STOCHASTIC MAXIMUM-LIKELIHOOD DOA ESTIMATION IN THE PRESENCE OF UNKNOWN NONUNIFORM NOISE

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ABSTRACT

This paper investigates the direction-of-arrival (DOA) estimation of multiple narrowband sources in the presence of nonuniform white noise with an arbitrary diagonal covariance matrix. While both the deterministic and stochastic Cramér-Rao Bound (CRB) and the deterministic Maximum-Likelihood (ML) DOA estimator under this model have been derived in [1], the stochastic ML DOA estimator under the same setting is still not available in the literature. In this paper, a new stochastic ML DOA estimator is derived. Its implementation is based on an iterative procedure which stepwise concentrates the log-likelihood function with respect to the signal and noise nuisance parameters. A modified inverse iteration algorithm is also presented for the estimation of the noise parameters. Simulation results have shown that the proposed algorithm is able to provide significant performance improvement over the conventional uniform ML estimator in nonuniform noise environments and require only a few iterations to converge to the nonuniform stochastic CRB.

Index Terms— Direction of arrival estimation, Maximum likelihood estimation

1. INTRODUCTION

Direction of arrival (DOA) estimation has been one of the central problems in array signal processing. While a wide variety of high performance DOA estimators have been proposed in the past few decades, the Maximum Likelihood (ML) estimator plays an important role among these techniques. Many of the proposed ML estimators are derived from the *uniform white noise assumption*[2, 3], in which the noise process of each sensor is assumed to be spatially uncorrelated white Gaussian with identical unknown variance. It is shown that under this assumption the estimates of the nuisance parameters can be expressed as a function of DOAs[4, 5, 6], and therefore the number of independent parameters to be estimated can be substantially reduced.

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This uniform white noise assumption may be unrealistic. For a sparsely placed sensor array, the noise at each sensor can be assumed to be spatially uncorrelated but the noise power of each sensor can still be different due to the nonuniformity of sensor noise or the imperfection of array calibration. In this case, the noise covariance matrix should be modeled as a general diagonal matrix with non-identical diagonal elements.

The deterministic and stochastic CRBs for the considered DOA estimation problem are first derived by Pesavento *et al.*[1]. In order to mitigate the burden of the straightforward implementation of the maximum likelihood DOA estimator, a new deterministic MLE based on the stepwise concentration is also proposed[1]. However, to the best of our knowledge, no similar work has been carried out for the stochastic case. In this paper, we propose a new stochastic maximum-likelihood DOA algorithm under the same noise model and in some sense "complete" the understanding of this problem.

Throughout this paper, we denote the superscripts ^{T}, *, ^{H}, and † as transpose, conjugate, conjugate transpose, and pseudo inverse of a matrix. The operator $\mathrm{diag}\{\mathbf{v}\}$ denotes the diagonal matrix with the vector $\mathbf{v}$ as the the main diagonal, and $\mathrm{diag}\{\mathbf{X}\}$ denotes the column vector formed from the elements of the main diagonal of square matrix $\mathbf{X}$. In addition, $\mathrm{tr}\{\cdot\}$ and $|\cdot|$ denote the trace and determinant of a square matrix respectively.

2. SIGNAL MODEL

Let there be M narrowband sources in the far field of a P-element sensor array. For simplicity, we assume the sources and the array lie in the same plane, and denote θ_m as the DOA of the mth source with respect to the array centroid, where $m = 1, \cdots, M$. The array output $\mathbf{y}(t)$, observed at the tth snapshot can then be modelled as

$$\mathbf{y}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{e}(t),\ t = 1, \cdots, N, \quad (1)$$

where $\mathbf{x}(t) = [x_1(t), \cdots, x_M(t)]^T$ and and $\mathbf{e}(t)$ are the signal and the noise vector observed at the tth snapshot respectively, and $\mathbf{A} = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_M)]$ is the steering matrix. It will be assumed that the geometry of the sensor array is such that $\mathbf{A}$ is full rank for all the DOAs of interest. As a result, $\mathbf{A}^H\mathbf{A}$ will always be positive definite.

