FIRST-ORDER ANALYSIS OF THE PRIOR-BASED KNOWLEDGE MINNORM ALGORITHM

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ABSTRACT

In the context of the Direction-Of-Arrival (DOA) estimation problem, we can sometimes assume the *a priori* knowledge of $M-S$ DOA among M. Some authors have propose to incorporate this *a priori* knowledge to better estimate the DOA of interest (*ie.*, the unknown ones). In a previous work, the authors have proposed two prior MinNorm schemes based on oblique projection which allow the integration of this prior-knowledge. In particular, numerical and theoretical expressions of the variances have been derived. In this work, we go further into the analysis already given. We first focus on the asymptotic (large number of sensors) behavior of the standard, the constrained and the prior versions of the MinNorm algorithm and we show that in this case the exploitation of a prior-knowledge is not beneficial. Next, we derive closed-form approximations of the variance of these algorithms in case of two closely-spaced sources for small/moderate number of sensors and we show that the prior-MinNorm algorithms based on oblique projection is much more insensitive to the proximity of the DOA as compared to the standard and the constrained MinNorm algorithms. Finally, these theoretical analysis are checked against computer simulations by means of Monte-Carlo runs.

Keywords: Direction of arrival estimation

1. INTRODUCTION

Directions-Of-Arrival (DOA) of narrow-band sources estimation is one of the central problems in passive radar, sensor sonar, radio-astronomy, and seismology. This problem has received considerable attention in the last 30 years, and a variety of techniques for its solution have been proposed. We assume that we have the a prior-knowledge of $M-S$ DOA among a total number of M. Remark, we can encounter this type of prior-knowledge in biomedical applications [4] or in RADAR processing [6]. Consequently, the underlying model can be viewed as a signal of interest plus interference model [1]. Based on this model, we can distinguish two main approaches to integrate this prior-knowledge into estimation schemes: (i) *orthogonal projection* of the noisy observation on the interference subspace [6, 4] referred as an orthogonal deflation of the signal subspace and (ii) *oblique projection* of the noisy observation onto the interference subspace [2, 8, 9] referred as oblique deflation of the signal subspace. Results point out in [8, 9] show that the most suitable approach to integrate this type of prior-knowledge is the oblique deflation. So, the authors have proposed in a previous work [9] two prior-knowledge MinNorm schemes and they have derived theoretical variances of these algorithms. In this work, we go further into the analysis of these schemes and explain with more detail why the oblique deflation is efficient in the context of closely-spaced DOA for small/moderate number of sensors. In particular, we propose in this paper additional theoretical studies for (i) asymptotic analysis, *i.e.* when the number of sensors grows to infinite, and for (ii) a closely-spaced analysis for two DOA.

2. PARAMETRIC MODEL

Assume there are M narrowband plane waves (sources) simultaneously incident on an L sensor Uniform Linear Array (ULA). Define S signals of interest and the $M-S$ interfering signals, considered as structured interferences. Let $x_\ell(t)$ be the noisy observation on the ℓ-th sensor for the t-th snapshot. Then, the array response, $x(t) = [x_1(t)\dots x_L(t)]^T$, for the t-th snapshot is

$$x(t) = \underbrace{A\alpha(t)}_{\text{Signal of interest}} + \underbrace{B\beta(t)}_{\text{Structured Interference}} + \underbrace{n(t)}_{\text{Unstructured noise}} \quad (1)$$

