CONSTRAINED LINEAR LEAST SQUARES APPROACH FOR TDOA LOCALIZATION: A GLOBAL OPTIMUM SOLUTION

Yifeng Zhou and Louise Lamont

Communications Research Center
Ottawa, Ontario, Canada K2H 8S2

ABSTRACT

In this paper, we formulate the time-difference-of-arrival (TDOA) emitter localization problem as a quadratically constrained linear least squares problem. We show that the constrained least squares problem has a unique global minimum and develop a computationally efficient algorithm for finding the emitter location estimates that corresponds to the global minimum. The approach is robust and more resilient to moderate and large sensor measurement noise than other existing TDOA location techniques. Computer simulations are used to demonstrate the effectiveness and performance of the proposed algorithm.

Index Terms— TDOA, emitter location, nonlinear least squares, Lagrangian multiplier, global minimum

1. INTRODUCTION

TDOA based emitter location is a classical problem and has been under study for many years [1]. Navigation systems such as Loran use time-of-arrival (TOA) measurements to locate aircraft and ships at sea. More recently, it has attracted a lot interest the in wireless communications system for locating an mobile unit. This is mainly due to the adoption of certain regulations by the Federal Communications Commission [2] that require all wireless carriers and cell phone manufacturers to incorporate position location capability in order to provide the Enhanced-911 service. Beside emergency service, mobile location techniques can also be used in vehicle navigation and network optimization for resource management. Another emerging application of TDOA emitter location is acoustic source localization and tracking with microphone arrays [4].

In the past, many TDOA emitter localization techniques have been developed, which include the maximum likelihood method, the closed-form least squares solutions and the nonlinear least squares approaches. The maximum likelihood estimator is known to provide asymptotically unbiased estimates with covariances that achieve the Cramér-Rao bound under Gaussian assumption. The main drawback of the maximum likelihood estimator is that it is computationally intensive and lacks the capability of global convergence. Due to the highly nonlinear nature of the likelihood criteria, numerical optimization techniques are needed for finding the optimal solutions. The numerical procedures always require an initial estimate sufficiently close to the actual solution; otherwise the algorithms will converge to a local minimum instead of the desired global one. In practice, this requirement imposes difficulties on the use of the maximum likelihood estimator since accurate initial estimates are in general not available to us.

The closed-form solutions include the spherical interpolation (SI) method [5][6], the spherical interaction (SX) method [7] and the two-stage approach by Chan and Ho [8]. These methods minimize an equation error that is introduced based the squares of distances between the sensors and the emitter. The equation error is linear both in the emitter location vector and its norm. The SI and SX methods are based on an intermediate least squares solution that treats the emitter location vector and its norm as independent. The use of the intermediate least squares solution is to eliminate either the emitter location vector or its norm to make them solvable. The SI and SX methods minimize the equation error that is projected onto a subspace with a reduced dimension [6]. Since the projection has caused the elimination of the constraint between the emitter location vector and its norm, the SI and SX solutions are necessary but not sufficient for the original least squares formulation. They are biased and suboptimal solutions in general. To counter the constraint problem, Chan and Ho [8] proposed a two-stage approach, in which an unconstrained least squares solution is first obtained and then refined by applying the constraint. The two-stage approach is shown to attain the Cramér-Rao bound when the measurement noise is sufficiently small. However, for moderate or large measurement noise, the approximations involved in the two-stage approach may not be valid and will lead to degraded estimation performance. In addition, the SX method requires additional information on the region of interest to resolve ambiguity.

In this paper, we formulate the TDOA emitter localization as a linearly constrained least squares problem, and apply the Lagrangian multiplier method to solve the minimization problem as in [9][10]. We discuss the properties of the constrained linear least squares problem and show that it has unique global minimum. An algorithm for finding the global minimum solution is developed. It should be noted that, although there are algorithms in linear algebra for constrained linear least squares minimization problems with quadratic equality constraints [11][12][13], they can only deal with convex constraints and are not applicable to TDOA emitter location problems since the constraint is non-convex. The rest of the paper is organized as follows. In Section 2, the TDOA emitter location problem is formulated. In Section 3, the quadratically constrained linear least squares solution for TDOA emitter location is developed. The existence and uniqueness of a global minimum is discussed. Finally, in Section 4, computer simulations are used to demonstrate the effectiveness and performance of the proposed algorithm. Comparisons are made to other methods and the Cramér-Rao bound.

