CRAMER-RAO LOWER BOUND ON DOPPLER FREQUENCY OF COHERENT PULSE TRAINS

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ABSTRACT

This paper derives the Cramer-Rao lower bound (CRLB) on estimates of the frequency of a coherent pulse-train passively intercepted at a moving antenna. Such estimates are used to locate the transmitting radar. Although frequency estimation algorithms for pulse trains have been proposed, no results were previously available for the CRLB; thus, it has been impossible to assess the complete potential of location. The derived CRLB is compared to previously published algorithm accuracy results. A general rule of thumb is found that the CRLB depends inversely on pulse on-time, number of pulses, variance of pulse times, and the product of signal-to-noise-ratio and sampling frequency; pulse shape and modulation have negligible impact on the result. When K pulses are equally spaced by the pulse repetition interval (PRI), then the CRLB decreases as 1/PRI^2 and as $1/K^3$.

Index Terms— Source Location, Frequency Estimation, Doppler Measurements

1. INTRODUCTION

Frequency-based passive location of a coherent, stationary radar with unknown frequency from a single moving platform has been investigated [1] – [5]. The approach is to estimate the Doppler-shifted frequency of an intercepted radar pulse train at a set of times and use them to estimate the emitter's location. Thus, to evaluate the location accuracy it is necessary to evaluate the accuracy of the frequency estimation. No results have been available for the Cramer-Rao lower bound (CRLB); thus, it has been impossible to assess the complete potential of this location method. The only accuracy results previously available were: simulation results in [7], experimentally obtained accuracy levels reported by corporations developing actual systems [2], and an approximate accuracy analysis [6]. This paper derives the CRLB for estimating the frequency of a received coherent pulse train and compares the bound to published accuracy results; this shows that the existing algorithm in [7] does not meet the CRLB and that the reported experimental accuracy [2] also does not meet the CRLB.

Note that the CRLB for estimating Doppler with a matched filter in a radar processor is known (e.g., see [8]); in contrast, the result given here is for a *passive* system that can not exploit a matched filter.

2. DERIVATION OF THE CRLB

Many modern radars emit so-called coherent pulse trains where there is a single underlying sinusoid that is turned on and off by a multiplicative baseband pulse train [6], [7]. It is the coherency that can be exploited to yield frequency estimates accurate enough for use in locating the emitter [7]. A complex-valued coherent pulse train consisting of K pulses received from an emitter can be modeled by

$$r(t) = e^{j(\omega t+\phi)} \sum_{k=0}^{K-1} p(t-T_k) \tag{1}$$

where $p(t)$ is a single pulse (possibly complex), ω is the Doppler-shifted frequency, ϕ is a phase offset, and the T_k are called the pulse times. Without loss of generality we can model the complex $p(t)$ as

$$p(t) = A(t)e^{j[\omega_p t+\phi(t)]}; \tag{2}$$

no constant phase is included because it is consumed within ϕ in (1). Using (2) in (1) gives

$$r(t) = e^{j[(\omega+\omega_p)t+(\phi+\phi_p)]} \sum_{k=0}^{K-1} A(t-T_k) e^{j\phi(t-T_k)} e^{-j\omega_p T_k}\ , \tag{3}$$

from which we see that in order for (3) to reduce to (1) with a <u>shifted</u> pulse inside the summation, albeit with ω in (1) replaced by $\omega+\omega_p$, we need that $\omega_p = l_k 2\pi / T_k$ for all k where each l_k is an integer. We take as our fundamental definition of a coherent pulse train the forms in (1) and (2) together with this constraint on ω_p. So in other words, the definition of coherence places a strict constraint on the frequency offset ω_p of the pulse $p(t)$. In practice, the pulse $p(t)$ is generated as a baseband pulse and therefore assuming that $\omega_p = 0$ is generally valid and we assume that here; this is consistent with Class III in [6] and describes modern pulse Doppler radars. These conditions ensure the decomposition

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in (1) is unique and yields a well-posed frequency estimation problem for the parameter ω in (1).