where the L-sensor steering vector is defined by $p(\omega) = \begin{bmatrix}1 & e^{i\omega} & \dots & e^{i(L-1)\omega}\end{bmatrix}^T$ in which $\omega = -2\pi\frac{d}{c}\sin(\theta)$ is the spatial pulsation with θ the DOA, d the inter-sensor distance, c the wavelength. The sources are stacked in two vectors $\alpha(t) = [\alpha_1(t)\dots\alpha_S(t)]$ and $\beta(t) = [\alpha_{S+1}(t)\dots\alpha_{M-S}(t)]$ where $\alpha_m(t)$ denotes the m-th sources for the t-th snapshot and $A = [p(\omega_1)\dots p(\theta_S)]$ and $B = [p(\omega_{S+1})\dots p(\omega_{M-S})]$ are the corresponding steering manifolds. The noise vector, $n(t) = [n_1(t)\dots n_L(t)]^T$, in which each $n_\ell(t)$ is the contribution of the noise on the ℓ-th sensor which is assumed to be a complex zero-mean temporally and spatially white Gaussian process of variance σ^2. The number of sources, M, is assumed to be known or previously estimated. Finally, the model for T snapshots is

$$X = A\Lambda_\alpha + B\Lambda_\beta + N \quad (2)$$

where $X = [x(1)\dots x(T)]$, $\Lambda_\alpha = [\alpha(1)\dots\alpha(T)]^T$, and $\Lambda_\beta = [\beta(1)\dots\beta(T)]$. As $S < M$, matrix A (respectively B) is a rank-S (rank-$(M-S)$) matrix. We name $\mathcal{R}(A)$ the deflated signal subspace since its dimension is $M-S$ which is smaller than the dimension of the signal subspace $\mathcal{R}(Z)$ where $Z = [A\ B]$. We have $\mathcal{R}(A) \subseteq \mathcal{R}(Z)$.

We assume that the sources associated to the known and to the unknown parts are uncorrelated. Consequently, the spatial covariance matrix is block-diagonal and is defined according to

$$R_X = \mathbb{E}(XX^H) = S_X + \sigma^2 I \tag{3}$$

where $S_X = S_A + S_B$, with $S_A = AR_{\Lambda_\alpha}A^H$ and $S_B = BR_{\Lambda_\beta}B^H$ where $R_{\Lambda_\alpha} = \mathbb{E}(\Lambda_\alpha \Lambda_\alpha^H)$, $R_{\Lambda_\beta} = \mathbb{E}(\Lambda_\beta \Lambda_\beta^H)$ and $\mathbb{E}(.)$ denotes the mathematical expectation.

3. WEIGHTED PRIOR-MINNORM (WP-MINNORM), PRIOR MINNORM (P-MINNORM) AND CONSTRAINED MINNORM (C-MINNORM) ALGORITHMS

The optimization criterion of the WP-MinNorm, P-MinNorm and C-MinNorm algorithms are

$$\arg\min_{\omega} |f_{\mathtt{WP}}(\omega)|^2 \quad \text{where} \quad f_{\mathtt{WP}}(\omega) = 1 - e_1^T E_{(A\ B)} p(\omega) \tag{4}$$

$$\arg\min_{\omega} |f_{\mathtt{P}}(\omega)|^2 \quad \text{where} \quad f_{\mathtt{P}}(\omega) = e_1^T P_A^{\perp} p(\omega) \tag{5}$$

$$\arg\min_{\omega} |f_{\mathtt{C}}(\omega)|^2 \quad \text{where} \quad f_{\mathtt{C}}(\omega) = e_1^T P_Z^{\perp} p(\omega) \tag{6}$$

with $E_{(A\ B)}$ the oblique projector [1] on $\mathcal{R}(A)$ along $\mathcal{R}(B)$ and $P_Z^{\perp} = I - ZZ^{\dagger}$, $P_A^{\perp} = I - AA^{\dagger}$ where $.^{\dagger}$ denotes the Moore-Penrose pseudo-inverse and e_1 is the first column of the identity matrix. Note that expression (6) provides a new interpretation of the C-MinNorm, initially presented in [7].
Obviously, projectors $E_{(A\ B)}$, $P_Z^{\perp}$ and $P_A^{\perp}$ are unknown quantities as long as $\mathcal{R}(A)$ is unknown. In [9], a subspace-based algorithm which assumes that an available estimation of $\mathcal{R}(B)$ is presented. This approach is relied to the Singular Value Decomposition of matrix $P_B^{\perp} R_X$ where $P_B^{\perp} = I - BB^{\dagger}$.

