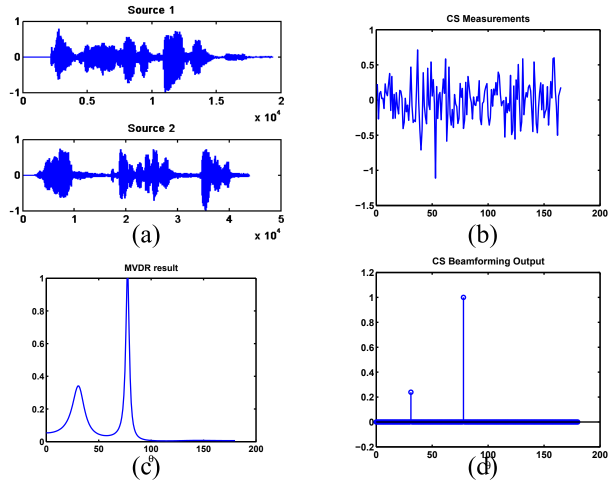

Fig. 2. (a) Source signals, (b) Noisy compressive measurements from all sensors, (c) MVDR result, (d) Compressive beamformer result.

5. REFERENCES

[1] D. H. Johnson and D. E. Dudgeon, *Array Signal Processing: Concepts and Techniques*, Prentice Hall, 1993.

[2] E. J. Candes, J. Romberg, and T. Tao, "Robust uncertanity principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. on Information Theory*, vol. 52, pp. 489–509, 2006.

[3] R. Baraniuk and P. Steeghs, "Compressive radar imaging," in *IEEE Radar Conf.*, 2007, pp. 128–133.

[4] J. J. Fuchs, "On the application of the global matched filter to DOA estimation with uniform circular arrays," *IEEE Trans. Signal Processing*, vol. 49, 2001.

[5] J. J. Fuchs, "Linear programming in spectral estimation. application to array processing," in *ICASSP*, 1996, vol. 6, pp. 3161–3164.

[6] E. Candes and T. Tao, "The Dantzig Selector: Statistical estimation when p is much larger than n," *To appear in Annals of Statistics*.

[7] V. Cevher, A.C. Gurbuz, James H. McClellan, and R. Chellappa, "Compressive wireless arrays for bearing estimation," in *submitted to ICASSP*, 2008.