# IDENTIFIABILITY OF THE PARAFAC MODEL FOR POLARIZED SOURCE MIXTURE ON A VECTOR SENSOR ARRAY

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## ABSTRACT

By means of the Parallel Factor (PARAFAC) decomposition, we present a novel method working on a vector-sensor array for blind separation of polarized sources in virtue of their distinct spatial and temporal signatures. Identifiability is studied, and explicit constraints on the sources are derived to ensure the data model identifiable. We show, by numerical simulations, that the estimation performance can approach that of non-blind estimation by optimally designing the source polarizations.

***Index Terms***— Multidimensional signal processing, Array signal processing, Polarization, Direction of arrival estimation, Identification

## 1. INTRODUCTION

Following the recent development of the electromagnetic vector sensor technology, it is now possible to consider polarization as an additional diversity for blind beamforming as proposed by [1]. An electromagnetic sensor is composed of 6 spatially collocated but diversely polarized antennas, measuring all 6 components of the incident electromagnetic field [2]. Regardless of the mutual interference and the noise effects, these measurements are shown to fit a three-way Parallel Factor (PARAFAC) model. In this paper, we address the problem of model identifiability using polarization diversity. We precisely state the physical constraints, under which this model is identifiable. Optimizing source polarizations is proposed to enhance the performance of this blind beamforming strategy. Furthermore, we illustrate by simulations the performance of source signal estimation comparing to those of non-blind source separation methods [3].

## 2. DATA MODEL

Consider a uniform array built up with $M$ identical sensors spaced by $\Delta x$ along the $x$-axis, collecting narrow-band signals emitted from $N$ ($N \leq M$ and known *a priori*) far-field sources. For the $n$th incoming wave, the direction of arrival (DOA) is determined by the elevation angle $\theta_n \in [0, \pi]$ (measured from $+z$-axis) and the azimuth angle $\phi_n \in [0, \pi]^1$ (measured from $+x$-axis), as defined in [3]. Under the far-field assumption, the steering vector for the $n$th impinging wave can be modeled in a Vandermonde structure as

$$\boldsymbol{a}_n \triangleq \left[1,\ a_n,\ \ldots,\ a_n^{M-1}\right]^T, \tag{1}$$

where $a_n = \exp(jk_0 \Delta x \sin\theta_n \cos\phi_n)$ is the inter-sensor phase shift and $k_0$ is the wave number.

We use a $2 \times 1$ complex vector

$$\boldsymbol{g}_n = \begin{bmatrix} \cos\alpha_n & -\sin\alpha_n \\ \sin\alpha_n & \cos\alpha_n \end{bmatrix} \begin{bmatrix} j\sin\beta_n \\ \cos\beta_n \end{bmatrix}$$

to describe the polarization state of the $n$th signal in terms of the orientation angle $\alpha_n \in (-\pi/2, \pi/2]$ and the ellipticity angle $\beta_n \in [-\pi/4, \pi/4]$ [4]. Suppose the signals are completely polarized, and the propagation takes place in an isotropic, homogeneous medium. The normalized electric and magnetic fields of the $n$th incoming wave, $\boldsymbol{e}_n$ and $\boldsymbol{h}_n$, can be put together in a $6 \times 1$ vector $\boldsymbol{b}_n$ in Cartesian coordinates [2]:

$$\boldsymbol{b}_n \triangleq \begin{bmatrix} \boldsymbol{e}_n \\ \boldsymbol{h}_n \end{bmatrix} = \begin{bmatrix} \cos\theta_n\cos\phi_n & -\sin\phi_n \\ \cos\theta_n\sin\phi_n & \cos\phi_n \\ -\sin\theta_n & 0 \\ -\sin\phi_n & -\cos\theta_n\cos\phi_n \\ \cos\phi_n & -\cos\theta_n\sin\phi_n \\ 0 & \sin\theta_n \end{bmatrix} \boldsymbol{g}_n. \tag{2}$$

For notation compactness, we use $\boldsymbol{\Pi}_n$ to denote the $6 \times 2$ matrix on the righthand side of (2), hence we have $\boldsymbol{b}_n = \boldsymbol{\Pi}_n \boldsymbol{g}_n$.