2. PROBLEM FORMULATION

Assume that M sensors measure the TOAs of signal from a same emitter. The emitter and sensor locations are denoted by $\underline{x}_s$ and $\{\underline{\xi}_m\}$, respectively, where $m = 1, 2, \ldots, M$. Let t_0 denote the signal departure time. The sensor TOA measurements can be modeled by

$$t_m = t_0 + \frac{1}{c} r_m + w_m, \text{ for } m = 1, 2, \ldots, M \tag{1}$$

where $r_m = \|\underline{\xi}_m - \underline{x}_s\|$ denotes the distance between the emitter and the mth sensor, c denotes the signal propagation speed and w_m is the sensor measurement noise. The measurement noise is assumed to be zero mean Gaussian distributed random process [14]. The Gaussian assumption is often used for mathematical convenience and motivated by the Central Limit Theorem. We use the first sensor as the reference without loss of generality. Define $r_{m1} = r_m - r_1$. The TDOA measurements with reference to the first sensor can be written as

$$d_{m1} = \frac{1}{c} r_{m1} + w_{m1}, \text{ for } m = 2, \ldots, M, \tag{2}$$

where $w_{m1} = w_m - w_1$. The objective of TDOA location is to find the estimate of $\underline{x}_s$ given TDOA measurements $\{d_{m1};\ m = 1, 2, \ldots, M\}$. In TDOA emitter location problems, each TDOA measurement corresponds to a line of position that is defined by a hyperbola with foci at the two sensor sites. This line of position is called an isochrone since the points on it will result in a constant difference in distance to the two sensor sites. When multiple TDOA measurements are available, the emitter location can be obtained by finding the intersection point of the isochrones. In practice, since measurement noise always exists, the set of isochrones will not intersect at a single point, and some criteria, such as the maximum likelihood, must be used for finding the optimal emitter location estimate that would minimize its distance to the isochrones.

An equation error model can also be formulated based the squares of the TOA measurements. This is the model that the SI, the SX and the two-stage approach by Chan and Ho are based on. Without loss of generality, we choose the first sensor site as the origin of the system coordinates. The following relationship can be obtained [5]

$$\epsilon_m = \frac{1}{2}[r_m^2 - d_{m1}^2] - r_s - \underline{\xi}_m^T \underline{x}_s, \tag{3}$$

for $m = 2, 3, \ldots, M$, where r_s is the distance of the emitter location to the origin, and ϵ_m is a equation error introduced to represent the sensor measurement noise. In matrix form, (3) can be written as

$$\underline{\epsilon} = \underline{\delta} - r_s \underline{d} - \tilde{A} \underline{x}_s, \tag{4}$$

where $\underline{d} = [d_{21}, d_{31}, \ldots, d_{M1}]^T$,

$$\underline{\delta} = \frac{1}{2} \begin{bmatrix} r_2^2 - d_{21}^2 \\ r_3^2 - d_{31}^2 \\ \vdots \\ r_M^2 - d_{M1}^2 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \\ \vdots & \\ \xi_{M1} & \xi_{M2} \end{bmatrix}, \tag{5}$$

and ξ_{m1} and ξ_{m2} are the xy coordinates of the mth sensor location. Note that, in (4), the equation error ϵ is linear in $\underline{x}_s$ and r_s. However, when considering the constraint relationship between $\underline{x}_s$ and its norm r_s, ϵ is nonlinear in $\underline{x}_s$. The emitter location can be estimated by minimizing the norm of the equation error with respect to $\underline{x}_s$ [5][6][7][8].

3. CONSTRAINED LINEAR LEAST SQUARE APPROACH

Under the Gaussian assumption, we formulate the minimization of the norm of the equation error as the following constrained generalized linear least squares minimization problem

$$\min_{\underline{x}} \|A\underline{x} - \underline{b}\| \text{ subject to } \underline{x}^T C \underline{x} = 0. \tag{6}$$

where $C = \text{diag}\ [1, 1, -1]$, $A = W^{1/2}\hat{A}$ and $\underline{b} = W^{1/2}\underline{\delta}$, and W is a weighting matrix. The equality constrained optimization problem (6) can be typically solved by using Lagrange multiplier method. Note that, in [9][10], similar formulation and the use of Lagrange multiplier method were also considered. However, the fundamental question about the global optimum is not properly discussed including its existence and uniqueness. In the following, due to length limit of the paper, we will only consider the case where matrix A is of full rank .