In this paper, we investigate the case where the sensor noise of each channel is a spatially uncorrelated white Gaussian random process with covariance $\mathbf{Q} = \mathrm{diag}\{\sigma_1^2, \cdots, \sigma_P^2\}$, where σ_p^2 is the noise variance of the pth sensor. The signal waveform $\mathbf{x}(t)$ is modeled as a zero-mean Gaussian process with covariance matrix $\mathbf{R_x}$, and it follows immediately that $\mathbf{y}(t)$ is also a zero mean Gaussian random process with covariance $\mathbf{R_y} = \mathbf{A}\mathbf{R_x}\mathbf{A}^H + \mathbf{Q}$.

3. MAXIMUM-LIKELIHOOD DOA ESTIMATION

Under the signal model defined in the previous section, we can derive the joint log-likelihood function of the unknown parameters $\mathbf{\Psi} = \{\mathbf{\Theta}, \mathbf{R_x}, \mathbf{Q}\}$ as

$$L(\mathbf{\Psi}) = g(\mathbf{S}, \mathbf{R_y}) = -N\{\log|\mathbf{R_y}| + \mathrm{tr}(\mathbf{R_y}^{-1}\mathbf{S})\}, \quad (2)$$

where $\mathbf{S} = \frac{1}{N}\sum_{t=1}^{N}\mathbf{y}(t)\mathbf{y}(t)^H$ is the sample covariance matrix of the array output.

The maximum-likelihood estimate for $\mathbf{\Psi}$ can then be obtained by solving

$$\hat{\mathbf{\Psi}} = \arg\max_{\mathbf{\Psi}} L(\mathbf{\Psi}). \quad (3)$$

Clearly the search space of solving (3) is huge (of dimension $M + M^2 + P$), and therefore we seek to reduce the optimization problem since the estimation of $\mathbf{\Theta}$ is our only interest.

We first maximize (3) with respect to $\mathbf{R_x}$ for fixed $\mathbf{\Theta}$ and $\mathbf{Q}$. Taking partial derivatives of $L(\mathbf{\Psi})$ with respect to $[\mathbf{R_x}]_{i,j}$ and setting it to zero, we obtain a necessary condition for the extremum point

$$\mathbf{A}^H\mathbf{R_y}^{-1}[\mathbf{S} - \mathbf{R_y}]\mathbf{R_y}^{-1}\mathbf{A} = \mathbf{0}. \quad (4)$$

For more detailed derivation, see [5].

Following a similar derivation as [7], the optimal $\mathbf{R_x}$ that solves (4) is obtained as

$$\hat{\mathbf{R}}_\mathbf{x} = [\tilde{\mathbf{A}}^H\tilde{\mathbf{A}}]^{-1}[\tilde{\mathbf{A}}^H\tilde{\mathbf{S}}\tilde{\mathbf{A}} - \tilde{\mathbf{A}}^H\tilde{\mathbf{A}}][\tilde{\mathbf{A}}^H\tilde{\mathbf{A}}]^{-1}, \quad (5)$$

where $\tilde{\mathbf{A}} = \mathbf{Q}^{-1/2}\mathbf{A}$ and $\tilde{\mathbf{S}} = \mathbf{Q}^{-1/2}\mathbf{S}\mathbf{Q}^{-1/2}$.

Substituting $\mathbf{R_y}$ of (2) by $\mathbf{A}\hat{\mathbf{R}}_\mathbf{x}\mathbf{A}^H + \mathbf{Q}$, the maximum likelihood estimate for $\mathbf{\Theta}$ and $\mathbf{Q}$ can be obtained by maximizing

$$\begin{aligned} L(\mathbf{\Theta}, \mathbf{Q}) &= -N\log|\mathbf{Q}^{1/2}\{\mathbf{P}_{\tilde{\mathbf{A}}}\tilde{\mathbf{S}}\mathbf{P}_{\tilde{\mathbf{A}}} - \mathbf{P}_{\tilde{\mathbf{A}}} + \mathbf{I}_P\}\mathbf{Q}^{1/2}| \\ &\quad -N\mathrm{tr}\{\tilde{\mathbf{S}}\} + N\mathrm{tr}\{\mathbf{P}_{\tilde{\mathbf{A}}}\tilde{\mathbf{S}}\}, \end{aligned} \quad (6)$$

where $\mathbf{P}_{\tilde{\mathbf{A}}} = \tilde{\mathbf{A}}[\tilde{\mathbf{A}}^H\tilde{\mathbf{A}}]^{-1}\tilde{\mathbf{A}}^H$. Further concentration over $\mathbf{Q}$ seems to be analytically impossible [7] which prevents us from further simplifications.