Let T_{OT} be the total pulse "on-time" with $p(t)$ time-limited to $[0, T_{OT}]$ with $T_{OT} < T_k - T_{k-1}$. The received signal is corrupted by noise and sampled at interval Δ to result in

$$\begin{aligned} x[n] &= r[n] + w[n] \\ &= e^{j\omega\Delta n} e^{j\phi} \sum_{k=0}^{K-1} p(\Delta n - T_k) + w[n], \end{aligned} \tag{4}$$

where $w[n]$ is complex zero-mean white Gaussian noise with variance σ^2. From the signal data $x[n]$ we wish to estimate $\boldsymbol{\theta} = [\omega \quad \phi \quad T_0 \quad T_1 \quad \cdots \quad T_{K-1}]$, but the key parameter to be estimated is ω, with the others as nuisance parameters. Note that pulse $p(t)$ is generally unknown; however, the CRLB will, in general, depend on the form of $p(t)$ so numerical results for the CRLB might depend on the specific pulse $p(t)$ assumed to have been intercepted.

The elements of the Fisher Information Matrix (FIM), $\mathbf{J}$, for this case can be found using [9]

$$[\mathbf{J}]_{ij} = 2\,\mathrm{Re}\left\{ \frac{\partial \mathbf{r}^H}{\partial \theta_i} \frac{\partial \mathbf{r}}{\partial \theta_j} \right\}. \tag{5}$$

where $\mathbf{r} = [r[0] \quad r[1] \quad \cdots \quad r[N-1]]^T$. For later use, define:

$$S_m = \frac{1}{N} \sum_{n=0}^{N-1} n^m \left| p(n\Delta) \right|^2, \quad m = 0, 1, 2, \tag{6}$$

$$C_m = \mathrm{Im}\left\{ \frac{1}{N} \sum_{n=0}^{N-1} n^m p(\Delta n) p'^*(\Delta n) \right\} \quad m = 0, 1, \tag{7}$$

$$B = \frac{1}{S_0 N} \sum_{n=0}^{N-1} \left| p'(n\Delta) \right|^2 \quad R_m = \frac{1}{K} \sum_{k=0}^{K-1} (T_k)^m, \quad m = 1, 2. \tag{8}$$

Then the FIM is found to be

$$\mathbf{J} = \frac{2N}{\sigma^2} \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & S_0 B \mathbf{I}_{K \times K} \end{bmatrix} \tag{9}$$

where the 2×2 matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} \Delta^2 K S_2 + 2\Delta K R_1 S_1 + K R_2 S_0 & \Delta K S_1 + K R_1 S_0 \\ \Delta K S_1 + K R_1 S_0 & K S_0 \end{bmatrix} \tag{10}$$

and the 2×K matrix $\mathbf{C}$ is given by

$$\mathbf{C} = \begin{bmatrix} \Delta C_1 + T_0 C_0 & \Delta C_1 + T_1 C_0 & \cdots & \Delta C_1 + T_{K-1} C_0 \\ C_0 & C_0 & \cdots & C_0 \end{bmatrix}. \tag{11}$$

Using results for inverting symmetric partitioned matrices yields the 2×2 CRLB matrix for frequency and phase as

$$\mathbf{CRLB}_{\omega,\phi} = \frac{\sigma^2}{2N} \left(\mathbf{A} - \frac{\mathbf{C}\mathbf{C}^T}{S_0 B} \right)^{-1}, \tag{12}$$

and then finding the 1,1-element of this 2×2 inverse gives

$$CRLB_\omega = \frac{1}{2NK\ SNR\left(1 - (C_0^2 / \tilde{S}_0^2 B)\right)\left(\Delta^2 \tilde{D} + \tilde{R}_2\right)}, \tag{13}$$