Assume no power loss during the wave propagation and neglect all the mutual coupling effects inner- or inter-sensors. Focus on the $m$th ($m = 1, 2, \ldots, M$) vector-sensor, which outputs 6 parallel discrete-time baseband-equivalent data flows simultaneously. Let $p$ ($p = 1, 2, \ldots, 6$) index the six field components. The $p$th output, $x_{m,p}(t_k)$ ($k = 1, 2, \ldots, K$ and $K \geq N$), is obtained by summing up all the contributions from the $N$ wavefronts, *i.e.*,

$$x_{m,p}(t_k) = \sum_{n=1}^{N} a_n^{m-1} [\boldsymbol{b}_n]_p s_n(t_k) + n_{m,p}(t_k), \tag{3}$$

with the following notations:

[1] We assume the sources are all coming from the $+y$ side of the $x - z$ plane.

$[\,\cdot\,]_p$ the $p$th component of a vector;
$s_n(t_k)$ $k$th temporal sample of the $n$th source signal;
$n_{m,p}(t_k)$ additive white noise, i.i.d. for $m$, $p$, and $t_k$.

Let us define respectively

$$\mathbf{A} \triangleq [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N] \tag{4}$$

$$\mathbf{B} \triangleq [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_N] = [\boldsymbol{\Pi g}_1, \ldots, \boldsymbol{\Pi g}_N] \tag{5}$$

$$\mathbf{S} \triangleq \begin{bmatrix} s_1(t_1) & \cdots & s_N(t_1) \\ \vdots & \ddots & \vdots \\ s_1(t_K) & \cdots & s_N(t_K) \end{bmatrix} \tag{6}$$

as the $M \times N$ array response, the $6 \times N$ polarization-dependent response of each vector-sensor, and $K \times N$ source signal matrix. We also define the $M \times K$ matrix

$$\tilde{\mathbf{X}}_p \triangleq \begin{bmatrix} x_{1,p}(t_1) & \cdots & x_{1,p}(t_K) \\ \vdots & \ddots & \vdots \\ x_{M,p}(t_1) & \cdots & x_{M,p}(t_K) \end{bmatrix} \tag{7}$$

collecting the compact data measured on the $p$th component of all $M$ sensors. If the entire data set $\{\tilde{\mathbf{X}}_p \mid p = 1, \ldots, 6\}$ is organized in an $M \times K \times 6$ tensor $\tilde{\mathcal{X}}$, the data model formulated in (3) can be re-expressed in the form

$$\tilde{\mathbf{X}}_p = \mathbf{A}\mathrm{D}_p(\mathbf{B})\mathbf{S}^T + \mathbf{N}_p, \quad p = 1, \ldots, 6., \tag{8}$$

where $\mathrm{D}_p(\mathbf{B}) = \mathrm{diag}\left([b_{p,1}, \ldots, b_{p,N}]\right)$ is a diagonal matrix which takes the $p$th row of $\mathbf{B}$ as its diagonal, and $b_{p,n} = [\boldsymbol{b}_n]_p$ is the $(p,n)$th entry of $\mathbf{B}$.

Equations (8) clearly expresses a 3-way PARAFAC structure of the recorded data [5].

## 3. IDENTIFIABILITY AND BLIND BEAMFORMING PERFORMANCE

### 3.1. Identifiability of Noise-Free Data Model

Let $\mathcal{X} \triangleq \tilde{\mathcal{X}} - \mathcal{N}$ be the noise-free data, then the $p$th slice, $\mathbf{X}_p$, is equal to

$$\mathbf{X}_p = \mathbf{A}\mathrm{D}_p(\mathbf{B})\mathbf{S}^T, \quad p = 1, \ldots, 6, \tag{9}$$

and $\mathcal{X}$ can be unfolded into a $6M \times K$ matrix:

$$\begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_6 \end{bmatrix} = \begin{bmatrix} \mathbf{A}\mathrm{D}_1(\mathbf{B})\mathbf{S}^T \\ \vdots \\ \mathbf{A}\mathrm{D}_6(\mathbf{B})\mathbf{S}^T \end{bmatrix} = (\mathbf{B} \odot \mathbf{A})\mathbf{S}^T \tag{10}$$

where $\odot$ is the Khatri-Rao (column-wise Kronecker) product.

