BOUNDS AND ALGORITHMS FOR TIME DELAY ESTIMATION ON PARALLEL, FLAT FADING CHANNELS

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ABSTRACT

We study time delay estimation (TDE) on parallel channels with flat fading. Several models for the channel gains are considered, and for each case we present the the maximum likelihood estimator (MLE), the Cramér-Rao bound (CRB), and the Ziv-Zakai bound (ZZB). The bounds facilitate an analysis of the effects of fading and diversity on TDE accuracy over parallel channels. Computer simulations of the mean-squared error of the MLEs confirm the validity of the bounds.

Index Terms— Delay estimation, fading channels, maximum likelihood estimation, error analysis, diversity methods.

1. INTRODUCTION

Time delay estimation (TDE) has been studied extensively for several decades, e.g., see [1], with most of the attention focused on signals received over a single channel. In this paper, we study the problem of estimating a time delay (TD) parameter based on processing signals received on multiple, parallel channels. An example of TDE on parallel channels is a waveform containing multiple frequency subbands that is received through a frequency-selective channel. The channel may be known or unknown, as well as deterministic or randomly fading. The subbands may be viewed as parallel channels, and the objective is to jointly process the signals on each channel to estimate the (common) TD parameter. If the parallel channels include random fading, then interesting questions include an analysis of diversity gain with respect to TDE accuracy, and tradeoffs between signal energy per channel and the number of channels. TDE on parallel channels also arises with a frequency-hopping waveform in which multiple hops are processed to estimate the TD [2].

In this paper, we formulate a model for TDE on parallel channels in Section 2. Each channel is modeled as frequency nonselective (flat), so it is characterized by a complex gain. We consider several models for the channel gain, including deterministic (known and unknown) and random (Rayleigh fading). In Section 3, each channel model is analyzed, and we present the maximum-likelihood estimator (MLE), Cramér-Rao bound (CRB), and Ziv-Zakai bound (ZZB) for the TD parameter. Section 4 contains Monte Carlo simulation results that compare the mean-squared error (MSE) of the MLEs to the bounds. We note that bounds for TDE on a convolutive (frequency selective) channel have been developed recently in [3].

2. SIGNAL MODEL

We model the complex-valued signals received on N channels as

$$r_i(t) = \gamma_i \, s_i(t-\tau) \, + \, n_i(t), \quad i = 1, \ldots, N. \tag{1}$$

The $s_i(t)$ are known waveforms, $n_i(t)$ are complex, AWGN processes with zero mean and two-sided PSD $\mathcal{N}_o$, and each γ_i is a complex scalar representing the gain of the channel. We consider five models for the vector of complex channel gains, $\boldsymbol{\gamma}$.

(1) $\boldsymbol{\gamma}$ is known and deterministic.

(2) $\boldsymbol{\gamma}$ is unknown and deterministic.

(3) Each γ_i is a complex, circular, Gaussian random variable with zero mean and variance $E\left\{|\gamma_i|^2\right\} = \sigma_s^2$, independent for $i = 1, \ldots, N$. This corresponds to Rayleigh fading, denoted by $\boldsymbol{\gamma} \sim \mathrm{CN}\left(\mathbf{0}, \sigma_s^2 \boldsymbol{I}\right)$. The results in this paper are identical whether σ_s^2 is modeled as a known or unknown parameter.

(1A) Case (1) averaged over the Rayleigh fading distribution, so $\boldsymbol{\gamma} \sim \mathrm{CN}\left(\mathbf{0}, \sigma_s^2 \boldsymbol{I}\right)$. This corresponds to a fading channel and a receiver that has perfect knowledge of $\boldsymbol{\gamma}$, so the TDE is estimated with a coherent combination of the diversity channels.

(2A) Case (2) averaged over $\boldsymbol{\gamma} \sim \mathrm{CN}\left(\mathbf{0}, \sigma_s^2 \boldsymbol{I}\right)$. This corresponds to (imperfect) estimation of $\boldsymbol{\gamma}$ on each realization, and then averaging over the Rayleigh fading distribution.

Case 3 is different from case 2A in that $\boldsymbol{\gamma}$ is not estimated in case 3 since it is modeled as a random vector.