where $SNR \triangleq S_0 / \sigma^2$, and

$$\tilde{R}_2 \triangleq R_2 - R_1^2 \quad \tilde{D} \triangleq \left(\tilde{S}_2 / \tilde{S}_0\right) - \left(\tilde{S}_1 / \tilde{S}_0\right)^2 \tag{14}$$

with

$$\begin{aligned} &\tilde{S}_0 \triangleq S_0 - (C_0^2 / S_0 B) \quad \tilde{S}_1 \triangleq S_1 - (C_o C_1 / S_0 B) \\ &\tilde{S}_2 \triangleq S_2 - (C_1^2 / S_0 B); \end{aligned} \tag{15}$$

the units for $\tilde{R}_2$ and $\Delta^2 \tilde{D}$ are sec². Note that $\tilde{D}$ is a "mean-normalized" measure of the pulse duration and $\tilde{R}_2$ is a "mean-normalized" measure of the time spread of the pulse times. These mean-normalized measures ensure that the choice of time origin has no effect on the CRLB for the Doppler estimate. The S_m quantities defined previously are replaced here by alternatives that now include the effect of the so-called skew measures given by C_0 and C_1.

The $CRLB_\omega$ above depends on C_0, which is evaluated here using an integral approximation given by

$$C_0 \approx \frac{1}{N} \mathrm{Im}\left\{ \frac{1}{\Delta} \int_0^{T_{OT}} p(t) p'^*(t) dt \right\}. \tag{16}$$

Clearly, C_0 is zero if $p(t)$ is real; that is if the pulse has no phase/frequency modulation on it. But C_0 is zero or at least small in most if not all other cases. Converting from (2) with $\omega_p = 0$ gives

$$p(t) = \underbrace{A(t)\cos[\phi(t)]}_{\triangleq a(t)} + j \underbrace{A(t)\sin[\phi(t)]}_{\triangleq b(t)}, \tag{17}$$

where $A(t)$ is the real-valued envelope function and $\phi(t)$ is the real-valued phase function gives

$$C_0 \approx \frac{1}{N} \frac{1}{\Delta} \int_0^{T_{OT}} \left[a'(t) b(t) - a(t) b'(t) \right] dt, \tag{18}$$

which after some manipulation gives

$$C_0 \approx -\frac{1}{N} \frac{1}{\Delta} \int_0^{T_{OT}} A^2(t) \omega_i(t) dt, \tag{19}$$

where $\omega_i(t)$ is the instantaneous frequency.

Thus, for any pulse for which the instantaneous frequency varies symmetrically around zero then (19) implies that C_0 will be zero if $A(t)$ is symmetric around its time center; this includes typical linear FM pulses. Although it is difficult to make a wide-sweeping general conclusion here, it is also likely that most if not all typical phase modulations will give a small value of C_0 because the pulse's instantaneous frequency typically varies uniformly (at least approximately) above and below zero. Also, it should be noted that the effect of any non-zero value of C_0 will be deemphasized through the division in $\left[1-(C_0^2/\tilde{S}_0^2 B)\right]$. Thus, the parameter C_0 has little effect and the CRLB on Doppler-shifted frequency can be approximated as

$$CRLB_{\omega} \approx \frac{1}{2T_{OT} K \; SNR \; F_s \left[\Delta^2 \tilde{D} + \tilde{R}_2\right]}, \tag{20}$$

where $N\Delta = T_{OT}$ (the pulse on-time) and $F_s = 1/\Delta$.

3. DISCUSSION

Notice that the CRLB result in (20) depends inversely on $2T_{OT} \; SNR \; F_s$. Thus, doubling T_{OT} halves the CRLB. It may also appear that doubling the sampling rate F_s will also halve the CRLB but this is likely not true. Increasing F_s would require either (i) a commensurate increase in front-end bandwidth to maintain whiteness for $w[n]$, which will increase the noise power but not increase the signal power, or (ii) would correlate the noise samples and invalidate the result, which was derived under white noise conditions. The decreased SNR perfectly counters the increased Fs so that the product $SNR \; F_s$ remains the same. In case (i), changes in F_s will cause changes in $\Delta^2 D$ (and other similar terms) but those changes will be small for all F_s above the Nyquist rate.