4. ANALYSIS OF THE VARIANCES

Due to the finite number of snapshots, an estimation error is induced by considering the sample covariance. In reference [9], the theoretical variances of the WP-MinNorm, P-MinNorm, and C-MinNorm algorithms has been derived. We recall in the following these results.

$$\mathrm{Var}_{\mathtt{WP}}(\hat{\omega}_i) = \sigma^2 \frac{(e_1^T(P_Z^{\perp} + \Omega)e_1)\ (p(\omega_i)^H S_A^{\dagger} R_A S_A^{\dagger} p(\omega_i))}{2T\ |e_1^T E_{(A\ B)}\ p'(\omega_i)|^2} \tag{7}$$

$$\mathrm{Var}_{\mathtt{P}}(\hat{\omega}_i) = \sigma^2 \frac{(e_1^T P_A^{\perp} e_1)\ (p(\omega_i)^H S_A^{\dagger} R_A S_A^{\dagger} p(\omega_i))}{2T\left|e_1^T P_A^{\perp} p'(\omega_i)\right|^2} \tag{8}$$

$$\mathrm{Var}_{\mathtt{C}}(\hat{\omega}_i) = \sigma^2 \frac{(e_1^T P_Z^{\perp} e_1)\ (p(\omega_i)^H S_A^{\dagger} R_A S_A^{\dagger} p(\omega_i))}{2T\left|e_1^T P_Z^{\perp} p'(\omega_i)\right|^2} \tag{9}$$

where $\Omega = B\left(B^H P_A^{\perp} B\right)^{-1} B^H$ and $R_A = S_A + \sigma^2 I$ and $p'(\omega_i) = \frac{\partial p(\omega)}{\partial \omega}\Big|_{\omega=\omega_i}$.

In addition, the variance of the MinNorm algorithm can be found in [5] and in the sequel, we denote it by $\mathrm{Var}_{\mathtt{MN}}(\hat{\omega}_i)$. In this work, we go further than the analysis presented in [9], by providing more insights on expressions (7)-(9). More precisely, we study these expressions (i) for a large number of sensors, L and (ii) for a small/moderate L but for closely-spaced DOA.

4.1. Asymptotic Variances

The asymptotic variances, *i.e.*, for a large number of sensors of the considered algorithms are given in the following theorem.

Theorem 1 *For a large number of sensor (large L) and for a sufficient SNR, the asymptotic variance of the WP-MinNorm, P-MinNorm, C-MinNorm and MinNorm algorithms, are given by*

$$Var_{\mathtt{WP}}(\hat{\omega}_i) = Var_{\mathtt{P}}(\hat{\omega}_i) = \frac{2(L-S)}{TL^3\ SNR_i}, \tag{10}$$

$$Var_{\mathtt{C}}(\hat{\omega}_i) = Var_{\mathtt{MN}}(\hat{\omega}_i) = \frac{2(L-M)}{TL^3\ SNR_i} \tag{11}$$

where $SNR_i = \frac{\sigma_i^2}{\sigma^2}$ for $i \in [1:S]$ with σ_i^2 the variance of the i-th source.

Proof : see Appendix 7.1

Let us define

$$\rho = \frac{\mathrm{Var}_{\mathtt{WP}}(\hat{\omega}_i)}{\mathrm{Var}_{\mathtt{MN}}(\hat{\omega}_i)} = \frac{\mathrm{Var}_{\mathtt{P}}(\hat{\omega}_i)}{\mathrm{Var}_{\mathtt{MN}}(\hat{\omega}_i)} = \frac{\mathrm{Var}_{\mathtt{WP}}(\hat{\omega}_i)}{\mathrm{Var}_{\mathtt{C}}(\hat{\omega}_i)} = \frac{\mathrm{Var}_{\mathtt{P}}(\hat{\omega}_i)}{\mathrm{Var}_{\mathtt{C}}(\hat{\omega}_i)} = \frac{L-S}{L-M}.$$