The probability density function (pdf) for the model (1) is required for two cases: the conditional pdf, $p_c\left(\boldsymbol{R} \,|\, \boldsymbol{\gamma}; \tau\right)$, where $\boldsymbol{\gamma}$ is a fixed vector, and the unconditional pdf, $p_u\left(\boldsymbol{R} \,|\, \tau\right) = E_{\boldsymbol{\gamma}}\left\{p_c\left(\boldsymbol{R} \,|\, \boldsymbol{\gamma}; \tau\right)\right\}$, that is averaged over $\boldsymbol{\gamma} \sim \mathrm{CN}\left(\mathbf{0}, \sigma_s^2 \boldsymbol{I}\right)$. The signal autocorrelation functions are defined as

$$\rho_i(\xi) = \int s_i(t-\xi)^* \, s_i(t) \, dt, \quad i = 1, \ldots, N, \tag{2}$$

where * denotes complex conjugate and the limits on all integrals are $(-\infty, \infty)$. The coherent and noncoherent matched filters applied to the signal received on channel i are defined as follows,

$$Y_i^{\mathrm{COH}}(\xi) = \mathrm{Re} \int \gamma_i^* \, s_i(t-\xi)^* \, r_i(t) \, dt \tag{3}$$

$$Y_i^{\mathrm{NONCOH}}(\xi) = \left| \int s_i(t-\xi)^* \, r_i(t) \, dt \right|. \tag{4}$$

Let $\boldsymbol{r}_i$ be a vector representation of $r_i(t)$, and define $\boldsymbol{R} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N]$. Then the conditional pdf for $\boldsymbol{R}$ has the following form, where factors independent of $\boldsymbol{\gamma}, \tau$ are suppressed:

$$p_c\left(\boldsymbol{R} \,|\, \boldsymbol{\gamma}; \tau\right) \propto \prod_{i=1}^N \, \exp\left\{ \tfrac{1}{\mathcal{N}_o} \left[2 \, Y_i^{\mathrm{COH}}(\tau) - |\gamma_i|^2 \rho_i(0) \right] \right\}. \tag{5}$$

The unconditional pdf is obtained by observing that $\boldsymbol{\gamma} \sim \mathrm{CN}\left(\mathbf{0}, \sigma_s^2 \boldsymbol{I}\right)$ implies that $\boldsymbol{r}_i \sim \mathrm{CN}\left(\mathbf{0}, \boldsymbol{\Sigma}_i\right)$, where the covariance matrix is

$$\boldsymbol{\Sigma}_i = \sigma_s^2 \, \boldsymbol{s}_i(\tau) \boldsymbol{s}_i(\tau)^H + \mathcal{N}_o \, \boldsymbol{I} \tag{6}$$

$$\boldsymbol{\Sigma}_i^{-1} = \frac{1}{\mathcal{N}_o} \left[\boldsymbol{I} - \frac{\sigma_s^2 \, \boldsymbol{s}_i(\tau) \boldsymbol{s}_i(\tau)^H}{\mathcal{N}_o + \sigma_s^2 \, \rho_i(0)} \right], \tag{7}$$

where $\boldsymbol{s}_i(\tau)$ is the vector representation of $s_i(t-\tau)$ and ^{H} is the conjugate-transpose operation. Then, ignoring terms and factors that are independent of τ, the unconditional pdf is proportional to

$$p_u\left(\boldsymbol{R} \,|\, \tau\right) \propto \prod_{i=1}^N \exp\left\{ \frac{\sigma_s^2}{\mathcal{N}_o\left(\mathcal{N}_o + \sigma_s^2 \, \rho_i(0)\right)} Y_i^{\text{NONCOH}}(\tau)^2 \right\} \tag{8}$$

We make the following three assumptions.

A1 The signal autocorrelation functions in (2) are the same on each channel, $\rho_i(\xi) \triangleq \rho(\xi), i = 1, \ldots, N$, so the signal energy is

$$\rho_i(0) = \int |s_i(t)|^2 \, dt \triangleq \rho(0), \quad i = 1, \ldots, N. \tag{9}$$

A2 The signals have the property $\int s_i(t) \, \dot{s}_i(t)^* \, dt = 0$, where $\dot{s}_i(t) = \frac{d}{dt} s_i(t)$.

A3 The mean-squared signal bandwidth is the same on each channel and is denoted by B_s Hz, with definition

$$B_s^2 = \int \frac{|\dot{s}_i(t)|^2}{\rho(0) \, (2\pi)^2} \, dt = \int \frac{f^2 \, |S_i(f)|^2}{\rho(0)} \, df, \tag{10}$$

where $S_i(f)$ is the Fourier transform of $s_i(t)$.