The CRLB in (20) depends on the factor $\tilde{R}_2$, which is a measure of the temporal spread of the pulse times, and the factor $\Delta^2 \tilde{D}$, which is a measure of the temporal spread of the pulse; both are in units of sec^2. More widely spaced pulses increases $\tilde{R}_2$ which adds to $\Delta^2 \tilde{D}$ to decrease the CRLB. Because $\Delta^2 \tilde{D}$ is a measure of pulse duration and $\tilde{R}_2$ is a measure of pulse train duration, $\tilde{R}_2$ will be much larger than $\Delta^2 \tilde{D}$ and that leads to the following approximation:

$$CRLB_{\omega} \approx \frac{1}{2T_{OT} K \; SNR \; F_s \tilde{R}_2}. \tag{21}$$

This shows that in addition to the negligible effect of C_0 as discussed above, the approximation in (21) shows that C_1 also has negligible impact. Thus, the skew factors C_0 and C_1 have negligible impact on the CRLB for frequency.

As a general rule of thumb the CRLB depends inversely on: (i) the pulse on-time T_{OT}, (ii) the number of pulses K, (iii) the variance of pulse times $\tilde{R}_2$, and (iv) the product $SNR \times F_s$; the pulse shape and pulse modulation seem to have little or no impact on the frequency accuracy. As a special case consider the scenario where the intercepted pulses are equally spaced by T_{PRI}, the pulse repetition interval (PRI), then it is easily shown that $\tilde{R}_2 = T_{PRI}^2 (K^2-1)/12$ and the approximation in (21) can be re-written as

$$CRLB_{\omega} \approx \frac{6}{T_{OT} K (K^2-1) \; SNR \; F_s \; T_{PRI}^2}, \tag{22}$$

from which we see that the frequency CRLB varies as $O(1/T_{PRI}^2)$ and as $O(1/K^3)$. Thus, for all other parameters fixed, doubling the number of equally spaced pulses decreases the CRLB by 1/8$^{\text{th}}$, whereas doubling the pulse on-time only decreases the CRLB by ½.

Numerically computing the CRLB for typical values of pulse train parameters shows that it is possible to achieve frequency accuracy much lower than 1 Hz (1 Hz is the accuracy that is claimed experimentally in [2] and the accuracy for the simulation results in [7] are on the order of a few Hz). Thus, it is clear that the existing algorithms, while achieving quite good accuracies, are still far from achieving the CRLB. For comparison, Table 1 presents some simulation accuracy results published in [7] side-by-side with the corresponding coherent CRLB results computed using (22). Note that the results in [7] were given for the case of staggered PRI values within each pulse train (but each PRI approximately 1 ms); however, the computed results given in Table 1 were computed without the stagger for convenience; the stagger was slight and would not significantly change the computed CRLB values. The computed CRLB values are much smaller than the reported simulation accuracies. Also note that the discrepancy between simulation results and CRLB are worse at higher SNR; the algorithm's variance does not drop inversely with SNR as the CRLB result does (note that classic CRLB results for frequency estimation of a sinusoid also exhibit this inverse SNR dependence).

Table 1: Comparison to simulation results for rectangular pulses of width 1 μs with PRI of approximately 1 ms, and Fs = 100 MHz.