So, we have:

- If $M \gg S$, then $\rho \gg 1$ and
$$\mathrm{Var}_{\mathtt{WP}}(\hat{\omega}_i) = \mathrm{Var}_{\mathtt{P}}(\hat{\omega}_i) \gg \mathrm{Var}_{\mathtt{C}}(\hat{\omega}_i) = \mathrm{Var}_{\mathtt{MN}}(\hat{\omega}_i).$$
- If $M = O(S)$ or $L \gg M$ then $\rho \approx 1$ and
$$\mathrm{Var}_{\mathtt{WP}}(\hat{\omega}_i) = \mathrm{Var}_{\mathtt{P}}(\hat{\omega}_i) \approx \mathrm{Var}_{\mathtt{C}}(\hat{\omega}_i) = \mathrm{Var}_{\mathtt{MN}}(\hat{\omega}_i).$$

In conclusion, for large L, the exploitation of the a priori knowledge is not determinant.

4.2. Variance in the case of two closely-spaced DOA

Now, consider the two signals case (one of interest and one as interference) according to $A = p(\omega_1)$ and $B = p(\omega_2)$. We assume that the spatial pulsation ω_1 and ω_2 are close in the sense that

$$\Delta_\omega = \omega_2 - \omega_1 \ll 1.$$

Consequently, $p(\omega_2)$ can be well approximated by a first order Taylor expansion according to

$$p(\omega_2) \stackrel{1}{=} p(\omega_1) + \Delta_\omega p'(\omega_1)$$

where $p'(\omega_1)$ is the first order derivative of the steering vector considered at the spatial pulsation ω_1. So, based on the above first-order approximation and on expressions (7)-(9), it is straightforward[1] to derive the following theorem.

Theorem 2 *For two closely spaced DOA/spatial pulsation with $\Delta_\omega \ll 1$, we have the following variances:*

$$Var_{\mathtt{WP}}(\hat{\omega}_1) = \frac{12}{L(L^2-1)}\bar{\xi}_1 \tag{12}$$

$$Var_{\mathtt{P}}(\hat{\omega}_1) = \frac{4}{L(L-1)}\bar{\xi}_1 \tag{13}$$

$$Var_{\mathtt{C}}(\hat{\omega}_1) = \frac{144}{\Delta_\omega^2 L(L^2-1)(L-2)}\bar{\xi}_1 \tag{14}$$

$$Var_{\mathtt{MN}}(\hat{\omega}_1) = \frac{144}{\Delta_\omega^2 L(L^2-1)(L-2)}\xi_1 \tag{15}$$

[1] Due to the lack of space, the whole calculus are not given in this article.

where $\bar{\xi}_1 = \frac{1}{2T\,SNR_1}\left(1+\frac{1}{SNR_1}\frac{1}{L}\right)$ *and where* $\xi_1 = \frac{1}{2T\,SNR_1}\left(1+\frac{1}{SNR_1}\frac{12}{\Delta_\omega^2 L(L^2-1)}\right)$.

Proof : see Appendix 7.2

As we can see on the above expressions, the C-MinNorm and the standard MinNorm are very sensitive to quantity Δ_ω. This is not the case for the proposed schemes. Furthermore, it is straightforward to see that

$$\left.\begin{array}{lcl}\frac{\mathrm{Var}_{\mathtt{WP}}(\hat{\omega}_1)}{\mathrm{Var}_{\mathtt{C}}(\hat{\omega}_1)} & = & \frac{\mathrm{Var}_{\mathtt{WP}}(\hat{\omega}_1)}{\mathrm{Var}_{\mathtt{MN}}(\hat{\omega}_1)}\\ \frac{\mathrm{Var}_{\mathtt{P}}(\hat{\omega}_1)}{\mathrm{Var}_{\mathtt{C}}(\hat{\omega}_1)} & = & \frac{\mathrm{Var}_{\mathtt{P}}(\hat{\omega}_1)}{\mathrm{Var}_{\mathtt{MN}}(\hat{\omega}_1)}\end{array}\right\} = O(\Delta_w^2)$$