Then the log-likelihood functions corresponding to (5) and (8) are

$$\mathcal{L}_c\left(\boldsymbol{R} \,|\, \boldsymbol{\gamma}; \tau\right) = \frac{1}{\mathcal{N}_o} \sum_{i=1}^N \left[2 \, Y_i^{\text{COH}}(\tau) - |\gamma_i|^2 \rho(0) \right] \tag{11}$$

$$\mathcal{L}_u\left(\boldsymbol{R} \,|\, \tau\right) = \frac{\sigma_s^2}{\mathcal{N}_o\left(\mathcal{N}_o + \sigma_s^2 \, \rho(0)\right)} \sum_{i=1}^N Y_i^{\text{NONCOH}}(\tau)^2. \tag{12}$$

3. TDE ESTIMATORS AND BOUNDS

In this section, the MLEs, CRBs, and ZZBs are presented for the five channel model cases. Before presenting the details, a summary is provided in Table 1. In cases 1 and 1A, the MLE is a sum of *coherent* matched filters (MFs) because $\boldsymbol{\gamma}$ is known. In cases 2, 3, and 2A, the MLE is a sum of *noncoherent* matched filters. Although the MLEs are identical for several cases in Table 1, the *performance* of the MLE varies from case to case, except that the performance is the same in cases 2A and 3. The CRBs for cases 1, 2, and 3 are generally distinct. However, for the known signal waveforms considered here with assumption **A2**, the τ and $\boldsymbol{\gamma}$ parameters are decoupled in the Fisher Information Matrix (FIM), so $\mathrm{CRB}_2 = \mathrm{CRB}_1$. The CRB is not defined for cases 1A and 2A, but the so-called modified CRB (MCRB) [4, 5] and asymptotic CRB (ACRB) [6] definitions are similar to these cases. The ZZB is unique in four of the five channel model cases, and the ZZB is tractable in the four distinct cases.

3.1. Maximum likelihood estimators

The MLEs are sums of coherent and noncoherent matched filters,

$$Z^{\text{COH}}(\xi) = \sum_{i=1}^N Y_i^{\text{COH}}(\xi) \tag{13}$$

$$Z^{\text{NONCOH}}(\xi) = \sum_{i=1}^N Y_i^{\text{NONCOH}}(\xi)^2. \tag{14}$$

For case 1, the MLE is the value of τ that maximizes the LL in (11) when $\boldsymbol{\gamma}$ is known, so

$$\hat{\tau}_1 = \arg\max Z^{\text{COH}}(\xi). \tag{15}$$

For case 2, $\boldsymbol{\gamma}$ is an unknown, deterministic parameter, so the MLE requires the joint maximization of the LL in (11) with respect to τ and $\boldsymbol{\gamma}$. For a given TDE estimate $\widehat{\tau}$, the MLE of each γ_i is

$$\widehat{\gamma}_i = \frac{1}{\rho(0)} \int s_i(t - \widehat{\tau})^* \, r_i(t) \, dt, \quad i = 1, \ldots, N. \tag{16}$$

The estimates $\widehat{\gamma}_i$ in (16) are substituted for $\boldsymbol{\gamma}$ in (11), yielding

$$\hat{\tau}_2 = \arg\max Z^{\text{NONCOH}}(\xi). \tag{17}$$

For case 3, $\boldsymbol{\gamma} \sim \mathrm{CN}\left(\mathbf{0}, \sigma_s^2 \boldsymbol{I}\right)$ so the MLE for τ is the value that maximizes the unconditional LL in (12), which is identical to (17):

$$\hat{\tau}_3 = \arg\max Z^{\text{NONCOH}}(\xi). \tag{18}$$

For cases 1A and 2A, the MLEs are given by $\hat{\tau}_1$ in (15) and $\hat{\tau}_2$ in (17), respectively. The MLE performance is different in cases 1A and 2A (compared with cases 1 and 2 in which $\boldsymbol{\gamma}$ is fixed) due to averaging over realizations of values for $\boldsymbol{\gamma} \sim \mathrm{CN}\left(\mathbf{0}, \sigma_s^2 \boldsymbol{I}\right)$. In Table 1, "Coh MF" refers to (13) and "NonCoh MF" refers to (14).