To obtain a unique and valid solution for the inverse problem posed in (10), identifiability must be studied before separating the source signal mixture. Kruskal's condition is a sufficient condition for unique PARAFAC decomposition, relying on the concept defined as Kruskal-rank or $k$-rank [6].

*K-rank*: Given a matrix $\mathbf{A} \in \mathbb{C}^{I \times J}$, if every linear combination of $l$ columns has full column rank, but this condition does not hold for $l+1$, then the $k$-rank of $\mathbf{A}$ is $l$, written as $k_{\mathbf{A}} = l$.

Note that $k_{\mathbf{A}} \leq \mathrm{rank}(\mathbf{A}) \leq \min(I, J)$, and both the equalities hold when $\mathrm{rank}(\mathbf{A}) = J$.

Kruskal's condition was first established for trilinear decomposition of real-valued arrays [6], and later extended by [7] to complex-valued cases. In our context, this uniqueness condition can be formulated as follows.

*Kruskal's condition*: Consider a 3-way array $\mathcal{X}$ that can be unfolded into matrix form as in (10). Decomposition into three matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{S}$ is unique up to column permutation and scaling ambiguities, if but not necessarily

$$k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{S}} \geq 2(N+1). \tag{11}$$

In separating two sources, if one source comes from $(\theta, \phi)$ and the other from $(\pi - \theta, \phi)$, the linear independence between $\mathbf{A}$'s columns is violated, which makes the model unidentifiable. An $L$-shaped array is herein adopted to eliminate this ambiguity. The $L$-shaped array involved here is constructed by posing $M_z$ sensors along the $+z$-axis and the $M - M_z$ others along the $+x$-axis, resulting in the following expression for the steering vector:

$$\boldsymbol{a}_n = \left[a_{n,z}^{M_z - 1}, \ldots, a_{n,z}, 1, a_{n,x}, \ldots a_{n,x}^{M - M_z}\right]^T, \tag{12}$$

where $a_{n,z} = \exp(jk_0 \Delta x \cos\theta_n)$,
$a_{n,x} = \exp(jk_0 \Delta x \sin\theta_n \cos\phi_n)$.

Now we make the following assumptions on the mixture of unknown sources in order to to satisfy Kruskal's condition.

(A1) Sources have distinct DOAs, *i.e.*, any two sources have at least one different parameter, either $\theta$ or $\phi$.

(A2) Each source sequence is drawn from an unknown stochastic process of continuous distribution.

Due to the particular structure of $\mathbf{A}$'s columns depicted in (12), it is straightforward to show that (A1) guarantees $\mathrm{rank}(\mathbf{A}) = N$ and hence $k_{\mathbf{A}} = N$. Since (A2) imposes the full rank condition on the signal matrix $\mathbf{S}$, it follows that $k_{\mathbf{S}} = N$. If $k_{\mathbf{B}} \geq 2$ is further verified, then Kruskal's condition can be satisfied to achieve unique PARAFAC decomposition. We will prove this by contradiction.

Assume $k_{\mathbf{B}} < 2$, then there must exist at least one pair of linear dependent columns, namely $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, so that

$$|\boldsymbol{b}_1^H \boldsymbol{b}_2| = \|\boldsymbol{b}_1\| \|\boldsymbol{b}_2\|. \tag{13}$$