and thus the prior versions of the MinNorm algorithm have a much smaller variance than the C-MinNorm or the standard MinNorm. In addition, we have $\frac{\mathrm{Var}_{\mathtt{WP}}(\hat{\omega}_1)}{\mathrm{Var}_{\mathtt{P}}(\hat{\omega}_1)} \approx \frac{3}{L+1}$ and for $L > 2$ we have

$$\mathrm{Var}_{\mathtt{WP}}(\hat{\omega}_1) < \mathrm{Var}_{\mathtt{P}}(\hat{\omega}_1). \qquad (16)$$

Consequently, the WP-MinNorm has a better accuracy than the P-MinNorm. We will see that the numerical simulations confirm this fact.

5. NUMERICAL SIMULATIONS

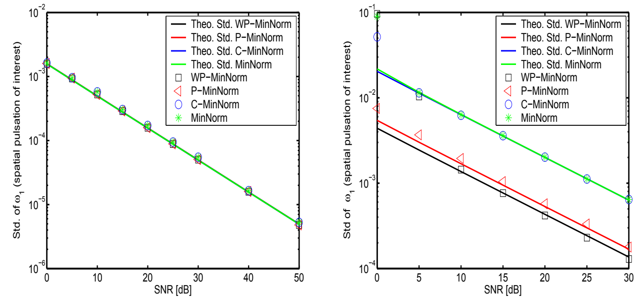
The context of the numerical simulation is a Uniform and Linear Array (ULA) with a half wavelength with $T = 100$ snapshots. For all experiments, we denote by $\omega = [\omega_1\ \omega_2]$ rad the vector of spatial pulsations in radian, where ω_2 is the known spatial pulsation and ω_1 the one of interest. Furthermore, for each experience we plot the Standard deviation (Std.), defined as the square root of the variance, of spatial pulsation of interest $\hat{\omega}_1$, by means of 500 Monte-Carlo trials.

5.1. Illustration of Theorem 1

In this simulation part we focus on the illustration of Theorem 1. To this end, we test the prior-knowledge algorithms with a large number of sensor ($L = 100$) and we compare the practical values to theoretical expressions. The analysis of Fig. 1-a shows the good fit between practical and theoretical values when the DOA are separated by 4 degrees. Therefore, this confirms the results given in Theorem 1, *i.e.* for asymptotic regime the prior-knowledge does not improved the estimation of the DOA of interest with respect to the C-MinNorm and the standard MinNorm. Nevertheless, we do recall that even if all the algorithms have the same accuracy in that regime, only the prior MinNorm algorithms are able to estimate only the DOA of interest. This is not the case for the other algorithms.

5.2. Illustration of Theorem 2

In the context of this simulation, we illustrate Theorem 2 where we have considered two sources with closely-spaced DOA (separated by 1 degree) for small/moderate number of sensors. The conclusions given by Fig. 1-b agree with those of Theorem 2 in the sense, the C-MinNorm is slightly more accurate than the standard MinNorm algorithm, showing $\bar{\xi}_1 < \xi_1$ in Theorem 2. So, for sufficient SNR, all practical simulations are in accordance with theoretical expressions. In addition, we can check that the WP-MinNorm is actually the best prior-knowledge algorithm since it has the highest accuracy. Finally, we can conclude that the prior-knowledge schemes based on oblique processing are less sensitive to the proximity of the DOA.