3.2. Cramér-Rao bounds

Next we consider the CRBs. In general, let us denote the deterministic parameters by the vector $\boldsymbol{\Theta}$ and the LL function by $\mathcal{L}(\mathbf{R} \,|\, \boldsymbol{\Theta})$. For cases 1, 2, and 3, $\boldsymbol{r}_i$ is a complex, Gaussian random vector, $\boldsymbol{r}_i \sim \mathrm{CN}(\boldsymbol{\mu}_i, \boldsymbol{C}_i)$, independent for $i = 1, \ldots, N$, so the Fisher information matrix (FIM) elements are given by [7]

$$[\boldsymbol{J}(\boldsymbol{\Theta})]_{m,n} = \sum_{i=1}^N \left\{ \text{trace}\left[\mathbf{C}_i^{-1} \frac{\partial \mathbf{C}_i}{\partial \theta_m} \mathbf{C}_i^{-1} \frac{\partial \mathbf{C}_i}{\partial \theta_n} \right] + 2 \cdot \mathrm{Re}\left[\frac{\partial \boldsymbol{\mu}_i}{\partial \theta_m}^H \mathbf{C}_i^{-1} \frac{\partial \boldsymbol{\mu}_i}{\partial \theta_n} \right] \right\}. \tag{19}$$

For cases 1 and 2, $\boldsymbol{\mu}_i = \gamma_i \boldsymbol{s}_i(\tau)$ and $\mathbf{C}_i = \mathcal{N}_o \boldsymbol{I}$, with $\boldsymbol{\Theta} = [\tau]$ for case 1 and $\boldsymbol{\Theta} = \left[\mathrm{Re}\left(\boldsymbol{\gamma}\right), \mathrm{Im}\left(\boldsymbol{\gamma}\right), \tau\right]$ for case 2. For case 3, $\boldsymbol{\mu}_i = \mathbf{0}$ and $\mathbf{C}_i = \boldsymbol{\Sigma}_i$ in (6), with $\boldsymbol{\Theta} = [\tau]$.

The CRB expressions are unified if we define the average SNR per channel in (20) for the deterministic cases 1 and 2, and the mean SNR per channel for cases 1A, 2A, and 3 in (21):

$$\mathrm{SNR}_{\text{det}}\left(\boldsymbol{\gamma}\right) = \left(\rho(0)/N\right) \sum_{i=1}^N \frac{|\gamma_i|^2}{\mathcal{N}_o} \tag{20}$$

$$\overline{\mathrm{SNR}} = \frac{\rho(0) \, E\left\{|\gamma_i|^2\right\}}{\mathcal{N}_o} = \frac{\rho(0) \, \sigma_s^2}{\mathcal{N}_o} = E_{\boldsymbol{\gamma}}\left\{\mathrm{SNR}_{\text{det}}\left(\boldsymbol{\gamma}\right)\right\}. \tag{21}$$

It can be shown that the FIM for case 2 is diagonal, so the CRB for τ is identical in cases 1 and 2:

$$\mathrm{CRB}_1\left(\widehat{\tau} \,|\, \boldsymbol{\gamma}\right) = \mathrm{CRB}_2 = \frac{1}{2\left(2\pi \, B_s\right)^2 \, N \, \mathrm{SNR}_{\text{det}}\left(\boldsymbol{\gamma}\right)}. \tag{22}$$

The notation on the left side of (22) emphasizes that the CRB depends on the channel gains, $\boldsymbol{\gamma}$. The FIM for case 3 is obtained from (19) with $\boldsymbol{\mu}_i = \mathbf{0}$, $\mathbf{C}_i = \boldsymbol{\Sigma}_i$ in (6), and $\boldsymbol{\Theta} = [\tau]$, yielding:

$$\mathrm{CRB}_3\left(\widehat{\tau}\right) = \frac{1 + \left(\overline{\mathrm{SNR}}\right)^{-1}}{2\left(2\pi \, B_s\right)^2 \, N \, \overline{\mathrm{SNR}}}. \tag{23}$$

	Channel Model				
	1	2	3	1A	2A
MLE	Coh MF	NonCoh MF	NonCoh MF	Coh MF	NonCoh MF
CRB	CRB_1	$\text{CRB}_2 = \text{CRB}_1$	CRB_3	MCRB	ACRB = MCRB
ZZB	ZZB_1	ZZB_2	ZZB_3	ZZB_{1A}	$\text{ZZB}_{2A} = \text{ZZB}_3$

Table 1. Relations between the MLEs, CRBs, and ZZBs for each channel model.