Let $(\theta_1, \phi_1)$ and $(\theta_2, \phi_2)$ denote the DOA of any two sources, and assume $\phi_2 > \phi_1$ with the difference $\Delta\phi = \phi_2 - \phi_1$. With $\Sigma$ and $\varepsilon$ respectively given by

$$\begin{aligned} \Sigma = &\{(\cos\theta_1 + \cos\theta_2)^2 \sin^2\Delta\phi \\ &+ [\sin\theta_1 \sin\theta_2 + (1 + \cos\theta_1 \cos\theta_2)\cos\Delta\phi]^2\}^{\frac{1}{2}} \end{aligned} \tag{14}$$

$$\varepsilon = \tan^{-1} \frac{(\cos\theta_1 + \cos\theta_2)\sin\Delta\phi}{\sin\theta_1 \sin\theta_2 + (1 + \cos\theta_1 \cos\theta_2)\cos\Delta\phi} \tag{15}$$

after tedious computations, one may have

$$\mathbf{\Pi}_1^H \mathbf{\Pi}_2 = \Sigma \begin{bmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{bmatrix}. \tag{16}$$

Denote $\Sigma_{\max}$ as the maximum of $\Sigma$, then the ratio $\Sigma/\Sigma_{\max}$ reflects the angle between the wavefronts of the two sources, and $\varepsilon$ depicts the angle between their electric/magnetic fields.

Define $\Omega_\varepsilon$ that satisfies $\cos\Omega_\varepsilon = \sin 2\beta_1 \sin 2\beta_2 + \cos 2\beta_1 \cos 2\beta_2 \cos 2(\alpha_2 - \alpha_1 + \varepsilon)$, and denote $\Omega_0 = \Omega_\varepsilon(\varepsilon = 0)$, which is the polarization separation[2] defined in [4], then we have

$$|\boldsymbol{b}_1^H \boldsymbol{b}_2| = |\boldsymbol{g}_1^H \mathbf{\Pi}_1^H \mathbf{\Pi}_2 \boldsymbol{g}_2| = \Sigma\sqrt{(1+\cos\Omega_\varepsilon)/2}. \tag{17}$$

Since $\Sigma$ is defined for $\theta_1, \theta_2 \in [0, \pi]$ and $\Delta\phi \in [0, \pi]$, the maximum of $\Sigma$ is achieved only if the partial derivatives

$$\frac{\partial\Sigma}{\partial\theta_1} = 0,\ \frac{\partial\Sigma}{\partial\theta_2} = 0,\ \frac{\partial\Sigma}{\partial\Delta\phi} = 0,$$

which result in $\Sigma_{\max} = 2$, when $(\theta_1, \phi_1) = (\theta_2, \phi_2)$ or $\theta_1 = \theta_2 = 0$, *i.e.*, the DOAs of the sources are identical; hence $\|\boldsymbol{b}_2\| = \|\boldsymbol{b}_1\| = \sqrt{|\boldsymbol{b}_1^H \boldsymbol{b}_1|} = \Sigma_{\max}^{1/2}$. On the contrary, when sources have distinct DOAs, *i.e.* (A1) holds, (17) yields

$$|\boldsymbol{b}_1^H \boldsymbol{b}_2| = \Sigma\sqrt{(1+\cos\Omega_\varepsilon)/2} \le \Sigma < \Sigma_{\max} = \|\boldsymbol{b}_1\|\|\boldsymbol{b}_2\|, \tag{18}$$

which contradicts (13) and fulfils the proof of $k_{\mathbf{B}} \ge 2$.

In general, the assumptions (A1) and (A2) can sufficiently satisfy Kruskal's condition (11) and ensure that the model is identifiable for most occasions.

### 3.2. Blind Beamforming Performance

To assess the noise effect on the beamforming performance, consider the average input SNR at the receiver defined as [7]

$$\mathrm{SNR} = 10\log_{10}\frac{\|\mathcal{X}\|_F^2}{E\|\mathcal{N}\|_F^2} \tag{19}$$

where $\|\cdot\|_F$ stands for the Frobenius norm, and $E(\cdot)$ denotes the statistical expectation. Note that the power of source signals $\|\mathbf{S}\|_F^2$ is determined by the transmitters, thus

$$\|\mathcal{X}\|_F = \|(\mathbf{B}\odot\mathbf{A})\mathbf{S}^T\|_F \le \|\mathbf{B}\odot\mathbf{A}\|_F \cdot \|\mathbf{S}\|_F \tag{20}$$

and equality holds only if all the columns of $\mathbf{B}\odot\mathbf{A}$ are orthogonal, that is,

$$(\mathbf{B}\odot\mathbf{A})^H(\mathbf{B}\odot\mathbf{A}) = (\mathbf{B}^H\mathbf{B}) \circ (\mathbf{A}^H\mathbf{A}) \propto \mathbf{I}_N, \tag{21}$$

where $\circ$ is the Hadamard (element-wise) product.