K	SNR (dB)	$\sqrt{CRLB_{\tilde{\omega}}}/2\pi$ (Hz)	$\sigma_{\tilde{f}}$ in [7] (Hz)
4	23	0.36	2.3
	37	0.07	1.5
5	23	0.25	2.1
	37	0.05	1.1
6	23	0.19	1.7
	37	0.04	1.0

Note that in (9), the lower-right corner shows that the Fisher information for the pulse times depends on B – a measure of pulse bandwidth that shows up in time-of-arrival estimation analyses [9]; pulses with large bandwidth admit more accurate estimation of time-of-arrival. It would seem that the accuracy with which one can estimate the pulse times should have a significant impact on the accuracy to which one can estimate the frequency of a coherent pulse train. Thus, it comes as unexpected that the parameter B that impacts the accuracy of the pulse time estimates does not show up in the CRLB for frequency in (21); recall, however, that it does show up (see (13) - (15)) but its negligible effect was shown to lead to (21).

However, the frequency estimate and the pulse time estimates are correlated. The full CRLB matrix is

$$\mathbf{CRLB} = \frac{\sigma^2}{2N}\begin{bmatrix} \mathbf{CRLB}_{\omega,\phi} & \mathbf{CRLB}_{\omega,\phi|[T_0 \;\cdots\; T_{K-1}]} \\ \mathbf{CRLB}^T_{\omega,\phi|[T_0 \;\cdots\; T_{K-1}]} & \frac{1}{S_0 B}\mathbf{I}_{K\times K} \end{bmatrix}, \quad (23)$$

where 2×2 matrix $\mathbf{CRLB}_{\omega,\phi}$ is as defined in (12) and the $K\times K$ matrix $\mathbf{CRLB}_{\omega,\phi|[T_0 \;\cdots\; T_{K-1}]}$ is found below. Using a standard result for the inverse of partitioned symmetric matrix we get that

$$\begin{aligned} CRLB^T_{\omega,\phi|[T_0 \;\cdots\; T_{K-1}]} &= -\frac{\sigma^2}{2N}\left(\mathbf{A} - \frac{\mathbf{C}\mathbf{C}^T}{S_0 B}\right)^{-1}\mathbf{C}\left(1/S_0 B\right) \\ &= -\frac{1}{S_0 B}\left[CRLB_{\omega,\phi}\right]\mathbf{C}, \end{aligned} \quad (24)$$

which then gives the cross-CRLB term between ω and T_k as

$$CRLB^T_{\omega|T_k} = -CRLB_{\omega}(\Delta C_1 + T_k)/S_0 B\,, \quad (25)$$

where we have used $C_0 = 0$.

Thus, there *is* correlation between the frequency estimate and the pulse time estimates; but notice that the correlation becomes closer to zero as B increases. However, as B changes, the CRLB result for frequency in (21) shows no (or at least very little) change. Such a situation is shown in the illustrative (i.e., not numerically computed for this case) results in Figure 1, which shows three error ellipses that change tilt due to changes in correlation while maintaining the same individual CRLB on ω (i.e., projection onto the ω axis – see [5] for more on projections of error ellipses).

In summary, we've shown that for a coherent pulse train the CRLB on the variance of the frequency estimate:

- varies inversely with pulse on-time
- does not depend on pulse shape or modulation
- varies inversely with "variance" of pulse times measured by $\tilde{R}_2$
- varies inversely with *SNR*
- varies as $O(1/T_{PRI}^2)$ and as $O(1/K^3)$ for equally-spaced pulses

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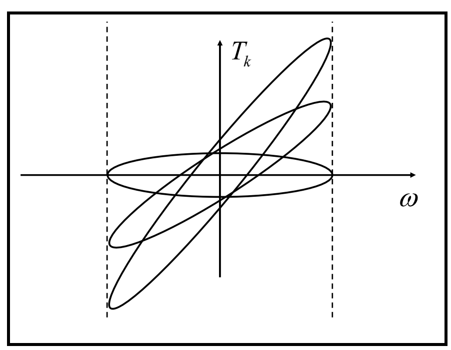

Figure 1: Illustrative error ellipses showing changing correlation between ω and T_k but constant CRLB on ω.