The CRB in (23) is the "true" CRB for the Rayleigh fading channel model. The derivation of (23) is tractable, but it requires significantly more computation than the CRB in (22) in which $\boldsymbol{\gamma}$ is a deterministic parameter. A number of bounds that are looser than the CRB have been studied for models with random nuisance parameters, such as the modified CRB (MCRB), hybrid CRB (HCRB), asymptotic CRB (ACRB), and others [4, 5, 6]. These looser bounds begin with a FIM in which γ is modeled as a deterministic parameter, and then an averaging operation is performed over $\boldsymbol{\gamma}$ on some function of the FIM. The CRB is not defined for channel models 1A and 2A in Table 1, but the MCRB and ACRB are closely related to cases 1A and 2A, respectively, and have the form

$$\text{CRB}_{1A}(\widehat{\tau}) \approx \text{MCRB}(\widehat{\tau}) \triangleq \frac{1}{2\,(2\pi\,B_s)^2\;N\,\overline{\text{SNR}}} \tag{24}$$

$$\text{CRB}_{2A}(\widehat{\tau}) \approx \text{ACRB}(\widehat{\tau}) \triangleq \frac{1}{2\,(2\pi\,B_s)^2\;N\,\overline{\text{SNR}}}. \tag{25}$$

Note from (23), (24), (25) that $\text{CRB}_3(\widehat{\tau}) \geq \text{MCRB}(\widehat{\tau}) = \text{ACRB}(\widehat{\tau})$ and that the three bounds coincide when $\overline{\text{SNR}} \gg 1$.

The CRBs in (22), (24), (25) are identical if we set $\text{SNR}_{\text{det}}(\boldsymbol{\gamma}) = \overline{\text{SNR}}$ in the deterministic and random channel models, and (23) has the same form when $\overline{\text{SNR}} \gg 1$. Thus the CRBs depend on the *total* signal power over the N channels, and the CRB analysis does not reveal any diversity advantage if more channels are used with proportionately less signal power per channel. We will show in the following subsection that the Ziv-Zakai bounds are quite sensitive to the channel model, and greater diversity (larger N) provides significant improvement in the threshold SNR at which the MLE performance is equal to the CRB.

3.3. Ziv-Zakai bounds

The CRB is a tight bound on the MLE performance only when the errors are small. As the errors get larger, a threshold behavior arises from ambiguities that are particular to the signal and its correlation. Several approaches have been used to develop bounds tighter than the CRB. Ziv and Zakai developed an approach based on hypothesis testing [8], and improvements were subsequently derived [9, 10]. In this section, we outline the approach used to find the improved ZZB for the model in (1) and present four distinct ZZB results, as summarized in Table 1. The detailed derivations are omitted due to space limitations.

The ZZB is derived from the following hypothesis test:

$$H_0: \quad r_i(t) = \gamma_i\, s_i(t-a) + n_i(t), \quad i = 1, \ldots, N \tag{26}$$

$$H_1: \quad r_i(t) = \gamma_i\, s_i(t-(a+\theta)) + n_i(t), \quad i = 1, \ldots, N. \tag{27}$$

The hypotheses are equally likely, τ is modeled as a uniform random variable on the interval $[-D/2, D/2]$, which represents *a priori* knowledge about the range of possible values for the TD parameter. Therefore in (26), (27), $\theta > 0$ and $a, (a+\theta) \in [-D/2, D/2]$. Let $P_e(a, a+\theta)$ denote the minimum probability of error in deciding between H_0 and H_1 for particular values of $a, a+\theta$. The $P_e(a, a+\theta)$ is derived from a likelihood ratio test (LRT), so it depends on the model for the channel gain vector, $\boldsymbol{\gamma}$. The conditional log-likelihood (LL) in (11) is used to find P_e for channel models 1 and 1A, with $\boldsymbol{\gamma}$ fixed in case 1 and averaged over $\boldsymbol{\gamma}$ in case 1A. For case 2, a generalized LRT formed with ML estimates of the channel gain parameters in (16) substituted into (11) reduces to a LRT based on (12), with $\boldsymbol{\gamma}$ fixed. Case 2A includes subsequent averaging over $\boldsymbol{\gamma}$. The P_e for case 3 is based on the unconditional LL in (12), and is found to be identical to case 2A. The ZZB for each case is obtained by using the corresponding P_e in the following expression,