[2]Polarization separation is defined when two sources have the same DOA in [4]. If two sources have different DOAs, $\mathbf{\Pi}_1^H\mathbf{\Pi}_2$ projects the wavefront of the 2nd source onto that of the 1st one, so that we can still quantify their polarization separation as if they were from the same DOA.

Restricted by $\|\mathbf{A}\|_F = M\sqrt{N}$ and $\|\mathbf{B}\|_F = \Sigma_{\max}\sqrt{N} = 2\sqrt{N}$, (21) holds only if both $\mathbf{A}$ and $\mathbf{B}$ are orthogonal, and the maximum of $\|\mathcal{X}\|_F$ can be achieved, written as

$$\max_{\mathbf{A},\mathbf{B}} \|\mathcal{X}\|_F = \max_{\mathbf{A},\mathbf{B}} \|\mathbf{B}\odot\mathbf{A}\|_F \cdot \|\mathbf{S}\|_F = 2M\sqrt{N}\|\mathbf{S}\|_F. \tag{22}$$

Normally, neither $\mathbf{A}$ nor $\mathbf{B}$ is orthogonal; however, given the same power of source signals and the same noise level, reducing the inner-products between their inter-columns can increase the average input SNR of the observations, and hence improves the performance of blind beamforming. The source polarizations are more flexible for performance enhancement compared to their DOAs. Recall (5) that relates the matrix $\mathbf{B}$ to all source polarizations $\boldsymbol{g}_1, \ldots, \boldsymbol{g}_N$. Since the diagonal of $\mathbf{B}^H\mathbf{B}$ equals $\Sigma_{\max}\mathbf{I}_N$, given $\mathbf{A}$, $\mathbf{S}$, and $E\|\mathcal{N}\|_F^2$, (19) can be optimized by selecting a set of polarizations from $\mathbf{G} = \{\boldsymbol{g}_n \mid \|\boldsymbol{g}_n\| = 1, n = 1, \ldots, N\}$ to fulfil

$$\min_{\boldsymbol{g}_1,\ldots,\boldsymbol{g}_N \in \mathbf{G}} \|\mathbf{B}^H\mathbf{B} - \Sigma_{\max}\mathbf{I}_N\|_F^2. \tag{23}$$

*i.e.*, minimizing the overall mutual interference power. Optimizing (23) can be accomplished via the Nelder-Mead simplex algorithm (implemented in MATLAB as "fminsearch" function) with respect to $[\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N]^T$, which is defined on a continuous convex domain rather than $\mathbf{G}$.

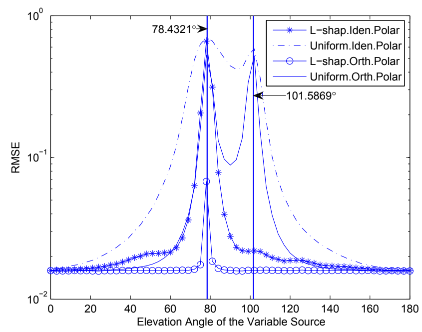
## 4. SIMULATIONS

The COMFAC algorithm [7] is adopted to achieve fast, accurate convergence for factorization of trilinear complex-valued tensors. Monte Carlo simulations are designed to evaluate the performance of the proposed algorithm in terms of the root mean square error (RMSE), as given by

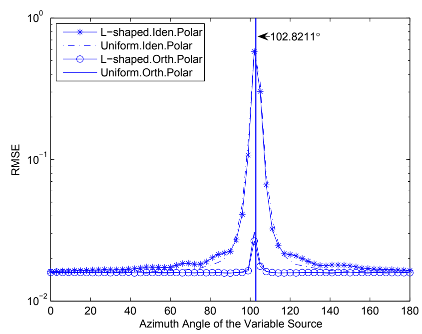
$$\mathrm{RMSE} = \sqrt{\frac{1}{LNK}\sum_{l=1}^{L}\|\hat{\mathbf{S}}_l - \mathbf{S}\|_F^2}, \tag{24}$$

where $\hat{\mathbf{S}}_l$ is the estimate of $\mathbf{S}$ obtained in the $l$th trial. $L = 500$ independent experiments contribute to each data point in these figures. All the noise is assumed to be Gaussian.