At the same time, we can approach the problem by fixing $\mathbf{\Theta}$ and $\mathbf{R_x}$ in (2) and solve for an estimator for $\mathbf{Q}$ that maximizes $L(\mathbf{\Psi})$. Denote $\mathbf{q} = \mathrm{diag}\{\mathbf{Q}\}$ and $\mathbf{q}_p$ as the pth element of $\mathbf{q}$, then the pth element of the gradient vector $\nabla_q L(\mathbf{\Psi})$ can expressed as

$$[\nabla_q L(\mathbf{\Psi})]_p \quad = \quad -N\mathrm{tr}\{[\mathbf{R_y}^{-1} - \mathbf{R_y}^{-1}\mathbf{S}\mathbf{R_y}^{-1}]\mathbf{E}_{p,p}\}, \quad (7)$$

where $\mathbf{E}_{i,j}$ is a $P \times P$ matrix with the (i, j)th element equals to 1 and 0 elsewhere. Setting $\nabla_q L(\mathbf{\Psi})$ to zero, a necessary condition for the extremum point is obtained as

$$[\mathbf{R_y}^{-1} - \mathbf{R_y}^{-1}\mathbf{S}\mathbf{R_y}^{-1}]_{p,p} = 0, \; p = 1, \cdots, P. \quad (8)$$

It appears that (8) is a rather complicated function of $\mathbf{Q}$ and therefore no analytical solution seems to be available.

To this end, a new maximum-likelihood DOA estimator based on the *stepwise concentration* is proposed. The idea of this technique is to numerically concentrate the log-likelihood function by the following iterative procedure.

Iterative procedure of the proposed ML DOA estimator

Step 1. Iter $= 1$. Compute the MLEs ($\hat{\mathbf{\Theta}}$,$\hat{\mathbf{R}}_\mathbf{x}$,$\hat{\mathbf{Q}}$) under the uniform white noise assumption, summarized as follows[6]:
$\hat{\mathbf{\Theta}} = \arg\max_{\mathbf{\Theta}} \log|\mathbf{A}\hat{\mathbf{R}}_\mathbf{x}\mathbf{A}^H + \hat{\sigma}^2\mathbf{I}_P|$,
$\hat{\mathbf{R}}_\mathbf{x} = \mathbf{A}^\dagger\mathbf{S}\mathbf{A}^{\dagger H} - \hat{\sigma}^2[\mathbf{A}^H\mathbf{A}]^{-1}$,
$\hat{\sigma}^2 = \mathrm{tr}\{(\mathbf{I}_P - \mathbf{A}\mathbf{A}^\dagger)\mathbf{S}\}/(P - M)$,
$\hat{\mathbf{Q}} = \hat{\sigma}^2\mathbf{I}_P$,
where $\mathbf{A}^\dagger = (\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$.

Step 2. Use ($\hat{\mathbf{\Theta}}$,$\hat{\mathbf{R}}_\mathbf{x}$,$\hat{\mathbf{Q}}$) as an initial estimate, and define $\hat{\mathbf{A}}$ as the steering matrix evaluated at $\hat{\mathbf{\Theta}}$.
Find a refined $\hat{\mathbf{Q}}$ so that the resultant $\hat{\mathbf{R}}_\mathbf{y} = \hat{\mathbf{A}}\hat{\mathbf{R}}_\mathbf{x}\hat{\mathbf{A}}^H + \hat{\mathbf{Q}}$ increases the log-likelihood function.