$$\text{ZZB}(\widehat{\tau}) \geq \frac{1}{D}\int_0^D \theta\, \mathcal{V}[(D-\theta)P_e(\theta)]\, d\theta, \tag{28}$$

where $\mathcal{V}[\cdot]$ is a nonincreasing function that fills the valleys of the bracketed function, and $P_e(a, a+\theta) = P_e(\theta)$ is independent of a.

The P_e expressions for each case are obtained with the aid of expressions from [11, pp. 882–886] and [12, p. 319 and pp. 619–624]. First, we define

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp(-u^2/2)\, du \tag{29}$$

$$R(x, N) = \left(\frac{1-x}{2}\right)^{2N-1} \sum_{i=0}^{N-1} \binom{2N-1}{i} \left(\frac{1+x}{1-x}\right)^i$$

$$\mu(\theta) = \left[1 + \frac{2}{\overline{\text{SNR}} \cdot (1 - \text{Re}\,[\rho(\theta)]\,/\rho(0))}\right]^{-1/2} \tag{30}$$

$$\nu(\theta) = \left[1 + \frac{2\left[1 + \left(\overline{\text{SNR}}\right)^{-1}\right]}{\left(\overline{\text{SNR}}/2\right) \cdot \left(1 - |\rho(\theta)/\rho(0)|^2\right)}\right]^{-1/2} \tag{31}$$

$$S(\alpha, \beta, N) = \frac{1}{2} + \frac{1}{2^{2N-1}} \sum_{i=1}^{N} \binom{2N-1}{N-i} [Q_i(\alpha,\beta) - Q_i(\beta,\alpha)]$$

$$Q_i(\alpha, \beta) = \text{ generalized } i^{th} \text{ order Marcum Q-function [12]}$$

$$a(\theta, \boldsymbol{\gamma}, N) = \left[\frac{N}{2} \cdot \text{SNR}_{\text{det}}(\boldsymbol{\gamma}) \cdot \left(1 - \sqrt{1 - |\rho(\theta)/\rho(0)|^2}\right)\right]^{1/2}$$

$$b(\theta, \boldsymbol{\gamma}, N) = \left[\frac{N}{2} \cdot \text{SNR}_{\text{det}}(\boldsymbol{\gamma}) \cdot \left(1 + \sqrt{1 - |\rho(\theta)/\rho(0)|^2}\right)\right]^{1/2}$$

Then the P_e expressions for cases 1, 2, 1A, and 2A = 3 are

$$P_{e,1}(\theta\,|\,\boldsymbol{\gamma}) = Q\left(\sqrt{N \cdot \text{SNR}_{\text{det}}(\boldsymbol{\gamma}) \cdot \left(1 - \frac{\text{Re}\,[\rho(\theta)]}{\rho(0)}\right)}\right) \tag{32}$$

$$P_{e,2}(\theta) = S\left(a(\theta, \boldsymbol{\gamma}, N), b(\theta, \boldsymbol{\gamma}, N), N\right) \tag{33}$$