Using the Lagrange multiplier, the Lagrangian function of the constrained problem (6) can be written as

$$L(\underline{x}, \lambda) = \|A\underline{x} - \underline{b}\|^2 + \lambda \underline{x}^T C \underline{x}. \tag{7}$$

where λ is the Lagrangian multiplier. According to the Lagrangian multiplier theorem [15], if $\underline{x}^*$ is a local minimum of (6), then there exists a unique λ such that the partial derivatives of the Lagrangian function with respect to x and λ, denoted by $L_x(\underline{x}, \lambda)$ and $L_\lambda(\underline{x}, \lambda)$, respectively, should be zero. A second-order necessary condition is given by

$$\underline{y}^T L_{xx}(\underline{x}^*, \lambda^*)\underline{y} \geq 0 \ \ \forall\ \underline{y} \in \{\underline{y} \neq 0;\ \underline{y}^T \underline{g}_x(\underline{x}^*) = \mathbf{0}\}, \tag{8}$$

where $L_{xx}(\underline{x}, \lambda)$ is Hessian and gradient of the Lagrangian function and $g(\underline{x}) = \underline{x}^T C \underline{x}$ with respect to $\underline{x}$, respectively

$$L_{xx}(\underline{x}, \lambda) = 2A^T A + 2\lambda C \text{ and } \underline{g}_x(\underline{x}) = 2C\underline{x}. \tag{9}$$

From the second-order necessary condition, we have the following property about $(A^T A + \lambda C)$ [16].

Property If $\underline{x}^*$ is a local minimum and λ^* is the optimal multiplier, then, the largest two eigenvalues of $A^T A + \lambda C$ must be nonnegative.

Consider the Hessian L_{xx}. Let $A = U\Gamma V^T$ be the SVD of A, and $D = \Gamma^T \Gamma$. Then L_{xx} can be written as

$$L_{xx} = 2(VDV^T + \lambda C) = 2(VD^{1/2})[I + \lambda P](VD^{1/2})^T, \tag{10}$$

where $P = D^{-1/2} V^T C V D^{-1/2}$. Let $P = Q\Sigma Q^T$ be the eigendecompostion of P. The Hessian can be further written as

$$L_{xx} = 2(VD^{1/2}Q)[I + \lambda\Sigma](VD^{1/2}Q)^T. \tag{11}$$

Consider the eigenvalues of P. The eigenvalue signs of a matrix are characterized by the inertia of the matrix . According to the Sylvester's law of inertia [3], since P and C can be shown to be congruent, the inertia of matrix P is the same as that of C. Since C has two positive and one negative eigenvalues, it follows that P also has two positive and one negative eigenvalues. Assume that $(A^T A + \lambda I)$ is invertible. From the first-order condition of the Lagrangian multiplier, we can obtain the normal equation as

$$\underline{x} = (A^T A + \lambda C)^{-1} A^T \underline{b}. \quad (12)$$

Substituting $\underline{x}$ into the quadratic constraint yields the secular equation

$$\phi(\lambda) = \sum_i^3 \frac{\beta_i^2 \sigma_i}{(1+\lambda\sigma_i)^2} = 0, \quad (13)$$

where $\underline{\beta} = \underline{b}^T U(:, 1:3)Q$ and $\{\sigma_i;\ i = 1, 2, 3\}$ denote the eigenvalues of Σ. The eigenvalues are arranged in a descending order. The optimal Lagrangian multiplier can be found from the roots of the secular function $\phi(\lambda)$, and corresponding $\underline{x}$ can be obtained from the normal equation (12). We have the following lemma for the optimal Lagrangian multiplier λ [16].

Lemma Let λ^* be the root of the secular function in $\mathcal{I}$, where

$$\mathcal{I} = (-1/\sigma_1, -1/\sigma_3).$$

Then, the corresponding $\underline{x}^*$ obtained from the normal equation (12) is the global minimizer of the constrained optimization problem (6).

The optimal Lagrangian multiplier is given by the root of the secular equation $\phi(\lambda)$ in $\mathcal{I}$. Since the secular function is nonlinear, numerical methods such as Newton's Method can be used. Newton's Method is known to be one of the most popular approaches for solving nonlinear equations. In our case, when the data matrix is well-conditioned, Newton's method is found to be efficient with a starting point of $\lambda = 0$. However, when A is not well-conditioned, the optimal λ may be close to one of the boundries of $\mathcal{I}$, which are poles of the secular function. In this case, Newton's Method becomes unstable and often divergent. To avoid to the divergence problem, we use a root searching algorithm provided in Matlab. The uses a combination of bisection, secant and inverse quadratic interpolation method. The implementation of the algorithm was based on [17], where a Fortran is described. The algorithm is very suitable to the application discussed here since the secular function has been shown to be continuous, decreasing in $\mathcal{I}$, and approaches ∞ and $-\infty$ at the left and right boundary, respectively. The algorithm is particularly efficient if an interval where the function values have different signs.