Step 3. Use the obtained $\hat{\mathbf{Q}}$ to find an improved $\hat{\mathbf{\Theta}}$ through $\hat{\mathbf{\Theta}} = \arg\max_{\mathbf{\Theta}} L(\mathbf{\Theta}, \hat{\mathbf{Q}})$ where $L(\mathbf{\Theta}, \mathbf{Q})$ is defined as in (6). Update $\hat{\mathbf{R}}_\mathbf{x}$ using the latest estimates ($\hat{\mathbf{\Theta}}$,$\hat{\mathbf{Q}}$) through (5).
Iter $=$ Iter $+ 1$.
Repeat Step 2 and Step 3 until the algorithm converges.

The parameter "Iter" denotes the index of iteration in the procedure. Many numerical algorithms can be used (or modified) to solve the DOA estimation problem in step 1 and step 3. The only remaining issue which has not been explained in this paper is how step 2 is implemented. The *modified inverse iteration algorithm* is proposed for this purpose.

4. MODIFIED INVERSE ITERATION ALGORITHM

In [8], a general method based on the inverse iteration algorithm for estimating a covariance matrix of a specified structure is proposed. We modified this algorithm to solve step 2 and referred to it as the "modified inverse iteration algorithm" in the following discussion.

The basic idea of the this algorithm is as follows. Let $\mathcal{D}$ denote the set of all $P \times P$ diagonal matrices. In each iteration we start with some initial estimates, $\hat{\mathbf{\Theta}}$, $\hat{\mathbf{R}}_{\mathbf{x}}$, and $\hat{\mathbf{Q}}$ and a $\Delta\mathbf{Q} \in \mathcal{D}$ is sought so that $(\mathbf{S} - \Delta\mathbf{Q}, \hat{\mathbf{R}}_{\mathbf{y}})$ satisfies (8), i.e.

$$[\hat{\mathbf{R}}_{\mathbf{y}}^{-1}(\hat{\mathbf{R}}_{\mathbf{y}} - (\mathbf{S} - \Delta\mathbf{Q}))\hat{\mathbf{R}}_{\mathbf{y}}^{-1}]_{p,p} = 0 \;, p = 1, \cdots, P, \quad (9)$$

where $\hat{\mathbf{R}}_{\mathbf{y}} = \hat{\mathbf{A}}\hat{\mathbf{R}}_{\mathbf{x}}\hat{\mathbf{A}}^H + \hat{\mathbf{Q}}$. Note $\Delta\mathbf{Q}$ is an improving direction since

$$\begin{aligned} \frac{[\nabla_q L(\hat{\mathbf{\Psi}})]^T \Delta\mathbf{q}}{N} &= tr\{[-\hat{\mathbf{R}}_{\mathbf{y}}^{-1} + \hat{\mathbf{R}}_{\mathbf{y}}^{-1}(\mathbf{S} - \Delta\mathbf{Q})\hat{\mathbf{R}}_{\mathbf{y}}^{-1}]\Delta\mathbf{Q}\} \\ &+ tr\{[\hat{\mathbf{R}}_{\mathbf{y}}^{-1}\Delta\mathbf{Q}\hat{\mathbf{R}}_{\mathbf{y}}^{-1}]\Delta\mathbf{Q}\} \quad (10) \\ &= tr\{[\hat{\mathbf{R}}_{\mathbf{y}}^{-1}\Delta\mathbf{Q}\hat{\mathbf{R}}_{\mathbf{y}}^{-1}]\Delta\mathbf{Q}\} \geq 0 \quad (11) \end{aligned}$$

The equality sign in (11) holds since the first term in (10) is zero by construction.

The condition (9) is equivalent to

$$tr\{[\hat{\mathbf{R}}_{\mathbf{y}}^{-1}\mathbf{S}\hat{\mathbf{R}}_{\mathbf{y}}^{-1} - \hat{\mathbf{R}}_{\mathbf{y}}^{-1}(\hat{\mathbf{R}}_{\mathbf{y}} + \Delta\mathbf{Q})\hat{\mathbf{R}}_{\mathbf{y}}^{-1}]\mathbf{E}_{p,p}\} = 0, \; p = 1, \cdots, P. \quad (12)$$