$$P_{e,1A}(\theta) = R\left(\mu(\theta), N\right) \tag{34}$$

$$P_{e,3}(\theta) = P_{e,2A}(\theta) = R\left(\nu(\theta), N\right). \tag{35}$$

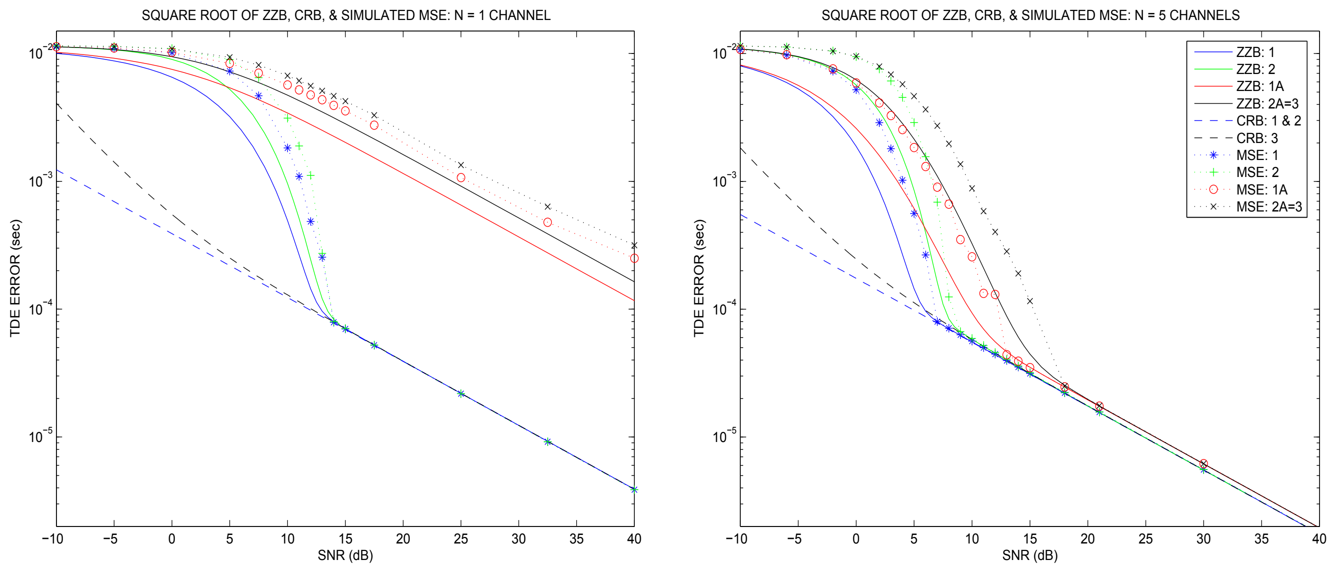

Fig. 1. Comparison of CRB, ZZB, and MSE for $N = 1$ and $N = 5$ channels.

Note that $P_{e,1A}$ and $P_{e,3} = P_{e,2A}$ have similar functional forms, so from (30) and (31) we can define an "SNR penalty",

$$\overline{\text{SNR}}_3 = \overline{\text{SNR}}_{2A} = \overline{\text{SNR}}_{1A} \cdot 2 \cdot \frac{1 - \text{Re}\left[\rho(\theta)\right]/\rho(0)}{1 - |\rho(\theta)/\rho(0)|^2}. \quad (36)$$

Case 1A assumes that the receiver has perfect knowledge of the channel gains, so the factor on the right side of (36) may be viewed as the additional SNR that is required in cases 2A and 3 to achieve the same P_e for a given θ as case 1A. This SNR penalty does not translate directly to the ZZB due to the integration over θ in (28).

It is possible to obtain simplified expressions for (32)-(35) at high SNR, leading to an analytical characterization of the SNR-threshold behavior of TDE on parallel channels. The details of this analysis will be elaborated elsewhere, but an interesting conclusion is as follows for the cases with random fading (1A and 2A=3). If the total signal power is held fixed as N is increased, so that $N \cdot \overline{\text{SNR}}$ remains fixed as N increases, then as long as the SNR per channel is $\gg 1$, $P_{e,1A}$ and $P_{e,3} = P_{e,2A}$ decrease as N increases. Therefore increased diversity reduces the SNR threshold for TDE on fading channels, while the increased diversity has no effect on the CRBs in (22)-(25) if $N \cdot \overline{\text{SNR}}$ is fixed.

4. NUMERICAL EXAMPLE AND SIMULATION

We present a numerical evaluation of the bounds and compare with the simulated mean-squared error (MSE) of the MLEs. The signal on each channel is a square-root, raised-cosine pulse with 0 excess bandwidth, period 10^{-3} sec, and unit energy. The MSEs are computed based on a minimum of 10,000 Monte Carlo runs at each SNR, with 22,000 runs for SNRs in the threshold region.

Figure 1 shows the bounds and simulated root-MSE for $N = 1$ and $N = 5$ channels for cases 1 (blue), 2 (green), 1A (red), and 3 (black). Note that for $N = 1$ channel (no diversity), the ZZBs with fading (1A and 2A = 3) are significantly larger than the CRB, and the CRB is not achievable an any SNR. For $N = 5$ channels, the ZZBs converge to the CRBs when the SNR is exceeds a threshold. For all cases, the SNR thresholds of the ZZB and MSE are similar.

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