4. PERFORMANCE ANALYSIS

In the simulation study, $M = 8$ sensors were used. The coordinates of the sensors in the xy plane are given by

$$(0, 0), (-5, 8), (4, 6), (-2, 4), (7, 3), (-7, 5), (2, 5), (-4, 2).$$

A near-field and a far-filed emitter were simulated at coordinates $(8, 22)$ and $(-50, 250)$, respectively. Additive Gaussian noise, which is assumed to be zero mean and have a common variance denoted by $\sigma_w^2/2$ for all sensors, was simulated and added to the sensor measurements. The noise is also assumed to be statistically independent among the sensors. In this case the covariance matrix of TDOA measurement noise $\underline{w}$ is given by an $(M-1) \times (M-1)$ matrix, whose diagonal off-diagonal elements are σ_w^2 and $0.5\sigma_w^2$, respectively. The ML method, the SI and the two-stage approach by Chan and Ho (referred to as the CH method) were tested for comparison. For the ML method, the true emitter location was used as an initial estimates in order to obtain good global convergence. For the SI method, $W = \Psi^{-1}$ was used as the weighting matrix according [5]. Since Ψ contains the unknown ranges from the sensors to the emitter, we approximate them by estimates from using $W = Q^{-1}$ similar to the iterative procedures used in [8].

We denote by QCLS (quadratically constrained least squares) the proposed approach for notational simplicity. Table 1 and 2 are the MSEs for different method for $\sigma_w^2 = 0.004/c^2$ and $\sigma_w^2 = 0.04/c^2$, respectively, under different numbers of sensors for the case of near-field emitters. When $\sigma_w^2 = 0.004/c^2$ is relatively small, the QCLS method is similar to Chan and Ho's method and the ML approach. Their MSEs are all close to the CRB. They outperformed the SI method for all numbers of sensors. When $\sigma_w^3 = 0.04/c^2$ increased, the QCLS method is seen to outperform the CH and SI method, especially when the number of sensors is small. Table 3 and 4 are the MSEs via different numbers of sensors for far-field emitter when $\sigma_2^w = 0.004/c^2$ and $\sigma_w^2 = 0.04/c^2$, respectively. For far-field emitters, the performance is dominated by the geometry of emitter and the sensors. In both cases, the QCLS performs slightly better than but close to the CH method. The QCLS, CH and the ML method outperformed the SI method, especially when the number of sensors is small.

Table 1. MSEs of Estimates by Different Approaches: Arbitrary array, Near-Field Source and $\sigma_w^2 = 0.004/c^2$.

	M=4	M=5	M=6	M=7	M=8
SI	6.3387	0.6356	0.5804	0.4839	0.4624
ML	2.6740	0.6038	0.5269	0.4612	0.4246
CH	2.5729	0.5894	0.5249	0.4650	0.4218
QCLS	2.5910	0.5904	0.5238	0.4613	0.4197
CRB	2.7534	0.5803	0.5337	0.4571	0.4215

Table 2. MSEs of Estimates by Different Approaches: Arbitrary array, Near-Field Source and $\sigma_w^2 = 0.04/c^2$.

	M=4	M=5	M=6	M=7	M=8
SI	1582.4	7.5806	6.1405	4.7459	4.6847
ML	162.30	7.4816	6.3505	5.0878	4.7365
CH	54.5805	6.3391	5.5393	5.0262	4.7198
QCLS	42.6554	6.2851	5.3518	4.7631	4.4785
CRB	27.5430	5.8032	5.3371	4.5709	4.2150

5. CONCLUSIONS

In this paper, the TDOA emitter localization was formulated as a quadratically constrained linear least squares problem, which was solved by the application of the Lagrangian multiplier method. We showed that the nonlinear least squares problem has a unique global minimum and an computationally efficient algorithm was developed

Table 3. MSEs of Estimates by Different Approaches: Arbitrary array, Far-Field Source and $\sigma_w^2 = 0.00005/c^2$.

	M=4	M=5	M=6	M=7
SI	7306.1	200.2814	46.4149	42.9571
ML	358.7286	139.5052	44.4287	40.2228
CH	473.7512	138.4030	44.5203	40.4216
QCLS	348.5150	137.4512	44.3042	40.2193
CRB	328.8220	143.9386	44.0635	38.5352

Table 4. MSEs of Estimates by Different Approaches: Arbitrary array, Far-Field Source and $\sigma_w^2 = 0.0001/c^2$.

	M=4	M=5	M=6	M=7
SI ($\times 10^6$)	9.6759	0.0029	0.0005	0.0004
ML ($\times 10^5$)	5.8486	1.8868	0.4719	0.4245
CH ($\times 10^4$)	3.3493	0.1614	0.0452	0.0425
QCLS ($\times 10^4$)	1.5718	0.1503	0.0438	0.0408
CRB ($\times 10^3$)	3.2882	1.4394	0.4406	0.3854

for finding the emitter location estimate corresponding to the global minimum. Another advantage of the approach is that, unlike the SX, it does not depend on additional information on the region of interest to resolve ambiguity of the location estimates. Computer simulations are used to demonstrate the effectiveness and performance of the proposed algorithm. It was shown that the proposed approach outperforms the existing methods especially when the sensor measurement noise is moderate or large.

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