Putting (12) into a matrix form, then $\Delta\mathbf{q}$, the vector of the diagonal elements of $\Delta\mathbf{Q}$ can be solved by the following linear equation

$$\begin{aligned} \mathbf{H}\Delta\mathbf{q} &= \mathbf{u}, \quad (13) \\ \mathbf{H}_{i,j} &= \mathrm{tr}\{\mathbf{E}_{j,j}\hat{\mathbf{R}}_{\mathbf{y}}^{-1}\mathbf{E}_{i,i}\hat{\mathbf{R}}_{\mathbf{y}}^{-1}\}, \quad (14) \\ \mathbf{u}_i &= \mathrm{tr}\{[\hat{\mathbf{R}}_{\mathbf{y}}^{-1}\mathbf{S}\hat{\mathbf{R}}_{\mathbf{y}}^{-1} - \hat{\mathbf{R}}_{\mathbf{y}}^{-1}]\mathbf{E}_{i,i}\}, \quad (15) \end{aligned}$$

for all $i, j = 1, \cdots, P$.

The overall procedure of the modified inverse iteration algorithm is summarized as follows.

Modified inverse iteration algorithm

Step 1. Given some initial estimates ($\hat{\mathbf{\Theta}}$,$\hat{\mathbf{R}}_{\mathbf{x}}$,$\hat{\mathbf{Q}}$), compute an improving direction $\Delta\mathbf{Q}$ by (13).

Step 2. **Backtracking Line Search**
Set $t = 1$.
while
$g(\mathbf{S}, \hat{\mathbf{R}}_{\mathbf{y}} + t\Delta\mathbf{Q}) < g(\mathbf{S}, \hat{\mathbf{R}}_{\mathbf{y}}) + \alpha t \nabla_q g(\mathbf{S}, \hat{\mathbf{R}}_{\mathbf{y}})^T \Delta\mathbf{q}$
or $\hat{\mathbf{Q}} + t\Delta\mathbf{Q} < 0$,
$t = \beta t$.
end

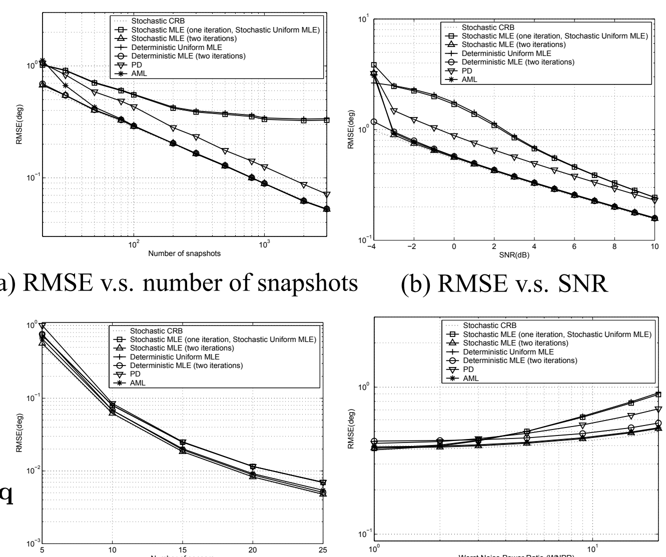
Step 3. Set new $\hat{\mathbf{Q}}$ as $\hat{\mathbf{Q}} + t\Delta\mathbf{Q}$
Repeat Step 1. to Step 3. until the algorithm converges.

α and β are two constants satisfying $0 < \alpha < 0.5$ and $0 < \beta < 1$.

5. SIMULATION RESULTS

In this section, we present the simulation results of the proposed stochastic ML estimator in comparison with the deterministic ML estimator [1], the Power Domain (PD) method [9], and the Approximate ML (AML) algorithm [10]. In order to make a fair comparison, we initialize the AML algorithm with the DOA estimates of the conventional stochastic uniform ML estimator (same as the 1st iteration estimate of the proposed stochastic ML algorithm). The nonlinear DOA estimation required in step 1 and step 3 of the proposed iterative procedure is solved through the Alternating Maximization (AM) algorithm[11], implemented by an initial 1° coarse grid search followed by the golden section fine search. Despite of the fact that AM guarantees only a local optimal solution, excellent global convergence has been observed in this type of DOA estimation problem[11].