(a) (b)

Fig. 1. (a) Std. Vs SNR with $\omega = [-0.43\ , -0.65]$ rad and $L = 100$ sensors, (b) Std. Vs SNR, with $\omega = [-0.43\ , -0.49]$ rad and $L = 18$ sensors.

6. CONCLUSION

In the context of the Direction-Of-Arrival (DOA) estimation problem, we can sometimes assume the *a priori* knowledge of $M - S$ DOA among M. In this paper, we compare two different ways to integrate this knowledge into the MinNorm algorithm. Namely, the orthogonal and the oblique deflation of the signal subspace. Toward this end, we derive theoretical closed-form expressions of the variance of the standard, constrained and prior versions of the MinNorm algorithms. We show that the oblique deflation is more suitable than orthogonal deflation to integrate a prior-knowledge into the MinNorm algorithm. Conversely, in the context of asymptotic analysis (large number of sensor), these two schemes are equivalent.

7. APPENDIX

7.1. Proof of Theorem 1

Notice that due to the lack of space, the proof given in the sequel is not exhaustive. We use now property on the asymptotic orthogonality of pure exponential, *i.e.*, $p^H(\omega_i)p(\omega_j) = L\delta_{i-j}$, $p^H(\omega_i)p'(\omega_j) \overset{L\to\infty}{\longrightarrow} i\frac{L(L-1)}{2}\delta_{i-j}$. Therefore, we have $e_1^T P_A^{\perp} e_1 \overset{L\to\infty}{\longrightarrow} \frac{L-S}{L}$, since the dimension of $\mathcal{R}(A)$ is S and $e_1^T P_Z^{\perp} e_1 \overset{L\to\infty}{\longrightarrow} \frac{L-M}{L}$ since the dimension of $\mathcal{R}(Z)$ is M. In addition, $E_{(A\ B)} \to P_A$ and $P_Z^{\perp} + \Omega \to I - P_B - P_A + P_B = P_A^{\perp}$ for large L. We have

$$\left.\begin{array}{l}|e_1^T P_A p'(\omega)|^2\\ |e_1^T P_Z p'(\omega)|^2\end{array}\right\} \overset{L\to\infty}{\longrightarrow} \frac{(L-1)^2}{4}.$$

Next, we use a result extracted from [3] to rewrite $p(\omega)^H S_A^{\dagger} R_A S^{\dagger} p(\omega)$ (common to (7)-(9)) according to SNR_i. We find, then

$$p(\omega)^H S_A^{\dagger} R_A S^{\dagger} p(\omega) = \frac{1}{\sigma_i^2} + \frac{1}{\sigma_i^2}\left[(A^H A)^{-1}\right]_{ii}\frac{1}{\mathrm{SNR}_i}. \qquad (17)$$

Using the above properties in expressions (7)-(9), it comes for the i-th pulsation of interest

$$\mathrm{Var}_{\mathtt{WP}}(\hat{\omega}_i) = \mathrm{Var}_{\mathtt{P}}(\hat{\omega}_i) = \frac{2(L-S)\left(1+\frac{1}{\mathrm{SNR}_i\ L}\right)}{TL^3\ \mathrm{SNR}_i} \qquad (18)$$

and

$$\mathrm{Var}_{\mathtt{C}}(\hat{\omega}_i) = \mathrm{Var}_{\mathtt{MN}}(\hat{\omega}_i) = \frac{2(L-M)\left(1+\frac{1}{\mathrm{SNR}_i\ L}\right)}{TL^3\ \mathrm{SNR}_i}. \qquad (19)$$

Finally, since $L \gg 1$ and for a sufficient SNR, then $\frac{1}{\mathrm{SNR}_i\ L} \ll 1$.