The effects on identifiability of the model while using the $L$-shaped sensor array are compared to those of a uniform array, as shown in Fig.1. A number of $M = 13$ sensors and $K = 50$ snapshots are used for both types of array. SNR is 20dB for all these simulations. One source is nominated the reference source, which has a set of fixed parameters as $\theta_1 = 78.4°$, $\phi_1 = 102.8°$, $\alpha_1 = 35.8°$, and $\beta_1 = 32.9°$; while the other one, namely the variable source, varies either on $\theta_2$ (Fig.1(a)) or on $\phi_2$ (Fig.1(b)). Fig.1(a) shows the advantage of using $L$-shaped array over the uniform array on the model identifiability by eliminating the ambiguity of the two sources having supplementary elevation angles. Fig.1 numerically verifies the efficiency of the assumptions (A1) and (A2) as sufficient conditions for the identifiability of this mixture model.




(a) $\phi_1 = \phi_2 = 102.8°$ varying $\theta_2$




(b) $\theta_1 = \theta_2 = 78.4°$ varying $\phi_2$

**Fig. 1**: RMSE of Signal Estimation vs. Angular Separation of Sources

Other than the $L$-shaped array manifold, the blind beamforming performance can also be enhanced by the optimal design of source polarizations, especially when the two sources are closely located, as shown in both Fig.1(a) and (b). Comparing the curves of identically polarized sources with those of orthogonally polarized, it shows that polarization separation becomes as an essential factor on the performance of blind source separation if lacking of angular separation between the two sources. As two sources are getting closer to each other, $\varepsilon \rightarrow 0$ and $\Sigma \rightarrow \Sigma_{\max}$, from (17), $|\boldsymbol{b}_1^H \boldsymbol{b}_2| \approx \Sigma_{\max}\sqrt{(1+\cos\Omega_0)/2}$, which indicates the source polarization separations can completely determine the column inner-products of $\mathbf{B}$, and hence the performance.

Assuming that the parameters are all known *a priori* or perfectly estimated, the source signals can be recovered under least mean square (LMS) criterium [3]. Fig.2 shows the performance of using our method compared to the non-blind LMS estimators, where $M = 6$ sensors are used to separate $N = 4$ closely posed sources. By optimally designing the source polarization according to (23), the performance of the proposed algorithm can be enhanced by approximately 10dB, approaching to that of the non-blind separation methods.

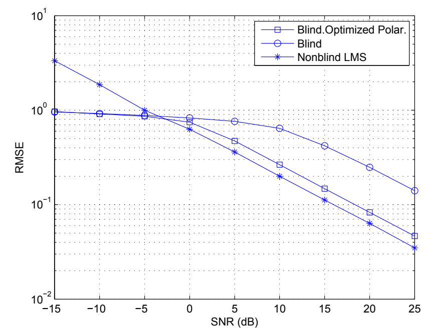



**Fig. 2**: Blind PARAFAC vs. Non-blind LMS

## 5. CONCLUSIONS

A link has been established between three-way PARAFAC decomposition and the vector-sensor array for blind beamforming for polarized signals. We proved that, with an $L$-shaped array, identifiability can always be achieved as long as the sources have distinct DOAs. Polarization is not essential to the identifiability of this model; however, optimal selection of the source polarizations yields performance improvement for the source separation. In particular, even if the sources are closely located, simulation results show that the performance of the proposed algorithm is still close to those of classical non-blind source separation methods by optimally designing the polarization of the sources.

## 6. REFERENCES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