The simulation settings and scenarios of this paper are chosen to be identical to the settings in [1] in which the DOA estimation of two uncorrelated narrow-band sources of equal power using a Uniform Linear Array (ULA) is considered. The DOAs of the narrowband sources are set to be 7° and 13° relative to the broadside respectively, and the ULA is assumed to be omnidirectional with half-wavelength inter-element spacing. In each simulation scenario, the Root-Mean-Square-Errors (RMSEs) of the DOA estimates are computed by averaging over 1000 Monte Carlo runs and plotted along with the stochastic CRB[1].



(a) RMSE v.s. number of snapshots (b) RMSE v.s. SNR

(c) RMSE v.s. number of sensors (d) RMSE v.s. WNPR

Fig. 1. Comparison of the DOA estimation RMSEs and the stochastic CRB versus (a)number of snapshots, (b)SNR, (c)number of sensors (d)WNPR.

In the first scenario, we consider the following noise covariance matrix

$$\mathbf{Q} = \sigma^2 \mathrm{diag}\{[10.0, 2.0, 1.5, 0.5, 8.0, 0.7, 1.1, 3.0, 6.0, 3.0]\}$$

and fix the array-SNR (ASNR)[1] to 5 dB. The RMSE performance of the tested algorithms with respect to N are investigated and plotted as Fig. 1(a).

In the second scenario, same noise covariance matrix is assumed. The number of snapshots are now fixed to 100 and the RMSEs are plotted in Fig. 1(b) with respect to the SNR.

In the next two scenarios, we investigate how the nonuniformity of the sensor noise affects the performance of the tested algorithms. In each Monte Carlo run, we fix the Worst-Power-Noise-Ratio (WNPR)[1] and randomly choose two sensor locations in the ULA, one with noise variance $\sigma^2_{\min}$ and the other $\sigma^2_{\max} = \text{WNPR}\sigma^2_{\min}$. The rest of the sensors are assigned their noise variances according to a uniform distribution $\mathcal{U}(\sigma^2_{\min}, \sigma^2_{\max})$.

The RMSEs versus the number of sensors are investigated in the third scenario. The ASNR, number of snapshots, and the WNPR are set to be 20 dB, 100, and 65 respectively and the simulation results are shown in Fig. 1(c).

In the last scenario, we set the ASNR, number of snapshots and the number of sensors to be 0 dB, 100, and 10 respectively. The RMSE performance versus the WNPR is investigated and plotted in Fig. 1(d).

From Fig. 1(a)-(d), it can be observed that all the tested nonuniform algorithms provide essential performance improvement over the conventional(uniform) ML estimator in the nonuniform noise environment. The performance improvement of the proposed stochastic nonuniform ML estimator becomes more significant when the number of freedoms (N times P) increases. For small N and P, the performance improvement is minor since the uniform ML estimator has fewer parameters to estimate (although mismatched).

Among the tested nonuniform algorithms, the PD method has the lowest complexity. However, the RMSE performance of the PD method appears to be the worst and approaches to that of the uniform ML estimator as the ASNR is sufficiently large. The RMSE performance and the complexity of the proposed stochastic nonuniform ML estimator is comparable to that of the deterministic nonuniform ML estimator[1] since both algorithms implement the idea of stepwise-concentration and both require only two iterations to converge to a solution close to the CRB in the large sample and high ASNR cases. The AML algorithm has a concentrated form yet a higher complexity comparing to the complexity of PD and the nonuniform ML estimators. However, despite of the high complexity, the RMSE performance of the AML algorithm is slightly higher than the deterministic and stochastic ML estimator which is probably due to the asymptotic approximations made in the derivations.

6. CONCLUSIONS

In this paper, we address the problem of estimating the DOAs of multiple narrowband sources in the presence of unknown nonuniform sensor noise. A new stochastic ML DOA estimator has been derived and the performance of the proposed algorithm is studied through extensive computer simulations. In all settings, the proposed algorithm asymptotically converges to the CRB within 2 iterations and therefore the complexity is only a few times higher than the conventional uniform ML estimator. Simulation results also demonstrate the performance improvement in comparison with other nonuniform algorithms for a variety of scenarios (number of snapshots, SNR, number of sensors, and etc.).

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