7.2. Proof of Theorem 2

First, note

$$\begin{aligned} p(\omega_2)^H p(\omega_2) &= L + \Delta_\omega^2 \frac{L(L-1)(2L-1)}{6} \\ p^H(\omega_1)(p(\omega_1) + \Delta_\omega p'(\omega_1)) &= L + i\Delta_\omega \frac{L(L-1)}{2} \\ p(\omega_2)^H p'(\omega_1) &= i\frac{L(L-1)}{2} + \Delta_\omega \frac{L(L-1)(2L-1)}{6}. \end{aligned}$$

Using the above expressions, we can characterize the following terms involved in (7)-(9) according to

$$\begin{aligned} |e_1^T P_Z^\perp p(\omega_1)^H|^2 &= \frac{\Delta_\omega^2 (L-1)^2(L-2)^2}{144}, && (20) \\ e_1^T P_Z^\perp e_1 &= \frac{(L-1)(L-2)}{L(L+1)}, && (21) \\ e_1^T P_A^\perp e_1 &= \frac{L-1}{L} && (22) \\ |e_1^T P_A^\perp p'(\omega_1)|^2 &= \frac{(L-1)^2}{4} && (23) \end{aligned}$$

In addition, we have to evaluate the following quantity:

$$e_1^T E_{(p(\omega_1)\ p(\omega_2))} p'(\omega_1) = \underbrace{e_1^T p(\omega_1)}_{=1} \underbrace{\left(p(\omega_1)^H P_{p(\omega_2)}^\perp p(\omega_1)\right)^{-1}}_{\to(i)} \underbrace{p(\omega_1)^H P_{p(\omega_2)}^\perp p'(\omega_1)}_{\to(ii)} \qquad (24)$$

where

- The term indexed by (i) in the above expressions involves

$$\left(p(\omega_2)^H p(\omega_2)\right)^{-1} \stackrel{1}{=} \frac{1}{L}\left(1 - \Delta_\omega^2 \frac{(L-1)(2L-1)}{6}\right) \quad (25)$$

  and thus

$$\left(p(\omega_1)^H P_{p(\omega_2)}^\perp p(\omega_1)\right)^{-1} \stackrel{1}{=} \frac{12}{\Delta_\omega^2 L(L^2-1)}. \qquad (26)$$

  In addition, remark that $e_1^T \Omega e_1 = \left(p(\omega_1)^H P_{p(\omega_2)}^\perp p(\omega_1)\right)^{-1}$.

- The second term, indexed by (ii) is

$$p(\omega_1)^H P_{p(\omega_2)}^\perp p'(\omega_1) \stackrel{1}{=} \Delta_\omega \frac{L(L^2-1)}{12}. \qquad (27)$$

Finally, a first-order approximation of the square modulus of expression (24) is given by

$$|e_1^T E_{(A\ B)} p'(\omega)|^2 = \left(\frac{12}{\Delta_\omega^2 L(L^2-1)} \Delta_\omega \frac{L(L^2-1)}{12}\right)^2 = \frac{1}{\Delta_\omega^2}. \quad (28)$$

Due to the proximity between ω_1 and ω_2, matrix $\left(Z^H Z\right)^{-1}$ involved in $p(\omega_1)^H S_X^\dagger R_X S_X^\dagger p(\omega_1)$ is no more equal to $\frac{1}{L}I$, conversely $(p(\omega_1)^H p(\omega_1))^{-1}$ remains unchanged and equal to $\frac{1}{L}$. So, by using 2×2 inversion matrix formula, we give the expansion of $(Z^H Z)^{-1}$ and it yields

$$\left(Z^H Z\right)^{-1} \stackrel{2}{=} \frac{12}{\Delta_\omega^2 L(L^2-1)} \begin{bmatrix} 1 & -1 - i\Delta_\omega \frac{L-1}{2} \\ -1 + i\Delta_\omega \frac{L-1}{2} & 1 \end{bmatrix}.$$

Therefore, we substitute $\left[(Z^H Z)^{-1}\right]_{(11)} = \frac{12}{\Delta_\omega^2 L(L^2-1)}$ into (17) generalized to the whole subspace $\mathcal{R}(Z)$, and we obtain

$$p(\omega_1)^H S_X^\dagger R_X S_X^\dagger p(\omega_1) = \frac{1}{\sigma_1^2}\left(1 + \frac{12}{\Delta_\omega^2 L(L^2-1)} \frac{1}{\mathrm{SNR}_1}\right) = \frac{2T}{\sigma^2}\xi_1.$$

Since $(A^H A)^{-1}$ remains unchanged in (17) we have straightforwardly

$$p(\omega_1)^H S_A^\dagger R_A S_A^\dagger p(\omega_1) = \frac{1}{\sigma_1^2}\left(1 + \frac{1}{L\ \mathrm{SNR}_1}\right) = \frac{2T}{\sigma^2}\bar{\xi}_1.$$

The last process to do is to substitute all the terms defined into theoretical expressions (7)-(9) and it comes

$$\begin{aligned} \mathrm{Var}_{\mathtt{WP}}(\hat{\omega}_1) &= \left(\Delta_\omega^2 \frac{12}{L(L^2-1)\Delta_\omega^2} + \frac{(L-1)(L-2)}{L(L+1)}\right)\bar{\xi}_1 \\ \mathrm{Var}_{\mathtt{P}}(\hat{\omega}_1) &= \frac{4(L-1)}{L(L-1)^2}\bar{\xi}_1 \\ \mathrm{Var}_{\mathtt{C}}(\hat{\omega}_1) &= \frac{144(L-1)(L-2)}{\Delta_\omega^2 L(L+1)(L-1)^2(L-2)^2}\bar{\xi}_1 \\ \mathrm{Var}_{\mathtt{MN}}(\hat{\omega}_1) &= \frac{144(L-1)(L-2)}{\Delta_\omega^2 L(L+1)(L-1)^2(L-2)^2}\xi_1. \end{aligned}$$

Finally, a basic manipulation completes the proof.

8. REFERENCES

[1] R.T. Behrens and L.L. Scharf, "Signal processing applications of oblique projection operators", *IEEE Trans. on Signal Processing*, Vol. 42, Issue 6, June 1994.

[2] M.L. McCloud and L.L Scharf, "A new subspace identification algorithm for high-resolution DOA estimation", *IEEE Transactions on Antennas and Propagation*, Vol. 50, Issue 10, Oct. 2002, pp.1382–1390.

[3] P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramér-Rao bound", *IEEE Trans. on Signal Processing*, Vol. 37, Issue 5, May 1989, pp.720-741.

[4] H. Chen, S. Van Huffel, D. Van Ormondt and R. De Beer, "Parameter estimation with prior knowledge of known signal poles for the quantification of NMR spectroscopy data in the time domain", *Journal of Magnetic Resonance A*, vol. 119, 1996, pp. 225-234.

[5] H. Krim, P. Forster and J.G. Proakis, "Operator approach to performance analysis of root-MUSIC and root-min-norm", *IEEE Trans. on Acoustics, Speech, and Signal Processing*, Vol. 40, Issue 7, July 1992.

[6] R.D. DeGroat, E.M. Dowling, D.A. Linebarger, "The Constrained MUSIC Problem", *IEEE Trans. on Signal Processing*, Volume 41, Issue 03, March 1993.

[7] M. Kaveh and A.J. Barabell, "The Statistical Performance of the MUSIC and the Minimum-Nom Algorithms in Resolving Plane Waves in Noise", *IEEE Trans. ASSP*, ASSP-34:331-341, April 1986.

[8] R. Boyer and G. Bouleux, "Cramér-Rao Lower Bound for Prior-Subspace Estimation", *Proceeding of the IEEE workshop on Sensor Array and Multi-channel processing (SAM-2006)*, July 2006.

[9] G. Bouleux and R. Boyer, "Analysis of Prior-Subspace Estimation Schemes", *Proceeding of the IEEE workshop on Sensor Array and Multi-channel processing (SAM-2006)*, July 2006.